HW 2 #3(f) Show that $\lim_{n\to\infty} \frac{\sqrt{n^2+1}}{n!} = 0$

proof: Let $\varepsilon > 0$

Note that $\left|\frac{\sqrt{n^2+1}}{n!} - 0\right| = \left|\frac{\sqrt{n^2+1}}{n!}\right| = \frac{\sqrt{n^2+1}}{n!} \underset{\substack{\uparrow \\ 1\le n \\ \text{so } 1\le n^2}}{\le} \frac{\sqrt{n^2+n^2}}{n!}$

$= \frac{\sqrt{2n^2}}{n!} = \frac{\sqrt{2}\,n}{n!} = \frac{\sqrt{2}\,\cancel{n}}{\cancel{n}(n-1)!} = \frac{\sqrt{2}}{(n-1)!} \underset{\uparrow}{\le} \frac{\sqrt{2}}{2^{n-2}}$

$(n-1)! = (n-1)(n-2)\cdots(3)(2)(1)$

If $n \ge 3$ then

$(n-1)! \ge \underbrace{2\cdot 2\cdot 2\cdots 2}_{n-2 \text{ of them}} = 2^{n-2}$

so if $n \ge 3$ then $\frac{1}{(n-1)!} \le \frac{1}{2^{n-2}}$

$n = 4$

$(n-1)! = 3\cdot 2\cdot 1$

2

$n = 5$

$(n-1)! = 4\cdot 3\cdot 2\cdot 1$

3

Suppose $n \ge 3$, then $\frac{\sqrt{2}}{2^{n-2}} < \varepsilon$ iff $\frac{\sqrt{2}}{\varepsilon} < 2^{n-2}$

$\underset{\uparrow}{\text{iff}}$ $\log_2\left(\frac{\sqrt{2}}{\varepsilon}\right) < n-2$ iff $\log_2\left(\frac{\sqrt{2}}{\varepsilon}\right) + 2 < n$

$x < y$ iff $\log(x) < \log(y)$

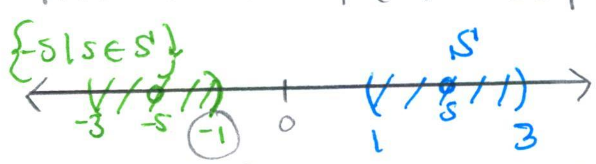
Set $N > \max\{3, \log_2\left(\frac{\sqrt{2}}{\varepsilon}\right) + 2$. If $n \ge N$, then

$\left|\frac{\sqrt{n^2+1}}{n!} - 0\right| \underset{\substack{\uparrow \\ n\ge 3}}{<} \frac{\sqrt{2}}{2^{n-2}} \underset{\substack{\uparrow \\ n \ge \log_2\left(\frac{\sqrt{2}}{\varepsilon}\right)+2}}{<} \varepsilon$ ▨

Test 1 (B) Let S be a non-empty subset of $\mathbb{R}$ that is bounded from below. Prove that $\inf(S) = -\sup\{-s \mid s \in S\}$

Ex: $S = (1,3)$

$\inf(S) = 1$



$-\sup\{-s \mid s \in S\} = -(-1) = 1 = \inf(S)$

proof: Since $S \ne \emptyset$ and S is bounded from below, $\inf(S)$ exists, let $x = \inf(S)$

Let's show $-x = \sup\{-s \mid s \in S\}$

1. Since $x = \inf(S)$ we know $x \le s \ \forall s \in S$

   Then $-x \ge -s \ \forall s \in S$.

   so $-x$ is an upper bound for $\{-s \mid s \in S\}$.

② Let's show that $-x$ is the least upper bound for $-S = \{-s \mid s \in S\}$

Let c be an upper bound for $-S$

Then, $-s \leq c \quad \forall\, s \in S$

so $\quad s \geq -c \quad \forall\, s \in S$

So $-c$ is a lower bound for S.

since $x = \inf(S)$, ie the greatest lower bound of S,

then $-c \leq x$.

so $c \geq -x$. so, $-x$ is the least upper bound of $-S$ ☒

② (another way to show that $-x = \sup(-S)$)

We already know from part ① that $-x$ is an upper bound for $-S$.

