On $(d, 1)$-Total Numbers of Graphs

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Abstract

A $(d, 1)$-total labelling of a graph G assigns integers to the vertices and edges of G such that adjacent vertices receive distinct labels, adjacent edges receive distinct labels, and a vertex and its incident edges receive labels that differ in absolute value by at least d. The span of a $(d, 1)$-total labelling is the maximum difference between two labels. The $(d, 1)$-total number, denoted $\lambda_d^T(G)$, is defined to be the least span among all $(d, 1)$-total labellings of G. We prove new upper bounds for $\lambda_d^T(G)$, compute some $\lambda_d^T(K_{m,n})$ for complete bipartite graphs $K_{m,n}$, and completely determine all $\lambda_d^T(K_{m,n})$ for $d = 1, 2, 3$. We also propose a conjecture on an upper bound for $\lambda_d^T(G)$ in terms of the chromatic number and the chromatic index of G.

Key words: channel assignment, $L(2, 1)$-labelling, $(d, 1)$-total labelling, chromatic number, chromatic index

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1 Introduction

Let d be a positive integer and $G(V, E)$ be a finite graph without loops or multiple edges. We always assume that G has at least one edge without explicitly saying so. A $(d, 1)$-*total labelling* of G is an integer-valued function f defined on the set $V(G) \cup E(G)$ such that

$$|f(x) - f(y)| \geqslant \begin{cases} 1 & \text{if vertices } x \text{ and } y \text{ are adjacent;} \\ 1 & \text{if edges } x \text{ and } y \text{ are adjacent;} \\ d & \text{if vertex } x \text{ and edge } y \text{ are incident.} \end{cases}$$

We may require $|f(x)-f(y)|$, for adjacent elements x and y, be greater than or equal to s, instead of 1, in the above defining inequality for some given positive integer s to get a more general notion of a (d, s)-*total labelling*; nevertheless we concentrate our attention only to the special case $s = 1$ in this paper. A $(d, 1)$-total labelling taking values in the set $\{0, 1, \ldots, k\}$ is called a $[k]$-$(d, 1)$-*total labelling*. The *span* of a $(d, 1)$-total labelling is the maximum difference between two labels. The minimum span, i.e. the minimum k, among all $[k]$-$(d, 1)$-total labellings of G, denoted $\lambda_d^T(G)$, is called the $(d, 1)$-*total number* of G.

A $(d, 1)$-total labelling of G is a generalization of an $L(2, 1)$-labelling of the subdivision of G studied in Whittlesey, Georges, and Mauro [15]. The notion of an $L(2, 1)$-labelling was motivated by an interference avoidance problem, introduced in Hale [7], in the assignment of radio frequency bands to transmitters. An $L(2, 1)$-labelling of G assigns nonnegative integer labels to the vertices of G so that vertices at distance two receive distinct labels and adjacent vertices receive labels that differ in absolute value by at least 2. Griggs and Yeh [6] initiated a systematic study into $L(2, 1)$-labellings of graphs that has been intensively developed ever since. The reader is referred to Yeh [17] for a recent survey of results and generalizations of $L(2, 1)$-labellings. The *subdivision* G^S of a graph G is the graph obtained by inserting one new vertex to each of the edges of G. If we define the span of an $L(2, 1)$-labelling to be the maximum difference between two labels, then the minimum span among all $L(2, 1)$-labellings of G^S is precisely $\lambda_2^T(G)$.

Havet and Yu [8] first introduced the notion of a $(d, 1)$-total labelling and their results have published only recently in [9]. Let $\Delta(G)$ denote the maximum degree of G. Havet and Yu proposed the following conjecture.

$(d, 1)$-Total Labelling Conjecture. $\lambda_d^T(G) \leqslant \min\{\Delta(G)+2d-1, 2\Delta(G)+d-1\}$.

In addition to [9], positive evidence to this conjecture has also been given in [1], [4], and [13]. Note that $\lambda_1^T(G) + 1$ is equal to the *total chromatic number* $\chi''(G)$ of the graph G and the $(d, 1)$-total labelling conjecture for the case $d = 1$ is equivalent to the well-known *Total Coloring Conjecture* proposed by Behzad [2] and independently by Vizing [14].

