California State University – Los Angeles
Mathematics
Masters Degree Comprehensive Examination

Complex Analysis   Spring 2009

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Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\mathrm{Log}\, z$ denotes the principal branch of $\log z$.
$\mathrm{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z;r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Spring 2009 # 1. Describe and sketch each of the following sets of complex numbers. Make sure you label enough points or other objects so that your sketch is not ambiguous.

a. $\left\{z \in \mathbb{C} : \text{Im}(z) \leq (\text{Re}(z))^2\right\}$
b. $\left\{z \in \mathbb{C} : 1 \leq |\, z \,| < 2 \text{ and } \frac{\pi}{4} \leq \arg z \leq \pi\right\}$
c. $\left\{z \in \mathbb{C} : \left|\dfrac{z-2}{z-1}\right| = 2\right\}$

Spring 2009 # 2. Suppose w_o and z_o are in $\mathbb{C}$ and ϕ is a fixed angle with $0 \leq \phi < 2\pi$. Show that the distance from the point z_o to the line parametrized by $z(t) = w_o + te^{i\phi}$, $t \in \mathbb{R}$ is

$$\left|\, \text{Im}\left((z_o - w_o)e^{-i\phi}\right) \right|$$

Spring 2009 # 3. **a. (14 pts)** Suppose $g : \mathbb{C} \to \mathbb{C}$ is analytic on $\mathbb{C}$ with $\text{Re}(g(z)) = \text{Im}(g(z))$ for all z. Show that z must be constant on $\mathbb{C}$.

b. (6 pts) Let $f : \mathbb{C} \to \mathbb{C}$ is analytic on $\mathbb{C}$ with $u(x, y) = \text{Re}(f(x + iy))$ and $v(x, y) = \text{Im}(f(x + iy))$. Show that if

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

everywhere, then f must be a polynomial of degree no more than 1.

Spring 2009 # 4. For each of the following real valued functions $u(x, y)$, decide whether there is a real valued function $v(x, y)$ such that $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is analytic with $f(0) = 2i$. If so, find $v(x, y)$. If not, explain how you know that there is no such function.

a. $u(x, y) = y^3 + 3x^2y + 3x$ **b.** $u(x, y) = y^3 - 3x^2y + 3x$

Spring 2009 # 5. Let $f(z) = \dfrac{1}{z(z-2)}$.

a. (14 pts) Find the Laurent series for f valid in each of the following regions.

i. $\{z \in \mathbb{C} : 0 < |\, z \,| < 2\}$ **ii.** $\{z \in \mathbb{C} : |\, z \,| > 2\}$

b. (6 pts) Find the residue of f at 0.

Spring 2009 # 6. Evaluate $\displaystyle\int_\gamma \left(\frac{e^{3z}}{z+2} + \frac{e^{2z}}{(z-4)^3}\right) dz$ for each of the following curves γ

a. The circle of radius 1 centered at 0 and travelled once counterclockwise.
b. The circle of radius 3 centered at 0 and travelled once counterclockwise.
c. The circle of radius 5 centered at 0 and travelled once counterclockwise.
d. The path formed by following straight line segments from $5 + i$ to $-5 - i$, from there to $-5 + i$, then to $5 - i$, and finally back to $5 + i$.

Spring 2009 # 7. Evaluate two of the following integrals. Sketch any curves and discuss any estimates needed to justify your method.

a. $\displaystyle\int_0^{2\pi} \frac{1}{2 + \cos\theta}\, d\theta$ **b.** $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4 + 16}\, dx$ **b.** $\displaystyle\int_0^{\infty} \frac{\sqrt{x}}{1 + x^2}\, dx$

End of Exam