FINDING EXACT VALUES FOR THE PARAMETER OF THE LONELY RUNNER CONJECTURE

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ABSTRACT

Finding Exact Values for the Parameter of the Lonely Runner Conjecture

By

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Consider D to be a finite set of positive integers. This thesis focuses on two parameters of D. The first parameter treats the elements of D as being the speeds of runners. This parameter, denoted $\kappa(D)$, is the value of the maximum possible distance a runner can be from any other runner on a track, given that all the runners have constant speeds and start at the same time from the same position. The second parameter, denoted $\mu(D)$, deals with the density of integral sequences with missing differences in D. This thesis studies the two parameters for the family of finite sets of positive integers $D = \{2, 3, x, y\}$, and finds values for the parameters for many x and y.

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CHAPTER 1

Introduction

Let D be a set of finite positive integers. In this thesis we focus on two parameters of D. First is the parameter involved in the Lonely Runner Conjecture, denoted $\kappa(D)$. Second is the parameter used in defining the density of sequences with missing differences, denoted $\mu(D)$. This thesis studies the values of $\kappa(D)$ and $\mu(D)$ for the family of sets $D = \{2, 3, x, y\}$.

The Lonely Runner Conjecture, as first given by Willis [18] and poetically named by Goddyn [3], is stated as follows. Consider k runners on a circular track. At $t = 0$, all runners are at the same position and start to run; the runner's speeds are pairwise distinct and all runners keep the same speed. A runner is said to be lonely at time t if he is at a distance of at least $\frac{1}{k}$ from every other runner at time t. The lonely runner conjecture states that each runner is lonely at some time. That is, for all finite sets D of positive integer speeds, the Lonely Runner Conjecture asserts that $\kappa(D) \geq \frac{1}{|D|+1}$. The assertion has been proven for $|D| \leq 6$ ([1], [2], [4], [8], [9]).

The elements in a finite D-set of integers can be treated as the speeds of objects traveling around some circle with circumference c, with $c \in \mathbb{Z}$. All of these elements begin from some common point, which we refer to as the "origin" of this circle. The elements traverse the circle as time progresses, so we evaluate their positions, denoted p, at any given time as being the remainder of their total distance traveled, denoted d', divided by c. This is in fact a modular calculation of their positions relative to the origin, with $p = d' \mod c$, and as such, it is possible to find patterns in the positions of all the elements for consecutive moments of time.

The parameter involved in the Lonely Runner Conjecture is defined as follows. For any real number x, let $||x||$ denote the minimum distance from x to an integer, that is, $||x|| = \min\{\lceil x \rceil - x, x - \lfloor x \rfloor\}$. For a given finite set of integers, D, and any real t, denote $||tD||$ as the smallest value of $||td||$ among all $d \in D$. $\kappa(D)$, is the supremum of $||tD||$ among all real t. That is,

$$\kappa(D) = \sup\{\alpha \in (0, 1/2) : ||td|| \geq \alpha \text{ for some } t \in (0, 1), \text{ for all } d \in D\}.$$

For a finite set D of positive integers, the parameter $\kappa(D)$ is closely related to another parameter of D called the "density of integral sequences with missing differences". For a finite set D of positive integers, a sequence S of non-negative integers is called a D-*sequence* if $|x - y| \notin D$ for any $x, y \in S$. Denote $S(n)$ as $|S \cap \{0, 1, 2, \cdots, n-1\}|$. The upper density $\bar{\delta}(S)$ and the lower density $\underline{\delta}(S)$ of S are defined, respectively, by $\bar{\delta}(S) = \overline{\lim}_{n\to\infty} S(n)/n$ and $\underline{\delta}(S) = \underline{\lim}_{n\to\infty} S(n)/n$. We say S has density $\delta(S)$ if $\bar{\delta}(S) = \underline{\delta}(S) = \delta(S)$.The parameter of interest is the *density of* D, $\mu(D)$, defined by

$$\mu(D) := \sup \{ \ \delta(S) : \ S \text{ is a } D\text{-sequence}\}.$$

It is known that for any set D [5]:

$$\mu(D) \geq \kappa(D).$$

The parameters $\kappa(D)$ and $\mu(D)$ are closely related to coloring parameters of distance graphs. Let D be a set of positive integers. The *distance graph generated by* D, denoted as $G(\mathbb{Z}, D)$, has all integers $\mathbb{Z}$ as the vertex set. Two vertices are adjacent whenever their absolute value difference falls in D. The *chromatic number* (minimum number of colors in a proper vertex-coloring) of the distance graph generated by D is denoted by $\chi(D)$. It is known that $\chi(D) \leq \lceil \frac{1}{\kappa(D)} \rceil$ for any set D [21].