It should be pointed out that a $(d, 1)$-total labelling is a special case ($r = s = 1$) of an $[r, s, d]$-coloring introduced and studied in [10], [11], and [12]. The $[1, 1, d]$-chromatic number $\chi_{1,1,d}(G)$ of a graph G defined there is exactly $\lambda_d^T(G) + 1$.

In Section 2, we will derive upper bounds for $\lambda_d^T(G)$. Based on these values, we propose an upper bound conjecture in terms of the chromatic number and the chromatic index of G.

In Section 3, we compute some values of $\lambda_d^T(K_{m,n})$ for complete bipartite graphs $K_{m,n}$ and completely determine all $\lambda_d^T(K_{m,n})$ for $d = 1, 2, 3$. These values give further support to the $(d, 1)$-total labelling conjecture.

2 Upper Bounds

We are going to present upper bounds for $\lambda_d^T(G)$ in terms of its maximum degree $\Delta(G)$, *chromatic number* $\chi(G)$, *chromatic index* $\chi'(G)$, and *list chromatic index* $\chi'_l(G)$. We will propose a conjecture on an upper bound of $\lambda_d^T(G)$ at the end of this section.

Let $\chi(G)$, or $\chi'(G)$, denote the smallest number of colors needed to color the vertices, respectively the edges, of G so that adjacent elements receive distinct colors. A vertex-coloring or an edge-coloring satisfying the above condition is said to be a *proper* vertex-coloring or edge-coloring. If each edge e of G is assigned a list $L(e)$ of possible colors and G has a proper edge-coloring ϕ such that $\phi(e) \in L(e)$ for all $e \in E(G)$, then we say that G is *L-edge-colorable.* The graph G is said to be *k-edge-choosable* if it is L-edge-colorable for every assignment L satisfying $|L(e)| = k$ for all $e \in E(G)$. Let $\chi'_l(G)$ denote the smallest k such that G is k-edge-choosable.

The following two lemmas were proved in Havet and Yu [9] and the case for $d = 2$ first appeared in Whittlesey, Georges, and Mauro [15].

Lemma 1 *For any graph G, $\lambda_d^T(G) \leqslant \chi(G) + \chi'(G) + d - 2$.*

Lemma 2 *For any graph G, $\lambda_d^T(G) \leqslant 2\Delta(G) + d - 1$.*

Throughout this paper, a proper vertex-coloring, or edge-coloring, using colors from the set $\{0, 1, \ldots, k - 1\}$ is said to be a *k-vertex-coloring*, or *k-edge-*

coloring. For integers $a \leqslant b$, we use $[a,b]$ to denote the set $\{a, a+1, \ldots, b\}$. For integers a and d, the set $[a-d+1, a+d-1]$ is denote by $[a]_d$.

Theorem 3 *For any graph G, $\lambda_d^T(G) \leqslant \chi'_l(G) + 4d - 3$.*

Proof. Since $\chi(G) \leqslant \Delta(G)+1$, we can give a proper vertex-coloring f_1 for G using colors $0, 1, \ldots, \Delta(G)$. For each edge $e = xy$, we define the list

$$L(e) = [0, \chi'_l(G) + 4d - 3] \setminus ([f_1(x)]_d \cup [f_1(y)]_d)\,.$$

As $|L(e)| \geqslant \chi'_l(G)$, there exists an L-coloring f_2 for the edges of G. Since $\chi'_l(G) \geqslant \chi'(G) \geqslant \Delta(G)$, we have $\chi'_l(G)+4d-3 \geqslant \Delta(G)$. Consequently, $f_1 \cup f_2$ forms a $[\chi'_l(G) + 4d - 3]$-$(d,1)$-total labelling of G. ∎

Borodin, Kostochka, and Woodall [3] proved that $\chi'_l(G) \leqslant \lfloor \frac{3}{2}\Delta(G) \rfloor$ for a multigraph graph G. Hence, by Theorem 3, the following upper bound for $\lambda_d^T(G)$ emerges.

