California State University – Los Angeles
Mathematics
Masters Degree Comprehensive Examination

Complex Analysis Spring 2019

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Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Spring 2019 # 1. Describe and sketch each of the following sets of complex numbers

a. $A = \{z \in \mathbb{C} : \left| z^2 \right| \geq 4\}$
b. $B = \{z \in \mathbb{C} : \mathrm{Re}(1/z) \geq 1/2\}$
c. $C = \{z \in \mathbb{C} : \mathrm{Re}(z^3) \geq 0\}$ (Suggestion: do the "equality" part first.)
d. $D = \{z \in \mathbb{C} : \mathrm{Im}(e^z) \geq 0\}$

Spring 2019 # 2. Let Ω be a nonempty open connected subset of $\mathbb{C}$. Show that if $f : \Omega \to \mathbb{C}$ is analytic and $\mathrm{Re}(f(z)) = (\mathrm{Im}(f(z)))^2$ for every z in Ω, then f must be constant on Ω.

Spring 2019 # 3. For each of the following real valued functions of two real variables $u(x, y)$, decide whether there is a real valued function $v(x, y)$ such that $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is analytic and $f(0) = 3i$. If there is such a v, find such a v. If not, explain why no such v exists.

a. $u(x, y) = x^3 - 3xy^2 + 4xy$ **b.** $u(x, y) = x^3 - xy^2 + 4xy$

Spring 2019 # 4. Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire, and suppose there is a positive constant M such that $\left| f^{(314)}(z) \right| < M$ for all z in $\mathbb{C}$.

a. Show that f is a polynomial.
b. What can you say about the degree of f?

Spring 2019 # 5. Evaluate each of the following integrals. Sketch any curves and discuss estimates needed to justify your method.

$$\textbf{a.}\quad \int_0^{2\pi} \frac{\sin^2 t}{5+4\cos t}\,dt \qquad \textbf{b.}\quad \int_0^{\infty} \frac{1+x^2}{1+x^4}\,dx$$

Spring 2019 # 6. Classify the singularities and find the residues of each of the following functions at the point $z_o = 0$.

$$\textbf{a.}\quad f(z) = \frac{e^z - 1}{z^2} \qquad \textbf{b.}\quad f(z) = \frac{e^z - 1}{\sin z}$$

$$\textbf{c.}\quad f(z) = \frac{\sin(z^2)}{z^2} \qquad \textbf{d.}\quad f(z) = z^3 \sin(1/z)$$

Spring 2019 # 7. Consider a linear fractional transformation (Möbius transformation) $T(z) = \dfrac{az+b}{cz+d}$, where a, b, c, and d are complex constants and $ad - bc \neq 0$. Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the extended complex plane.

a. Show that T extends to a bijective transformation $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$

b. A fixed point of T is a point $z_o \in \hat{\mathbb{C}}$ such that $T(z_o) = z_o$. Show that $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a linear fractional transformation other than the identity transformation, $T(z) = z$ for all z, then T has at least one, and at most two fixed points.

End of Exam