The *fractional chromatic number* of a graph G, denoted by $\chi_f(G)$, is the minimum ratio $\frac{m}{n}, m, n \in \mathbb{Z}^+$ of an $\frac{[m]}{n}$-coloring, where an $\frac{[m]}{n}$-coloring is a function of $V(G)$ to n-element subsets of $[m] = \{1, 2, ..., m\}$ such that if $uv \in E(G)$, then $f(u) \cap f(v) = \emptyset$, where $E(G)$ denotes the set of edges of the graph G, and $V(G)$ denotes the set of vertices of G. It is known that for any graph G, $\chi_f(G) \leq \chi(G)$ [21].

Denote the fractional chromatic number of $G(\mathbb{Z}, D)$ by $\chi_f(G)$. Chang et al. [6] proved that for any set of positive integers D, it holds that $\chi_f(D) = \frac{1}{\mu(D)}$. Together with the fact that $\chi_f(G) \leq \chi(G)$, we have

$$\frac{1}{\mu(D)} = \chi_f(D) \leq \chi(D) \leq \lceil \frac{1}{\kappa(D)} \rceil.$$

The chromatic number of distance graphs $G(\mathbb{Z}, D)$ with $D = \{2, 3, x, y\}$ was studied by several authors. For prime numbers x and y, the values of $\chi(D)$ for this family were first studied by Eggleton, Erdös [10] and was later completely solved by Voigt and Walther [16]. For general values of x and y, Kemnitz and Kolberg [13] and Voigt and Walther [17] determined $\chi(D)$ for some values of x and y. This problem was completely solved for all values of x and y by Liu and Setudja [14], in which $\kappa(D)$ was utilized as one of the main tools. In particular, it was proved in [14] that $\kappa(D) \geq \frac{1}{3}$ for many sets in the form $D = \{2, 3, x, y\}$. Hence, by [3], for those sets it holds that $\chi(D) = 3$.

In this thesis, the exact values of $\kappa(D)$ and $\mu(D)$ are studied for the family of sets $D = \{2, 3, x, y\}$. This thesis is organized as follows. In chapters 3, 4 and 5 we find $\kappa(D)$ for most x and y, of sets $D = \{2, 3, x, y\}$. chapters 6, 7 and 8 are dedicated to finding exact values and upper bounds for $\mu(D)$ for $D = \{2, 3, x, y\}$.

CHAPTER 2

Preliminary Concept and Examples

When investigating values of $\kappa(D)$, it is first necessary to understand how $\kappa(D)$ is found. Therefore the beginning of this chapter demonstrates some basic calculations that restrict possible values of $\kappa(D)$. This will be useful later when proceeding to the calculation portion of this chapter.

Let D be a finite set of positive integers and let $\kappa(D) = \frac{d}{c}$, such that $c, d \in \mathbb{R}$. We prove that c is the sum of two numbers from D, and that $\kappa(D) \in \mathbb{Q}$. Let the elements of D be treated as objects traveling around a circle, starting at the same time from some common point, which we call the origin. Let c be the circumference of this circle, and let δ be the position d distance away from the origin while $-\delta$ is the reflected position across the origin, with $-\delta = |c - d|$.

Assume that $\kappa(D)$ is the largest possible fraction such that all elements are at least an absolute distance d away from the origin at some time t. The positions of both δ and $-\delta$ must each be occupied by an element of the D-set traveling along the circle at t, with all elements being at least d distance away from the origin. Note that this means that all elements are in the interval $[\delta, -\delta]$.

To see why both positions must be simultaneously occupied, assume that all elements are at least an absolute distance of δ away from the origin, and that $-\delta$ is not occupied by an element. All elements are in the interval between $[\delta, -\delta]$ on the circle by our assumption. $-\delta$ not being occupied implies that there is some interval of distance in $[\delta, -\delta]$ between $-\delta$ and the nearest element. Define the element closest to $-\delta$ as k_1 and the position of k_1 in $[\delta, -\delta]$ as being p. All elements are in $[\delta, -\delta]$ at

t by assumption, so there is some time t', with $t' > t$, where the distance between the new position of k_1 and $-\delta$ is $\frac{|-\delta-p|}{2}$, while all other elements remain in $[\delta, -\delta]$. If we assess the positions of all of the elements at t', we find that the absolute distances of all of the other elements to the origin have increased, with the smallest of these values being larger than $\kappa(D)$, which contradicts our original assumption that $\kappa(D)$ is the largest possible fraction such that all elements are at least a distance of d away from the origin. Thus $-\delta$ must be occupied. Similar arguments may be used to show that δ must be occupied. Thus we know that both δ and $-\delta$ are occupied at the moment when all elements are within the interval $[\delta, -\delta]$.