Theorem 4 *For any graph G, $\lambda_d^T(G) \leqslant \lfloor \frac{3}{2}\Delta(G) \rfloor + 4d - 3$.*

Note that, for fixed d and sufficient large $\Delta(G)$, the upper bound for $\lambda_d^T(G)$ in Theorem 4 is better than the one in Lemma 2. In the rest of this section, we shall improve the bounds of Lemmas 1 and 2.

Theorem 5 *Let G be a graph with $\chi(G) = k$ and $\chi'(G) = k'$. If $k \geqslant 3d$, then $\lambda_d^T(G) \leqslant s + k' - 1$, where s is equal to $4d-2$ when $k = 3d$ or $3d+1$, and equal to $\lceil (k + 9d - 5)/3 \rceil$ when $k \geqslant 3d+2$.*

Proof. We choose a mapping $f : V(G) \cup E(G) \to [0, s + k' - 1]$ such that the restriction of f to $V(G)$ is a k-vertex-coloring and the restriction of f to $E(G)$ is a proper edge-coloring using colors in $[s, s+k'-1]$.

Let G' be the subgraph of G induced by the edges in $E' = \{e \in E(G) \mid f(e) \in [s, k+d-2]\}$. Then $\Delta(G') \leqslant k+d-s-1$. To any $e = xy \in E(G')$, we assign the list $L(e) = [0, k+d-2] \setminus ([f(x)]_d \cup [f(y)]_d)$. Then $|L(e)| \geqslant k - 3d + 1$. Since $\Delta(G') \leqslant k + d - s - 1$, G' is a disjoint union of edges when $k = 3d$, and is a disjoint union of paths and even cycles when $k = 3d+1$. It is well-known that $\chi'_l(G') \leqslant |L(e)|$ in these cases. When $k \geqslant 3d+2$, it follows from $s \geqslant (k+9d-5)/3$ that $3(k+d-s-1)/2 \leqslant k+3d+1$. Since $\chi'_l(G') \leqslant \lfloor 3\Delta(G)/2 \rfloor \leqslant 3(k+d-s-1)/2$, we have $\chi'_l(G') \leqslant |L(e)|$ again. Hence, there always exists an L-edge-coloring f' for G'. Re-labelling edges in G' by f' while keeping the rest of G unchanged, we get an $[s+k'-1]$-$(d,1)$-total labelling for G. ∎

By Theorem 5, the following conjecture holds for graphs with $\chi(G) \geqslant 3d$.

Conjecture 1 *Let a graph G satisfy $\chi(G) > \max\{2, d\}$. Then*

$$\lambda_d^T(G) \leqslant \chi(G) + \chi'(G) + d - 3.$$

The known values of $\lambda_d^T(K_n)$ for complete graphs K_n on n vertices that have been computed in [9] support the above conjecture. The following corollary also appeared in [9].

Corollary 6 *Let G be a bipartite graph. Then $\Delta(G) + d - 1 \leqslant \lambda_d^T(G) \leqslant \Delta(G) + d$ and $\lambda_d^T(G) = \Delta(G) + d$ when $d \geqslant \Delta(G)$ or G is regular.*

For a bipartite graph G, it is well-known that $\chi'(G) = \Delta(G)$. Hence, a consequence of Corollary 6 is $\lambda_d^T(G) = \Delta(G) + d = \chi(G) + \chi'(G) + d - 2$ for a bipartite regular graph G. This together with the fact $\lambda_4^T(K_4) = 9$ show that the assumption $\chi(G) > \max\{2, d\}$ in Conjecture 1 cannot be removed.

3 Complete Bipartite Graphs

The following can be easily derived when we examine the label of a vertex of maximum degree and the labels of its incident edges.

Lemma 7 *(1) $\lambda_d^T(G) \geqslant \Delta(G) + d - 1$.*

(2) If $\lambda_d^T(G) = \Delta(G) + d - 1$, then each vertex of maximum degree is labelled with 0 or $\Delta(G) + d - 1$ in any $[\Delta(G) + d - 1]$-$(d, 1)$-total labelling.