Let $\varepsilon > 0$

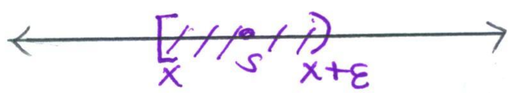
If we can find $s \in S$ with

$$-x - \varepsilon < -s \leq -x;$$

Then by the useful sup/inf fact $-x = \sup(-S)$

since $x = \inf(S)$ $\exists\, s \in S$ with $x \leq s < x + \varepsilon$

By the useful sup/inf fact



multiply by (-1) to get $-x - \varepsilon < -s \leq -x$ ☒

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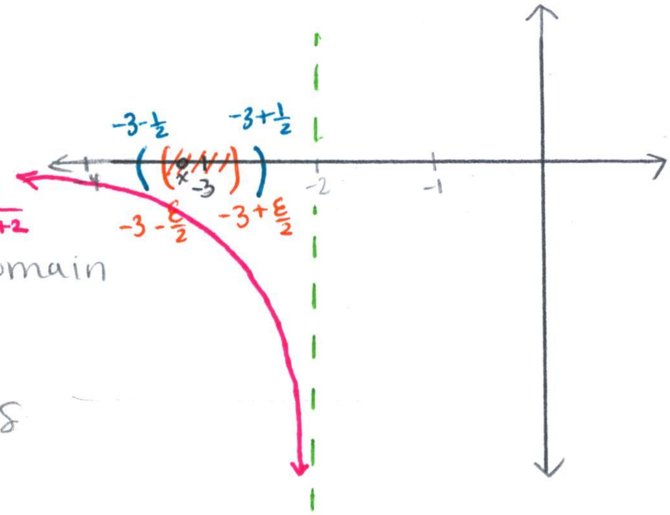
Limits Continued...

Ex: Prove that $\lim_{x \to -3} \frac{1}{x+2} = -1$.



Proof: let $\varepsilon > 0$

let $D = \mathbb{R} \setminus \{-2\}$ ← where D is the domain

We need to find $\delta > 0$

where if $x \in D$ and $0 < |x-(-3)| < \delta$

then $\left|\frac{1}{x+2} - (-1)\right| < \varepsilon$

Note that, $\left|\frac{1}{x+2} \underbrace{-(-1)}_{\frac{x+2}{x+2}}\right| = \left|\frac{1+(x+2)}{x+2}\right| = \left|\frac{x+3}{x+2}\right|$

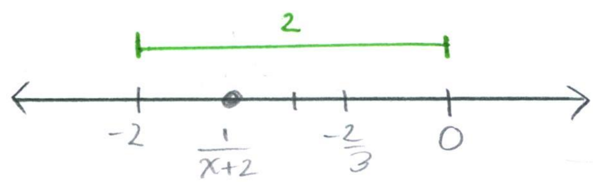
$= \underbrace{|x+3|}_{\text{we can control/bound with } \delta} \cdot \underbrace{\frac{1}{|x+2|}}_{\text{need to get rid of this guy by using a standing bound on } \delta}$

suppose $\delta \leq \frac{1}{2}$

let's try to bound $\frac{1}{|x+2|}$

If $\underbrace{|x+3|}_{|x-(-3)|} < \frac{1}{2}$, then $-\frac{1}{2} < x+3 < \frac{1}{2}$ so, $-\frac{3}{2} < x+2 < \frac{-1}{2}$

Then, $-\frac{2}{3} > \frac{1}{x+2} > -2$



Summarizing, if $|x+3| < \frac{1}{2}$, then $\left|\frac{1}{x+2}\right| = \frac{1}{x+2} < 2$

If $|x+3| < \frac{1}{2}$, then

$\left|\frac{1}{x+2} - (-1)\right| = |x+3| \cdot \frac{1}{|x+2|} < 2|x+3|$

Let $\delta = \min\{\frac{1}{2}, \frac{\varepsilon}{2}\}$. If $0 < |x+3| < \delta$, then

$\left|\frac{1}{x+2} - (-1)\right| \underset{\uparrow\, |x+3| < \frac{1}{2}}{<} 2|x+3| \underset{\uparrow\, |x+3| < \frac{\varepsilon}{2}}{<} 2\left(\frac{\varepsilon}{2}\right) = \varepsilon$

$\boxtimes$