California State University – Los Angeles
Mathematics
Masters Degree Comprehensive Examination

Complex Analysis Spring 2012

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Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\mathrm{Log}\, z$ denotes the principal branch of $\log z$.
$\mathrm{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z;r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Spring 2012 # 1. Sketch each of the following sets in $\mathbb{C}$.

a. $A = \left\{ z \in \mathbb{C} : \left| \dfrac{z+i}{z-3i} \right| \leq 1 \right\}$
b. $B = \{ z \in \mathbb{C} : |\, z - i \,| \leq \operatorname{Im}(z) \}$
c. $C = \{ z \in \mathbb{C} : |\, \operatorname{Arg}(z+1) \,| < \pi/4 \}$

Spring 2012 # 2. $f(z) = \dfrac{1}{z(z^2+1)}$.

a. On what set is f analytic?
b. Find the Laurent series for f valid for $0 < |\, z \,| < 1$.
c. Find the Laurent series for f valid for $|\, z \,| > 1$.
d. Find the residue of f at 0.

Spring 2012 # 3. Let γ be the closed path consisting of straight line segments from $2+2i$ to $-2-2i$, from there to $-2+2i$, from there to $2-2i$, and finally back to $2+2i$. Evaluate $\int_\gamma f(z)\, dz$ for each of the following functions giving reasons for your answers.

a. $f(z) = \dfrac{1}{z^2-1}$ **b.** $f(z) = \dfrac{e^{2z}}{(z-1)^2}$

Spring 2012 # 4. Let A be the closed unit disk $A = \{ z \in \mathbb{C} : |\, z \,| \leq 1 \}$.
Suppose f is an entire function whose Taylor series centered at the origin is $\sum_{k=0}^{\infty} a_k z^k$, and that f maps A into A.
Show that $|\, a_k \,| \leq 1$ for each k.

Spring 2012 # 5. For each of the following real valued functions $u(x,y)$, decide whether there is a real valued function $v(x,y)$ such that $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ is analytic with $f(0) = 2i$. If so, find $v(x,y)$. If not, explain how you know that there is no such function.

a. $u(x,y) = xy^3 - x^3y$ **b.** $u(x,y) = xy^3 + x^3y$

Spring 2012 # 6. Show that all the zeros of the polynomial $z^6 - 5z^2 + 10 = 0$ lie in the annulus $1 < |z| < 2$.

Spring 2012 # 7. Use complex variable methods to evaluate **two** of the following integrals. Sketch any contours and discuss estimates needed to justify your methods.

a. $\displaystyle\int_{-\infty}^{\infty} \frac{e^{-itx}}{1+x^2}\, dx$ for $t > 0$ **b.** $\displaystyle\int_0^{\pi} \frac{1}{3+\cos\theta}\, d\theta$ **c.** $\displaystyle\int_\gamma \frac{\cos(z^2)}{z^9}\, dz$

In part **c**, γ is the circle of radius 1 centered at 0 travelled once in the counterclockwise direction.

End of Exam