Cal State Los Angeles Department of Mathematics
Complex Analysis Comprehensive Examination
Spring 2021
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Directions: Do five of the following seven problems. If you turn in more than five, the best five will be used.

Spring 2021 # 1. Describe and sketch each of the following sets of complex numbers.

(a) $\left\{ z \;\middle|\; \overline{z} = \dfrac{1}{z} \right\}$

(b) $\left\{ e^z \;\middle|\; z = x + iy \text{ and } 1 < x < 2 \text{ and } \dfrac{3\pi}{4} < y \leq \dfrac{5\pi}{4} \right\}$

(c) $\{ z \;\mid\; |z - i| \leq \mathrm{Im}(z) \}$

Spring 2021 # 2. Compute the following integrals.

(a) $\displaystyle\int_\gamma \frac{e^{z^2}}{z^3}\,dz$ where γ is the unit circle oriented counter-clockwise

(b) $\displaystyle\int_0^\infty \frac{x^2}{1 + x^4}\,dx$

Spring 2021 # 3. Let $f(z) = \dfrac{\sin(z)}{(e^z - 1)^2}$

(a) Classify the singularity of f at $z_0 = 0$. That is, is it a removable singularity, a pole of order m, or an essential singularity?

(b) Compute the integral $\displaystyle\int_\gamma f(z)dz$ where γ is the unit circle oriented counter-clockwise

Spring 2021 # 4. Let $p(z) = z^4 + 3z^3 + 6$.

(a) Show that $p(z)$ has three zeros (counting multiplicity) in the set $\{z \mid |z| < 2\}$

(b) Show that $p(z)$ has one zero (counting multiplicity) in the set $\{z \mid 2 \leq |z| < 4\}$

Spring 2021 # 5. Prove that a sequence of complex numbers $\{z_n\}$ converges if and only if $\{z_n\}$ is Cauchy.

Note: You may use the fact that $\mathbb{R}$ is complete.

Spring 2021 # 6. We say a function $f : \mathbb{R} \to \mathbb{R}$ preserves orientation if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, and reverses orientation if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

If possible, find an entire function $g : \mathbb{C} \to \mathbb{C}$ such that

$$\operatorname{Im}[\, g(x + i0) \,] = 0 = \operatorname{Re}[\, g(0 + iy) \,],$$

and $f_1 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = g(x + i0)$ preserves orientation, while $f_2 : \mathbb{R} \to \mathbb{R}$ defined by $f_2(y) = g(0 + iy)$ reverses orientation. If not possible, prove that no such function g exists.

Spring 2021 # 7. If possible, find an entire function $g : \mathbb{C} \to \mathbb{C}$ such that

$$g'(z) = \begin{cases} z & \text{if } |z| < 1 \\ 2z & \text{if } |z| > 2 \end{cases}$$

If not possible, prove that no such function g exists.