Let the time of simultaneous occupation of both δ and $-\delta$ be called t, and let the two elements occupying the δ and $-\delta$ be called k_1 and k_2. Furthermore, let r_1 and r_2 represent the number of revolutions around the circle the elements k_1 and k_2 make before occupying δ and $-\delta$. Note that $k_1, k_2, r_1, r_2 \in \mathbb{Z}$. Multiplying the elements k_1, k_2 by this t, give the following equations.

$$(k_1)(t) = (r_1)(c) + \ \delta$$

$$(k_2)(t) = (r_2)(c) - \delta.$$

Adding these two equations results in

$$(k_1 + k_2)(t) = (r_1 + r_2)(c).$$

Solving for t results in $t = \frac{(r_1+r_2)(c)}{(k_1+k_2)}$. Reapplying this t value to our original multiplication of $(k_1)(t)$ gives us

$$(k_1)(\tfrac{(r_1+r_2)(c)}{(k_1+k_2)}) = \tfrac{(k_1)(r_1+r_2)(c)}{(k_1+k_2)}.$$

Thus, in order to attain $\kappa(D)$, where $\kappa(D) = \frac{d}{c}$, it is necessary for $c = k_1 + k_2$. Also, this gives $\kappa(D) \in \mathbb{Q}$, as desired, and furthermore that $t \in \mathbb{Z}$.

As we have established that the $c = (k_1 + k_2)$, for the remainder of this paper, it is only necessary to investigate bounds for $\kappa(D)$ that are equal to some fractional distance $\frac{d}{c}$, setting c equal to the addition of two of the elements in the D-set. Furthermore, it can be assumed that there is some element E not existing in D that has a value equal to c, which is always at the origin for all t, thus normalizing the frame of reference for the distances of all D-set elements.

Using the definition above, it is easy to calculate $\kappa(D)$ for a D-set. The procedure to finding $\kappa(D)$ is straightforward, if not slightly tedious. Let D be a finite set of positive integers. We create circles of circumference size c, where c is the addition of two of the elements from the D-set. For each time t, let $\frac{d}{c}$ be a fraction such that d is the minimum of the absolute distances for all elements to the origin of that circle with circumference c. When comparing the fractions generated by all t, there will always be a largest fraction generated for this circle. As there are many possible circles, the collection of these largest fractions can be thought of as a set of fractions, with each circle being represented by its single largest fraction. $\kappa(D)$ is the greatest element of this set.

For example, given $D = \{2, 3, 4, 5\}$, there are six possible denominator values to investigate, namely $\{5,\ 6,\ 7,\ 8,\ 9\}$. As 5 is an element of the D-set in addition to being a possible denominator, and as we know that we need only investigate $t \in \mathbb{Z}$, there is no need to calculate the minimum distance, as at every time iteration the minimum distance from all of the elements to the origin is 0. As two separate additions of elements within the D-set result in a value of 7, this circle and its largest possible fraction must only be investigated once. This leaves only four possibilities for examination. Investigating the circumference 6 gives the following table, where the upper left-hand corner contains the value of the circumference, and each row following the speed row contains the values of the time iteration and the resulting absolute distance of the corresponding element to the origin. Each row will have a smallest absolute distance to the origin, and the highlighted box represents the largest of these smallest distances for all times t. Note that the times being investigated stop once the time is equal to the circumference size, as at that point all elements have returned to the origin, and thus would merely repeat the table created.

Table 2.1: $c = 6$, $D = \{2, 3, 4, 5\}$

6				
t	2	3	4	5
1	2	3	2	1
2	2	0	2	2
3	0	3	0	3
4	2	0	2	2
5	2	3	2	1

Note that the maximum of all of the minimum distances for each time iteration is 1, thus making the maximum fractional value for this circumference $\frac{1}{6}$. The other 3 tables are calculated similarly, giving the following tables.

