California State University – Los Angeles Mathematics Masters Degree Comprehensive Examination

Complex Analysis Spring 2014

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Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\mathrm{Log}\, z$ denotes the principal branch of $\log z$.
$\mathrm{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z;r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Spring 2014 # 1. Evaluate the integral $\displaystyle\int_\gamma \frac{e^z}{(z-2)^2(z-4)}\,dz$ around each of the following curves.

a. $\gamma_a =$ the circle of radius 1 centered at 0 travelled once counterclockwise.
b. $\gamma_b =$ the circle of radius 1 centered at 2 travelled once counterclockwise.
c. $\gamma_c =$ the circle of radius 1 centered at 4 travelled once counterclockwise.

Give reasons for your answers.

Spring 2014 # 2. Use complex variable techniques to evaluate each of the following integrals, giving reasons for your answers, showing any curves and discussing any estimates needed to justify your method. (For full credit, include a discussion of why the improper integral exists.)

$$\textbf{a.}\quad \int_{-\infty}^{\infty} \frac{1}{x^2-2x+2}\,dx \qquad \textbf{b.}\quad \int_0^{2\pi} \frac{1}{8-2\sin\theta}\,d\theta.$$

Spring 2014 # 3. Let D be the disk $D = \{z \in \mathbb{C} : |\,z-2\,| < 2\}$

a. (8 points) Describe and sketch the image of the D under the transformation $f(z) = z/(2z-8)$.

b. (12 points) For which of the following sets is there a function f taking the set one-to-one analytically onto the disk D above? Give a clear yes or no answer for each with reasons for your answers.

i. $A = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$.
ii. $B = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0 \text{ and } \mathrm{Im}(z) > 0\}$.
iii. $C = \mathbb{C}$.

Spring 2014 # 4. Suppose $f : B \to \mathbb{C}$ is analytic on the disk $B = \{z \in \mathbb{C} : |\,z\,| < 2\}$, that $f(1) = 5$, and that $f(w) = f(w/2)$ for every w in B. Show that $f(z) = 5$ for all z in B.

Spring 2014 # 5. Determine the number of zeros, counting multiplicity, of the polynomial $p(z) = z^4 - 2z^3 + 9z^2 + z - 1$ inside the circle $C = \{z \in \mathbb{C} : |\,z\,| = 2\}$. Show your work.

Spring 2014 # 6. Consider $f(z) = \dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$.

a. (15 points) Find the Laurent series for $f(z)$ valid in each of the following regions.

i. $0 < |\,z\,| < 1$ **ii.** $1 < |\,z\,| < 2$ **iii.** $2 < |\,z\,| < \infty$

b. (5 points) Find the residue of f at 0.

Spring 2014 # 7. For each of the following real valued functions $u(x,y)$ find a real valued function $v(x,y)$ such that the function $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ is analytic or show that there can be no such function.

a. $u(x,y) = x^3 - 3xy^2 - 2xy$
b. $u(x,y) = x^3 - xy^2 - 2xy$.

End of Exam