California State University – Los Angeles
Mathematics
Masters Degree Comprehensive Examination

Complex Analysis   Fall 2008

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Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\mathrm{Log}\, z$ denotes the principal branch of $\log z$.
$\mathrm{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z; r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Fall 2008 # 1. A beginning student brings you the following:

$$-1 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1.$$

a. Explain what is going on in terms of branches of the square root function.
b. If the square root is defined for $z = re^{i\theta}$ with $0 \le \theta < 2\pi$ by $\sqrt{z} = r^{1/2}e^{i\theta/2}$, for what z and w is the equation $\sqrt{z}\sqrt{w} = \sqrt{z}\sqrt{w}$ true?

Fall 2008 # 2. Suppose $f : \mathbb{C} \to \mathbb{C}$ is analytic on $\mathbb{C}$ and that $\mathrm{Im}(f(z)) > 1$ for all z in $\mathbb{C}$. Show that f must be constant.

Fall 2008 # 3. Suppose $w_1, w_2, w_3, \ldots$ are points on the unit circle. (That is, $|\, w_k \,| = 1$ for each k.) Consider the infinite series

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \frac{1}{z - w_k}$$

Let $D = \{z \in \mathbb{C} : |\, z \,| < 1\}$.
a. Show that the series converges for each z in D.
b. Show that the sum of the series is an analytic function of z for z in D.

Fall 2008 # 4. Let γ be the unit circle $\{z \in \mathbb{C} : |\, z \,| = 1\}$. Suppose f is a function analytic on an open set containing γ and its interior, and that $|\, f(z) \,| < 1$ for all z on the circle γ. Show that f has exactly one fixed point inside γ. (That is, there is exactly one point z in the open unit disk with $f(z) = z$.) (Suggestion: Consider zeros of $g(z) = z - f(z)$.)

Fall 2008 # 5. Let $f(z) = \dfrac{z^2 + \pi^2}{1 + e^z}$.
a. Identify all of the singularities of f in $\mathbb{C}$ and classify each as removable, a pole (of what order), or essential. (Be sure to explain your answers.)
b. Explain why $f(z)$ has a series representation of the form $f(z) = \sum_{k=0}^{\infty} c_k z^k$ valid for z near 0, and find c_0, c_1, and c_2.
c. What is the radius of convergence of the series in part **(b)** ?
d. Discuss the relationship between f and the function g defined by

$$g(z) = \begin{cases} f(z) & \text{if } z \neq \pm\pi i \\ -2\pi i & \text{if } z = \pi i \\ 2\pi i & \text{if } z = -\pi i \end{cases}$$

in particular with respect to the answers to parts **(a)**, **(b)**, and **(c)**.

Fall 2008 # 6. Compute each of the following integrals

a. $\displaystyle\int_\gamma \left(\frac{e^z}{z-1} - \frac{e^{2z}}{(z+2)^2} \right) dz$ where γ is the circle of radius 3 centered at the origin and traveled once counterclockwise.

b. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\, dx$. Sketch any curves and explain estimates needed to justify your method.

Fall 2008 # 7.

a. Show that if $u(x, y)$ is harmonic on A and $v(x, y)$ is a harmonic conjugate for u on A, then $h(x, y) = u(x, y)v(x, y)$ is harmonic on A

b. Show that the function $\phi(x, y) = y^3 - 3x^2y$ is harmonic and find a harmonic conjugate.

End of Exam