California State University – Los Angeles Mathematics Masters Degree Comprehensive Examination

Complex Analysis Fall 2012

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Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\mathrm{Log}\, z$ denotes the principal branch of $\log z$.
$\mathrm{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z; r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a\cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a\cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\sin(a+b) = \sin a\cos b + \cos a \sin b \qquad \cos(a+b) = \cos a\cos b - \sin a\sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Fall 2012 # 1. Suppose a and z are in $\mathbb{C}$ with $|z| = 1$ and $|a| < 1$. Show that

$$\left| \frac{z-a}{1-\bar{a}z} \right| = 1.$$

Fall 2012 # 2. a. (4 points) Find a nonconstant function $f : \mathbb{C} \to \mathbb{C}$ which is analytic on all of $\mathbb{C}$ with $f(1/2) = 0$ and $f(1/3) = 0$.

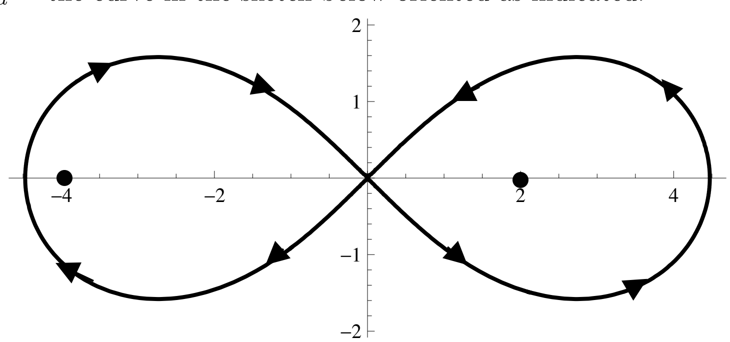
(This really is about as easy as it sounds. It is intended to get you started.)

b. (10 points) Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk. Find a nonconstant function $g : D \to D$ which is analytic on all of D with $g(1/2) = 0$ and $g(1/3) = 0$. Justify you answer.

c. (6 points) Can the function g in part **(b)** be a fractional linear (Möbius) transformation? (Why or why not)

Fall 2012 # 3. Consider the function $f(z) = e^{2z}\left(\frac{1}{(z+4)} + \frac{1}{(z-2)^2}\right)$. Evaluate $\int_\gamma f(z)\,dz$ for each of the following curves.

- **a.** γ_a = the circle of radius 1 centered at the origin and travelled once in the counterclockwise direction.
- **b.** γ_b = the circle of radius 3 centered at the origin and travelled once in the counterclockwise direction.
- **c.** γ_c = the circle of radius 5 centered at the origin and travelled once in the counterclockwise direction.
- **d.** γ_d = the curve in the sketch below oriented as indicated.



Fall 2012 # 4. Evaluate each of the following integrals. (Sketch curves and discuss any estimates needed to justify your answer.)

a. $\displaystyle\int_{-\infty}^{\infty} \frac{4}{x^4+1}\,dx$ **b.** $\displaystyle\int_0^{\pi} \frac{\cos\theta}{2+\cos\theta}\,d\theta$

Fall 2012 # 5. Consider the series $\sum_{n=1}^{\infty}(n+1)z^n$.

- **a.** For which complex numbers z does this series converge?
- **b.** For those z, let $f(z)$ be the sum of the series and find $f(z)$.
- **c.** Evaluate $f^{(5)}(0)$

(Suggestion: Think about derivatives or integrals.)

Fall 2012 # 6. For each positive real number t and complex number z, let

$$f(t, z) = f_t(z) = e^{2tz - z^2}$$

a. Explain why $f(t, z)$ has a representation of the form

$$f(t, z) = \sum_{n=0}^{\infty} \frac{H_n(t)}{n!} z^n$$

valid for z near 0.

b. Find $H_0(t)$, $H_1(t)$, and $H_2(t)$.

c. For which complex z is this expansion valid?

Fall 2012 # 7. Let $f(z) = \dfrac{z^2 - 2z}{(z+1)^2(z^2-4)}$.

a. Find all the singularities of f in $\mathbb{C}$, and classify each as removable, a pole (specify the order), or essential.

b. Find the residue of f at each of these singularities.

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End of Exam