Throughout this section, let $K_{m,n}$ $(m \geqslant n)$ denote the complete bipartite graph with parts $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$. By Corollary 6, $m+d-1 \leqslant \lambda_d^T(K_{m,n}) \leqslant m+d$. When a function f is defined over the edges of $K_{m,n}$, we write $f(i, j)$ for $f(x_i y_j)$. Furthermore, let $X_i = \{f(i, j) \mid 1 \leqslant j \leqslant m\}$ and $Y_j = \{f(i, j) \mid 1 \leqslant i \leqslant n\}$.

Theorem 8 *The following statements are equivalent.*

(1) $m \geqslant \min\{2n, n + 2d - 1\}$ and $m \geqslant n + d$.

(2) There exists an $[m + d - 1]$-$(d, 1)$-total labelling f for $K_{m,n}$ such that $f(x) = 0$ for all $x \in X$, or $f(x) = m + d - 1$ for all $x \in X$.

Proof. (1) $\Rightarrow$ (2). We are going to construct an $[m+d-1]$-$(d, 1)$-total labelling f for $K_{m,n}$ such that $f(x) = 0$ for all $x \in X$.

First assume that $m \geqslant 2n$. Let ρ be the composition of the two cyclic permutations $(1 \;\; 2 \;\; \cdots \;\; n)$ and $(n+1 \;\; n+2 \;\; \cdots \;\; m)$ on the set $[1, m]$. Let $f(x_i) = 0$ for all $1 \leqslant i \leqslant n$, $f(y_j) = m + d - 1$ for $1 \leqslant j \leqslant n$, $f(y_j) = 1$ for $n+1 \leqslant j \leqslant m$, and $f(i, j) = (d-1) + \rho^{i-1}(j)$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Since $m \geqslant 2n$, adjacent edges are labelled with distinct labels. We see that $Y_j = [d, d+n-1]$ when $1 \leqslant j \leqslant n$ and $Y_j \subseteq [d+n, d+m-1]$ when $n < j \leqslant m$. Since $1 \leqslant d \leqslant m - n$, the absolute difference between the label of any vertex and the label of any of its incident edge is at least d, hence f satisfies our requirements.

Next assume that $m \geqslant n+2d-1$. Let σ be the cyclic permutation $(1 \;\; 2 \;\; \cdots \;\; m)$ on the set $[1, m]$. For $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$, let $f(x_i) = 0$, $f(y_j) = (d-1) + \sigma^{n-1+d}(j)$, and $f(i, j) = (d-1) + \sigma^{i-1}(j)$. Adjacent edges are obviously labelled with distinct labels. Since $m \geqslant n+2d-1$, we see that $|\sigma^{n-1+d}(j) - \sigma^{i-1}(j)| \geqslant d$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$, hence f satisfies our requirements.

(2) $\Rightarrow$ (1). Assume there exists an $[m+d-1]$-$(d,1)$-total labelling f for $K_{m,n}$ such that $f(x) = 0$ for all $x \in X$. (By symmetry, we only need to show this case.)

Since $f(x_i) = 0$ for all i, we have $f(i, j) \geqslant d$ and $X_i = [d, m+d-1]$ for all i and j. Without loss of generality, we may assume that $d \in Y_j$ for $1 \leqslant j \leqslant n$, and hence $f(y_j) \geqslant 2d$. Let t_j denote the largest number in Y_j. Then $t_j \geqslant n+d-1$.

Assume that $t_p > f(y_p)$ for some $p \in [1, n]$. Then $[f(y_p)]_d \subseteq [d, t_p]$ and $[f(y_p)]_d \cap Y_p = \emptyset$. Moreover, since $|[f(y_p)]_d| = 2d-1$, $Y_p \in [d, t_p]$, and $|Y_p| = n$, it follows that $t_p - d + 1 \geqslant n + 2d - 1$. As $t_p \leqslant m + d - 1$, we conclude that $m \geqslant n + 2d - 1 \geqslant n + d$.

