California State University – Los Angeles
Mathematics
Masters Degree Comprehensive Examination

Complex Analysis Spring 2010

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Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.
Please

1. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
2. Write on one side of the paper only
3. Begin each problem on a new page
4. Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\mathrm{Log}\, z$ denotes the principal branch of $\log z$.
$\mathrm{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z; r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Spring 2010 # 1. Sketch (and describe as appropriately helpful) each of the following sets in $\mathbb{C}$.

a. $A = \{z \in \mathbb{C} : 2\,\mathrm{Re}(z) \leq |\,z\,|\}$
b. $B = \{z \in \mathbb{C} : 0 \leq |\,z\,| \leq \mathrm{Arg}(z) \leq \pi\}$
c. $C = \{z \in \mathbb{C} : \left|\dfrac{z-3}{z}\right| \geq 2\}$

Spring 2010 # 2. For each of the following real valued functions $u(x, y)$, decide whether there is a real valued function $v(x, y)$ such that $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is analytic with $f(0) = 2i$. If so, find $v(x, y)$. If not, explain how you know that there is no such function.

a. $u(x, y) = y^3 - 3x^2y + 2y$
b. $u(x, y) = y^3 + 3x^2y - 2y$

Spring 2010 # 3. **a.** Suppose p is a polynomial of degree greater than or equal to 2, and let $f(z) = 1/p(z)$. Show that the sum of the residues of f at all of its singularities in $\mathbb{C}$ is 0.

b. Evaluate $\displaystyle\int_\gamma \frac{1}{(z^4+1)(z-3)}\,dz$ if γ is the circle of radius 2 centered at the origin and travelled once in the counterclockwise direction.

Spring 2010 # 4. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges to a function which is analytic on the region $H = \{z \in \mathbb{C} : \mathrm{Re}(z) > 1\}$

Spring 2010 # 5. Let $f(z) = \dfrac{z}{e^z - 1}$

a. Find all singularities of f in $\mathbb{C}$ and classify each as removable, a pole (of what order), or essential.

b. Evaluate $\displaystyle\int_\gamma f(z)\,dz$ where γ is the circle of radius 7 centered at the origin and traveled once in the counterclockwise direction.

Spring 2010 # 6. Use complex variable methods to evaluate each of the following integrals. Sketch any contours and discuss estimates needed to justify your methods.

a. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+2x+3}\,dx$ **b.** $\displaystyle\int_{-\pi}^{\pi} \frac{\cos\theta}{2+\cos\theta}\,d\theta$ **c.** $\displaystyle\int_\gamma z^4 e^{1/z}\,dz$

In **c**, γ is the circle of radius 1 centered at 0 traveled once counterclockwise.

Question 7 is on the next sheet.

Spring 2010 # 7. Consider the function $f(z) = \dfrac{1}{1 - z - z^2}$

- **a.** Explain how you know that $f(z)$ has a series representation of the form $f(z) = c_0 + c_1 z + c_2 z^2 + \dots$ valid in a neighborhood of 0.
- **b.** Find $c_0, c_1, c_2, c_3, and c_4$.
- **c.** Find a relationship among c_n, c_{n-1}, and c_{n-2} valid for $n \geq 2$.
- **d.** (Not required) Do you recognize the sequence$c_0, c_1, c_2, c_3, \dots$?

End of Exam