Test 2, Math 209, Fall 2015 Name: Solutions

**Directions:** Show all of your work to get credit. No calculators. Good luck!

1. [5 points] Find the maximum and minimum values of

$$f(x,y) = 2x^2 + 3y^2 - 4x - 5$$

subject to the constraint $\underbrace{x^2 + y^2}_{g} = 16$.

$$\begin{array}{r} 36 \\ +11 \\ \hline 47 \end{array} \qquad \begin{array}{r} 32 \\ +16 \\ \hline 48 \\ -5 \\ \hline 43 \end{array}$$

$f_x = 4x - 4$ $\qquad$ $g_x = 2x$

$f_y = 6y$ $\qquad$ $g_y = 2y$

$$\boxed{\begin{array}{l} \nabla f = \lambda \nabla g \\ g = k \end{array}}$$

$$\boxed{\begin{array}{l} \langle 4x-4, 6y \rangle = \langle 2\lambda x, 2\lambda y \rangle \\ x^2 + y^2 = 16 \end{array}}$$

$$\boxed{\begin{array}{ll} 4x - 4 = 2\lambda x & (1) \\ 6y = 2\lambda y & (2) \\ x^2 + y^2 = 16 & (3) \end{array}}$$

**case 1: $y = 0$**

Plug this into (3) to get $x^2 = 16$ or $x = \pm 4$.

$(x,y) = (4,0), (-4,0)$

**case 2; $y \neq 0$**

Then (2) becomes $2\lambda = 6$ or $\lambda = 3$.

Then (1) becomes $4x - 4 = 2(3)x$ or $x = -2$.

Plug this into (3) to get $y^2 = 12$. or $y = \pm\sqrt{12}$.

So, $(x,y) = (-2, \pm\sqrt{12})$

**Plug in the points**

$f(4,0) = 2\cdot 4^2 + 3\cdot 0^2 - 4\cdot 4 - 5$

$= 32 - 16 - 5 = 11$

$f(-4,0) = 2\cdot(-4)^2 + 3\cdot 0^2 - 4(-4) - 5$

$= 32 + 16 - 5 = 43$

$f(-2, \pm\sqrt{12}) = 2(-2)^2 + 3(\pm\sqrt{12})^2 - 4(-2) - 5$

$= 8 + 3(12) + 8 - 5$

$= 47$

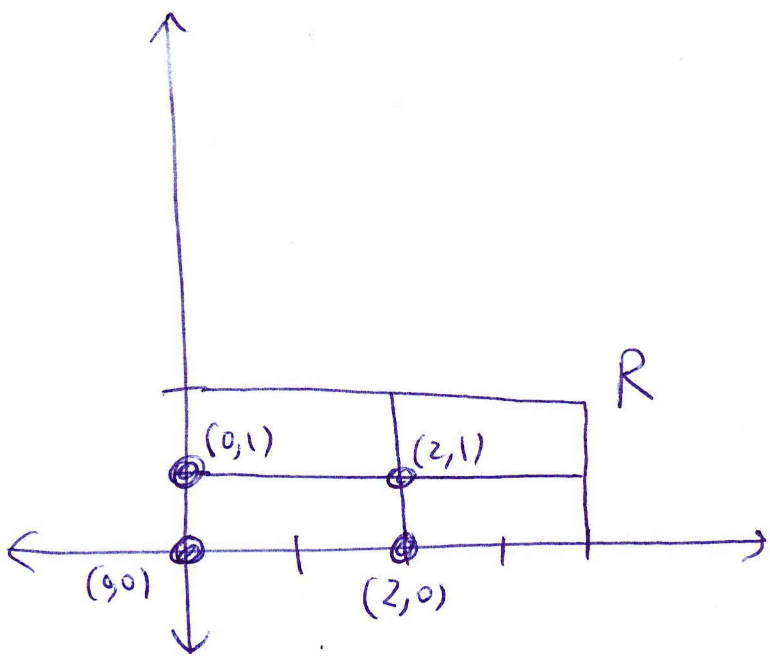
Answer

Max is at $(-2, \pm\sqrt{12})$

Min at $(4,0)$

2. [5 points] Estimate the volume of the solid that lies below the surface $z = x + 2y^2$ and above the rectangle $R = [0, 4] \times [0, 2]$. Use a Riemann sum with $m = 2$ and $n = 2$ and choose the sample points to be the lower left corners.