Assume $t_j < f(y_j)$ for all $j \in [1, n]$. Then we have $t_j \leqslant f(y_j) - d \leqslant m - 1$, implying $n + d - 1 \leqslant m - 1$. Therefore, $m \geqslant n + d$. Moreover, it also follows that $Y_j \subseteq [d, m-1]$ for $1 \leqslant j \leqslant n$. This implies that the edges that can be assigned labels from the set $[m, m+d-1]$ must be incident to y_j for some $j \in [n+1, m]$. Hence, $nd = \sum_{j=n+1}^{m} |Y_j \cap [m, m+d-1]| \leqslant (m-n)d$, implying $2n \leqslant m$. ∎

By Theorem 8, to further investigate the values of m and n such that $\lambda_d^T(K_{m,n}) = m + d - 1$, it remains to study the following two possibilities.

Case 1. $m \geqslant \min\{2n, n+2d-1\}$ and $m < n + d$, or equivalently, $2n \leqslant m < n + d$.

Case 2. $m < \min\{2n, n+2d-1\}$.

We shall deal with Cases 1 and 2 in Theorems 9 and 10, respectively. There is one more notation used in the proofs of Theorems 9 and 10. For any $[m+d-1]$-$(d,1)$-total labelling f for $K_{m,n}$, by Lemma 7, each vertex $x_i \in X$ is labelled with either 0 or $m+d-1$. Denote

$$I = \{i \mid f(x_i) = 0 \text{ and } 1 \leqslant i \leqslant n\}.$$

Then we have $X_i = [d, m+d-1]$ for each $i \in I$, while $X_i = [0, m-1]$ for each $i \notin I$.

Theorem 9 *If* $2n \leqslant m < n+d$, *then* $\lambda_d^T(K_{m,n}) = m+d$.

Proof. The assumption $2n \leqslant m < n+d$ implies that $n < d$ and $m < 2d$. Suppose to the contrary that $\lambda_d^T(K_{m,n}) = m+d-1$. Let f be an $[m+d-1]$-$(d,1)$-total labelling. By Theorem 8, $1 \leqslant |I| \leqslant n-1$. Since $d \in X_i$ for any $i \in I$, we have $d \in Y_j$ for some j. It implies that $2d \leqslant f(y_j) \leqslant m+d-2$ because $f(y_j) \notin \{0, m+d-1\}$, and hence $d \leqslant m-2$. It follows that $d \in X_i$ for any $i \notin I$. Now d belongs to all X_i's. Without loss of generality, we may assume that $d \in Y_j$ for $1 \leqslant j \leqslant n$.

Pick $i_0 \in I$. So $X_{i_0} = [d, m+d-1]$. Because all y_j, $1 \leqslant j \leqslant n$, are adjacent to x_{i_0}, there exists $w \geqslant n+d-1$ such that $w \in Y_{j_0}$ for some $j_0 \in [1,n]$. We know that $2d \leqslant f(y_{j_0}) \leqslant m+d-2$. If $\alpha \in [m-1, m+d-1]$, then $|f(y_{j_0}) - \alpha| < d$ since $m < 2d$. It follows that $Y_{j_0} \cap [m-1, m+d-1] = \emptyset$ and $n+d-1 \leqslant w \leqslant m-2$, contradicting the assumption $m < n+d$. ∎

Theorem 10 *Suppose that* $m < \min\{2n, n+2d-1\}$ *and* $\lambda_d^T(K_{m,n}) = m+d-1$. *Then all the following statements hold.*

(1) $m \geqslant 3d+1$.

(2) $(n-m+3d-1)(2n-m) \leqslant nd$.

(3) $m \geqslant n+d$.

(4) $n/m \leqslant (\alpha+1)/(\alpha+2)$, *where* $\alpha = \lfloor (m-d-2)/(2d-1) \rfloor$.

Proof. Assume $m < \min\{2n, n+2d-1\}$ and $\lambda_d^T(K_{m,n}) = m+d-1$. Let f be an $[m+d-1]$-$(d,1)$-total labelling. By Theorem 8, $1 \leqslant |I| \leqslant n-1$. Without loss of generality, we may assume that $\{d, m-1\} \subseteq Y_j$ for $1 \leqslant j \leqslant 2n-m$. It follows that $2d \leqslant f(y_j) \leqslant m-d-1$, and hence $m \geqslant 3d+1$. This completes the proof for (1).

