Compactness: $S \subseteq \mathbb{R}$ is compact if every open cover of S contains a finite subcover.

Heine-Borel Theorem:

let $K \subseteq \mathbb{R}$

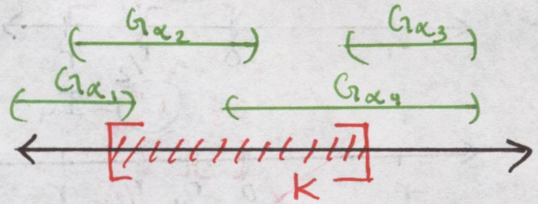
K is compact iff K is closed and bounded

proof:

($\Rightarrow$) See Handout.

($\Leftarrow$) suppose K is closed and bounded

Let $G = \{G_\alpha\}$ be an open cover of K. That is, each G_α is open and $K \subseteq \bigcup_\alpha G_\alpha$. We want to show that K is contained in some finite subcover from G.



note: this picture only has a finite # of G_α but there are infinitely many in G in proof.

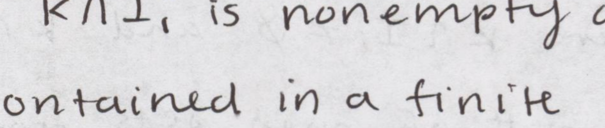
We prove this by contradiction.

Suppose that K is not contained in any finite subcover from G.

By hypothesis K is bounded.

So $K \subseteq [-r, r]$ for some $r > 0$

let $I_1 = [-r, r]$



Bisect I_1 into two intervals $I_1' = [r, 0]$ and $I_1'' = [0, r]$

Then at least one of $K \cap I_1'$ or $K \cap I_1''$ is nonempty and has the property that it is not contained in a finite subcover from G.