Recall that $m$ is how many times to subdivide the $x$ portion of $R$ and $n$ is the number of times to subdivide the $y$ portion of $R$.




$\boxed{f(x,y) = x + 2y^2}$

Area of each box is 2

$$\text{estimate} = 2f(0,0) + 2f(2,0) + 2f(0,1) + 2f(2,1)$$
$$= 2\left[(0+2\cdot 0^2) + (2+2\cdot 0^2) + (0+2\cdot 1^2) + (2+2\cdot 1^2)\right]$$
$$= 2\left[0+2+2+4\right] = 16$$

3. [5 points] Find the volume of the solid that lies under the hyperbolic paraboloid $z = 4 + x^2 - y^2$ and above the rectangle $R = [0, 1] \times [0, 2]$.

$$\int_0^1 \int_0^2 (4+x^2-y^2)\,dy\,dx$$

$$= \int_0^1 \left( 4y + x^2 y - \frac{y^3}{3} \right) \Big|_0^2 dx$$

$$= \int_0^1 \left[ \left( 4(2) + x^2(2) - \frac{2^3}{3} \right) - (0) \right] dx$$

$$= \int_0^1 \left( 8 + 2x^2 - \frac{8}{3} \right) dx \qquad \boxed{8 - \frac{8}{3} = \frac{24-8}{3} = \frac{16}{3}}$$

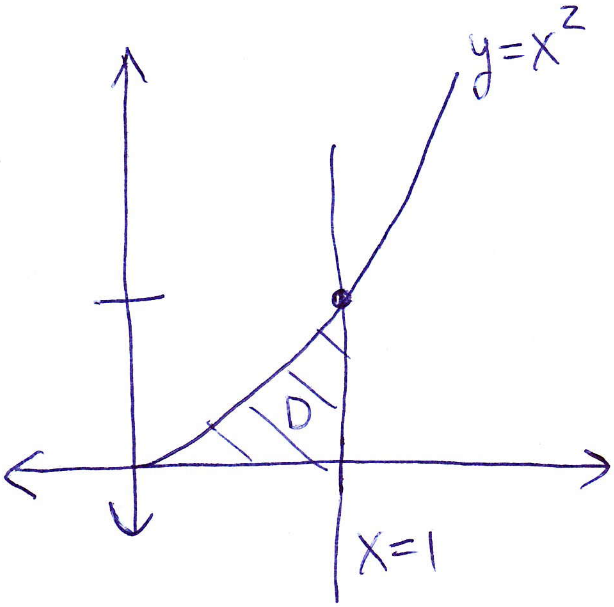
$$= \int_0^1 \left( 2x^2 + \frac{16}{3} \right) dx = \frac{2x^3}{3} + \frac{16}{3}x \Big|_0^1$$

$$= \left( \frac{2}{3} + \frac{16}{3} \right) - (0) = \frac{18}{3} = 6$$

4. [5 points] Calculate the double integral

$$\iint_D x\cos(y)\, dA$$

where $D$ is the region that is bounded by $y = 0$, $y = x^2$, and $x = 1$.