Since $2d \leqslant f(y_j) \leqslant m-d-1$ for $1 \leqslant j \leqslant 2n-m$, we have $|[d, m-1] \cap Y_j| \leqslant m-3d+1$. As $|Y_j| = n$, it follows that $|([0, d-1] \cup [m, m+d-1]) \cap Y_j| \geqslant n-m+3d-1$. Note that each label in $[0, d-1]$ is assigned to exactly $n-|I|$

edges, while each label in $[m, m+d-1]$ is assigned to exactly $|I|$ edges. We conclude that

$$\begin{aligned}(n-m+3d-1)(2n-m) &\leqslant \sum_{j=1}^{2n-m} |([0,d-1]\cup[m,m+d-1])\cap Y_j| \\ &\leqslant nd.\end{aligned}$$

This completes the proof for (2).

To prove (3), consider Y_j for $1 \leqslant j \leqslant 2n-m$. Since $2d \leqslant f(y_j) \leqslant m-d-1$ and $[f(y_j)]_d \cap Y_j = \emptyset$, we obtain that $n = |Y_j| = |[0, m+d-1] \cap Y_j| \leqslant m+d-(2d-1) = m-d+1$. Hence $m \geqslant n+d-1$. Suppose $m = n+d-1$. Then (2) implies $n \leqslant 2d-2$. This is impossible since $m = n+d-1 \geqslant 3d+1$ by (1).

It follows from (1) that the number α in (4) is positive and $\alpha(2d-1)+1 \leqslant m-d-1 \leqslant (\alpha+1)(2d-1)$. For each $j \in [1, 2n-m]$, since $2d \leqslant f(y_j) \leqslant m-d-1$ and $[f(y_j)]_d \cap Y_j = \emptyset$, the following statement holds: For each $s \in [1, \alpha]$, if $f(y_j) \in [s(2d-1)+1, (s+1)(2d-1)]$, then $s(2d-1)+d \notin Y_j$. For each $i \in [1, \alpha]$, let $t_i = |\{j \mid j \in [1, 2n-m] \text{ and } f(y_j) \in [i(2d-1)+1, (i+1)(2d-1)]\}|$. Because $t_1+t_2+\ldots+t_\alpha = 2n-m$, there exists some $k \in [1, \alpha]$ such that $t_k \geq (2n-m)/\alpha$. Therefore, $k(2d-1)+d$ does not belong to at least $(2n-m)/\alpha$ of the Y_j's for $1 \leqslant j \leqslant 2n-m$. Since the label $k(2d-1)+d$ belongs to exactly n of the Y_j's for $1 \leqslant j \leqslant m$, we conclude that $(2n-m)/\alpha \leqslant m-n$, hence (4) follows. ∎

The following is an immediate consequence of Theorem 9 and Theorem 10(3).

Corollary 11 *If $m < n+d$, then $\lambda_d^T(K_{m,n}) = m+d$.*

Now we are ready to give exact values of $\lambda_d^T(K_{m,n})$ for $d = 1, 2, 3$. The case for $d = 1$ is completely determined by the total chromatic number of $K_{m,n}$ and the reader is referred to Theorem 3.2 in Yap [16] for a proof.

Theorem 12 *Let $1 \leqslant n \leqslant m$. Then*

$$\lambda_1^T(K_{m,n}) = \chi''(K_{m,n}) - 1 = m + \delta_{m,n},$$

where $\delta_{m,n}$ denotes the Kronecker delta, i.e., its value is 1 if $m = n$ and is 0 otherwise.

Theorem 13 *Let* $1 \leqslant n \leqslant m$. *Then*

$$\lambda_2^T(K_{m,n}) = \begin{cases} m+2 & \textit{if } m \leqslant n+1, \textit{ or} \\ & m = n+2 \textit{ and } n \geqslant 3; \\ m+1 & \textit{otherwise.} \end{cases}$$

Proof. By Corollary 6, it suffices to consider the case for $m > n$. The results for $m \geqslant n+3$ follow from Theorem 8. For $m = n+1$, the result follows from Corollary 11. Assume $m = n+2$. The cases for $n = 1, 2$ follow from Theorem 8. The cases for $n = 3, 4$ follow from Theorem 10(1). The cases for $n = 5, 6$ follow from Theorem 10(4). All the remaining cases follow from Theorem 10(2). ∎

Theorem 14 *Let* $1 \leqslant n \leqslant m$. *Then*

$$\lambda_3^T(K_{m,n}) = \begin{cases} m+3 & \textit{if } m \leqslant n+2, \textit{ or} \\ & m = n+3 \textit{ and } n \geqslant 4, \textit{ or} \\ & m = n+4 \textit{ and } n = 5, 9, 10, 13, 14, 15; \\ m+2 & \textit{otherwise.} \end{cases}$$

Proof. By Corollary 6, it suffices to consider the case for $m > n$. The results for $m \leqslant n+2$ and $m \geqslant n+5$, respectively, follow from Corollary 11 and Theorem 8.

Assume $m = n+3$. The cases for $n = 1, 2, 3$ follow from Theorem 8. The cases for $n = 4, 5, 6$ follow from Theorem 10(1). The case for $n = 7$ follows from Theorem 10(4). The remaining cases for $n \geqslant 8$ follow from Theorem 10(2).

Finally assume $m = n+4$. The cases for $n = 1, 2, 3, 4$ follow from Theorem 8. The case for $n = 5$ follows from Theorem 10(1). The cases for $n \geqslant 17$ follow from Theorem 10(2). The cases for $n = 9, 10, 13, 14, 15$ follow from Theorem 10(4). In the appendix, we list $[n+6]$-$(3,1)$-total labellings obtained by *ad hoc* methods for each of the $K_{n+4,n}$, $n = 6, 7, 8, 11, 12, 16$. ∎

We conclude this paper with the following problem whose answer is positive for $d = 1, 2, 3$ from our results.

Problem. Under the assumption that $m < \min\{2n, n+2d-1\}$, are conditions (1) to (4) in Theorem 10 sufficient for $\lambda_d^T(K_{m,n}) = m+d-1$?

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A Appendix

When n is one of the numbers 6, 7, 8, 11, 12, or 16, an $[n+6]$-$(3,1)$-total labelling for $K_{n+4,n}$ is given below by a table. The notation used is as follows.

The label of the i-th row is assigned to the vertex $x_i \in X$.

The label of the j-th column is assigned to the vertex $y_j \in Y$.

The label at the (i,j) cell is assigned to the edge $x_i y_j$.

$K_{10,6}$	6	6	9	9	11	11	1	1	1	1
12	2	0	1	3	5	4	7	8	9	6
12	3	1	0	2	6	5	9	4	8	7
12	9	2	3	0	1	6	8	7	4	5
0	10	3	4	6	8	7	12	11	5	9
0	11	9	5	4	7	3	10	12	6	8
0	12	10	6	5	3	8	11	9	7	4

$K_{11,7}$	6	6	7	12	12	12	12	1	1	1	1
13	0	2	1	5	8	3	7	9	10	4	6
13	1	0	3	6	2	4	5	8	7	9	10
13	3	1	4	2	5	0	6	10	9	8	7
0	9	3	11	7	6	5	4	12	8	10	13
0	10	9	12	8	7	6	3	11	5	13	4
0	12	10	13	3	4	7	8	6	11	5	9
0	13	11	10	4	3	8	9	7	12	6	5

$K_{12,8}$	6	6	7	7	13	13	13	13	1	1	1	1
14	0	1	3	2	5	6	4	8	11	7	9	10
14	1	0	2	3	6	7	10	9	5	4	11	8
14	3	2	0	1	9	10	6	7	4	8	5	11
14	2	3	1	10	0	4	7	6	9	11	8	5
0	11	10	4	13	3	8	9	5	12	14	6	7
0	12	11	13	14	8	3	5	4	7	9	10	6
0	13	12	14	11	7	5	3	10	8	6	4	9
0	14	13	11	12	10	9	8	3	6	5	7	4