D is given by

$$\boxed{\begin{matrix} 0 \le x \le 1 \\ 0 \le y \le x^2 \end{matrix}}$$

OR

$$\boxed{\begin{matrix} 0 \le y \le 1 \\ \sqrt{y} \le x \le 1 \end{matrix}}$$

$$\int_0^1 \int_0^{x^2} x\cos(y)\,dy\,dx = \int_0^1 x\sin(y)\Big|_0^{x^2} dx$$

$$= \int_0^1 \left[x\sin(x^2) - x\sin(0)\right] dx$$

$$= \int_0^1 x\sin(x^2)\,dx = \int_0^1 \frac{1}{2}\sin(u)\,du =$$

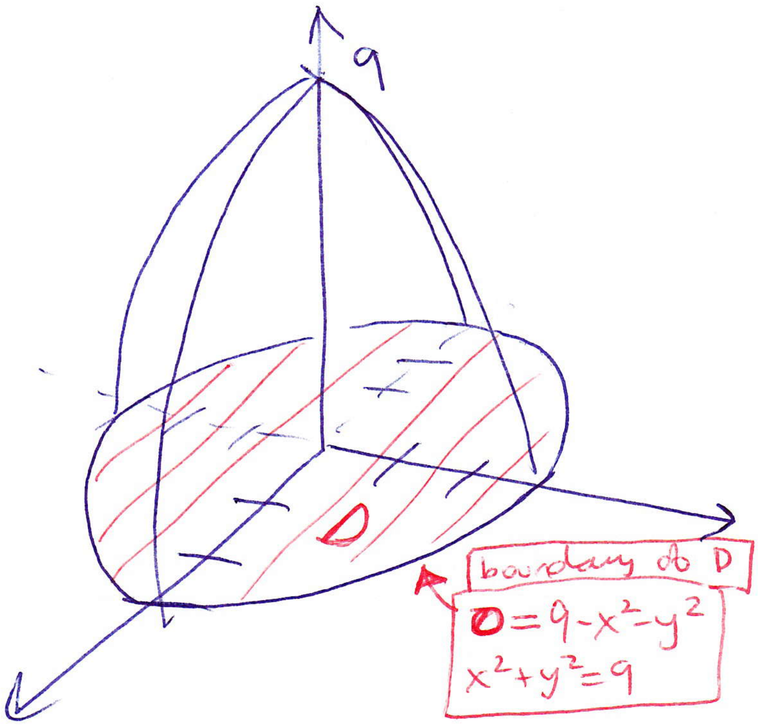
$$\boxed{\begin{matrix} u = x^2 \\ du = 2x\,dx \\ \frac{1}{2}du = x\,dx \\ x=0 \rightarrow u=0 \\ x=1 \rightarrow u=1 \end{matrix}}$$

$$= \frac{1}{2}\left(-\cos(u)\right)\Big|_0^1$$

$$= \frac{1}{2}\left[-\cos(1) - (-\cos(0))\right]$$

$$= \boxed{-\frac{1}{2}\cos(1) + \frac{1}{2}}$$

5. [5 points] Find the volume of the solid that lies under the paraboloid $z = 9 - x^2 - y^2$ and above the $xy$-plane.




D is given by

$0 \le r \le 3$

$0 \le \theta \le 2\pi$

$$\iint_D (9 - x^2 - y^2)\, dA = \int_0^{2\pi}\int_0^3 (9 - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^3 (9r - r^3)\, dr\, d\theta = \int_0^{2\pi} \left(\frac{9r^2}{2} - \frac{r^4}{4}\right)\Big|_0^3 d\theta$$

$$= \int_0^{2\pi} \left(\frac{9 \cdot 9}{2} - \frac{81}{4}\right) d\theta = \int_0^{2\pi} \underbrace{\left(\frac{81}{2} - \frac{81}{4}\right)}_{\frac{2 \cdot 81 - 81}{4} = \frac{81}{4}} d\theta = \int_0^{2\pi} \frac{81}{4}\, d\theta$$

$$= 2\pi\left(\frac{81}{4}\right)$$

$$= \boxed{\frac{81\pi}{2}}$$

6. [10 points - 5 each] Consider the function

$$f(x, y) = x^2 - 2xy + 2y$$

(a) Find the critical points of $f$ and determine if they are local minimums, local maximums, or saddle points. Recall that $D = f_{xx}f_{yy} - f_{xy}^2$.

$$\left.\begin{aligned} f_x &= 2x - 2y = 0 \\ f_y &= -2x + 2 = 0 \end{aligned}\right] \Longleftrightarrow x = 1$$

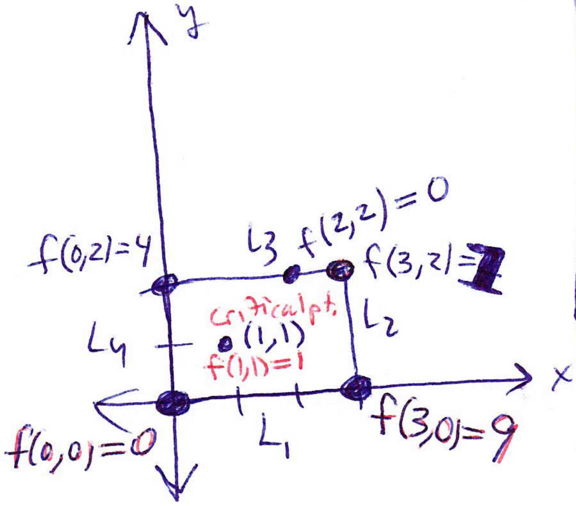
$f_x = 2x - 2y = 0 \longrightarrow 2(1) - 2y = 0$ (plug in $x=1$) $\quad y = 1$

So, $(x,y) = (1,1)$ is the only critical point.

$$\left.\begin{aligned} f_{xx} &= 2 \\ f_{yy} &= 0 \\ f_{xy} &= -2 \end{aligned}\right\} \quad D = (2)(0) - (-2)^2 = -4 < 0$$

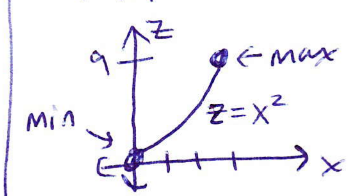
So, $(1,1)$ is a saddle point.

(b) Find the absolute maximum and absolute minimum of $f$ on the rectangle $D = [0, 3] \times [0, 2]$



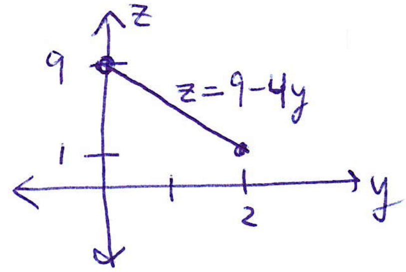

$L_1$ is given by $y=0$ and $0 \le x \le 3$

$f(x,0) = x^2$




$L_2$ is given by $x=3$ and $0 \le y \le 2$

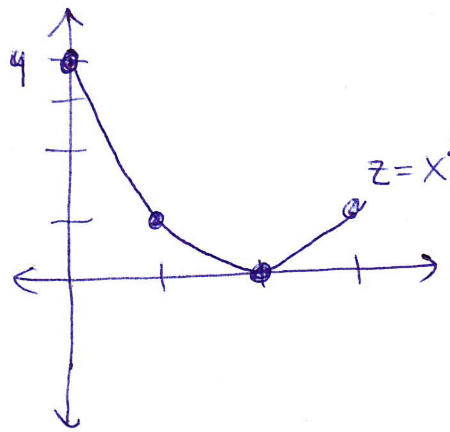
$f(3,y) = 9 - 6y + 2y = 9 - 4y$



6. continued... (extra space)

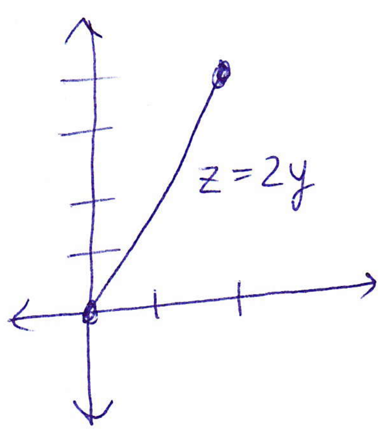
$L_3$ is given by $y=2,\ 0 \le x \le 3$

$$f(x,2) = x^2 - 4x + 4$$




$L_4$ is given by $x=0$ and $0 \le y \le 2$

$$f(0,y) = 2y$$




Answer