$K_{15,11}$	6	6	6	11	11	11	11	16	16	16	16	1	1	1	1
0	11	16	9	5	6	15	17	10	3	8	4	12	13	14	7
0	12	11	10	6	14	17	5	13	4	3	9	15	7	16	8
0	13	12	17	7	16	3	15	4	5	6	10	8	11	9	14
0	14	13	15	17	8	16	6	5	10	9	3	11	4	7	12
0	15	14	13	16	17	7	4	3	8	12	6	9	5	10	11
0	16	17	14	15	7	6	3	9	12	4	5	13	8	11	10
17	0	1	11	14	3	4	2	12	13	7	8	5	10	6	9
17	1	3	12	0	2	5	14	8	11	10	7	6	9	13	4
17	3	2	1	8	4	14	7	0	9	5	11	10	6	12	13
17	9	10	2	3	0	1	8	7	6	11	13	14	12	4	5
17	10	9	3	2	1	8	0	11	7	13	12	4	14	5	6

$K_{16,12}$	6	6	6	6	12	12	12	12	17	17	17	17	1	1	1	1
0	3	12	9	13	17	16	5	18	4	8	14	11	7	10	6	15
0	14	3	12	9	6	17	16	8	5	4	10	13	11	18	15	7
0	18	16	3	12	8	6	9	17	14	5	4	7	13	15	11	10
0	17	18	14	3	9	8	6	16	13	11	5	4	15	7	10	12
0	11	10	17	18	3	9	8	15	6	13	7	5	4	12	16	14
0	13	11	18	17	16	15	3	6	12	14	8	10	5	4	7	9
18	15	1	11	14	7	3	0	9	8	10	12	2	6	5	4	13
18	12	15	10	11	0	7	2	3	1	9	13	8	14	6	5	4
18	1	14	15	10	4	0	7	2	3	12	9	6	8	11	13	5
18	2	13	0	15	5	4	1	7	10	3	11	12	9	8	14	6
18	10	9	1	0	15	2	4	5	7	6	3	14	12	13	8	11
18	9	0	13	1	2	5	15	4	11	7	6	3	10	14	12	8

$K_{20,16}$	6	6	6	6	11	11	11	11	16	16	16	16	21	21	21	21	1	1	1	1
0	20	18	14	13	21	4	3	7	22	5	19	6	8	10	15	11	12	17	9	16
0	21	19	15	14	22	5	20	8	3	4	6	11	7	13	12	18	17	16	10	9
0	12	21	16	15	14	6	4	17	20	7	3	22	5	18	10	8	11	9	19	13
0	9	22	17	16	3	7	21	18	4	20	5	19	10	6	8	12	15	14	13	11
0	10	3	20	17	15	8	14	16	5	21	22	4	18	7	6	9	13	12	11	19
0	11	9	22	18	16	20	15	19	6	8	21	3	4	5	7	10	14	13	17	12
0	3	10	18	20	17	22	19	21	7	6	4	13	12	15	5	16	9	11	8	14
0	13	11	19	21	18	14	7	22	8	12	9	20	17	3	4	6	16	10	5	15
22	0	12	3	19	4	18	16	6	2	11	1	5	9	14	13	17	8	7	15	10
22	1	16	9	3	5	0	18	2	12	13	8	7	14	4	11	15	10	19	6	17
22	14	1	10	9	6	3	0	15	13	19	11	2	16	8	17	5	7	18	12	4
22	15	2	0	10	7	17	1	5	9	3	12	8	13	11	16	14	19	6	4	18
22	16	14	2	0	19	15	5	1	10	9	13	12	11	17	3	7	6	4	18	8
22	17	13	11	1	8	19	6	14	0	10	2	9	15	12	18	3	4	5	16	7
22	18	15	12	11	1	2	17	4	19	0	7	10	3	16	9	13	5	8	14	6
22	19	17	13	12	2	16	8	3	11	1	10	0	6	9	14	4	18	15	7	5

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