Table 2.2: $c = 7,\ D = \{2, 3, 4, 5\}$

7				
t	2	3	4	5
1	2	3	3	2
2	3	1	1	3
3	1	2	2	1
4	1	2	2	1
5	3	1	1	3
6	2	3	3	2

Table 2.3: $c = 8,\ D = \{2, 3, 4, 5\}$

8				
t	2	3	4	5
1	2	3	4	3
2	4	2	0	2
3	2	1	4	1
4	0	4	0	4
5	0	4	0	4
6	2	1	4	1
7	4	2	0	2

Table 2.4: $c = 9,\ D = \{2, 3, 4, 5\}$

9				
t	2	3	4	5
1	2	3	4	4
2	4	3	1	1
3	3	0	3	3
4	1	3	2	2
5	1	3	2	2
6	3	0	3	3
7	4	3	1	1
8	2	3	4	4

In the set of largest fractions for all circles, $\{\frac{1}{6}, \frac{2}{7}, \frac{2}{8}, \frac{2}{9}\}$, we see that the greatest is achieved with the circumference of 7, with the fraction being $\frac{2}{7}$. Thus, $\kappa(D) = \frac{2}{7}$.

Let $D = \{2, 3, x, y\}$. Note that in the above case, the x and y values were fixed as $x = 4$ and $y = 5$. Finding $\kappa(D)$ was not incredibly difficult with those values being assigned. The challenge lies in trying to generalize this into formulas such that $\kappa(D)$ may be found without having to go through this process. As the process is nothing more than an algorithm, the next logical step in the search for generalization is to write the algorithm used above into a computer program so that the first $\kappa(D)$ values can be found, systematically fixing an x value and then allowing the y to increase so that any patterns might make themselves apparent. The program generated $\kappa(D)$ $D = \{2, 3, x, y\}$, with $4 \leq x \leq 50$ and $x + 1 \leq y \leq 50$.

Figure 2.1: *Java Printout of Initial Results*

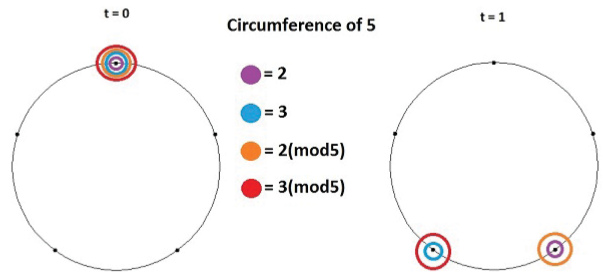
x	y:	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
4	||	1 (7) [2]	1 (8) [2]	2 (11) [3]	1 (6) [2]	1 (6) [2]	1 (6) [2]	2 (13) [4]	2 (14) [4]	3 (17) [5]	1 (6) [2]	1 (6) [2]	1 (6) [2]	3 (19) [6]	3 (20) [6]	4 (23)
5	||	0 (0) [0]	1 (8) [2]	2 (8) [2]	2 (13) [3]	1 (7) [2]	1 (7) [2]	1 (7) [2]	1 (7) [2]	2 (15) [4]	2 (16) [4]	2 (8) [2]	1 (7) [2]	1 (7) [2]	1 (7) [2]	1 (7)
6	||	0 (0) [0]	0 (0) [0]	2 (9) [3]	3 (14) [4]	2 (8) [2]	3 (13) [4]	2 (9) [3]	2 (9) [3]	4 (19) [5]	4 (17) [5]	2 (9) [3]	4 (18) [6]	5 (23) [7]	5 (21) [6]	5 (22)
7	||	0 (0) [0]	0 (0) [0]	0 (0) [0]	1 (5) [2]	3 (16) [5]	3 (13) [4]	3 (14) [5]	1 (5) [2]	1 (5) [2]	4 (21) [7]	2 (9) [3]	4 (19) [7]	1 (5) [2]	1 (5) [2]	5 (26)
8	||	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	2 (11) [4]	2 (12) [4]	4 (19) [6]	1 (5) [2]	1 (5) [2]	3 (16) [6]	3 (17) [6]	3 (18) [6]	1 (5) [2]	1 (5) [2]	4 (21)
9	||	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	2 (12) [4]	2 (13) [4]	2 (14) [4]	2 (11) [4]	2 (11) [4]	3 (17) [6]	3 (18) [6]	3 (19) [6]	5 (27) [9]	2 (11)
10	||	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	2 (13) [4]	2 (14) [4]	2 (15) [4]	2 (12) [4]	2 (12) [4]	2 (12) [4]	3 (19) [6]	3 (20) [6]	3 (13)
11	||	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	3 (14) [5]	5 (24) [7]	4 (17) [5]	4 (18) [6]	3 (14) [5]	3 (14) [5]	3 (20) [6]	5 (22)
12	||	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	1 (5) [2]	5 (26) [8]	4 (18) [6]	4 (19) [7]	1 (5) [2]	1 (5) [2]	6 (31)
13	||	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	3 (16) [6]	5 (28) [9]	6 (29) [9]	1 (5) [2]	1 (5) [2]	4 (21)
14	||	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	3 (17) [6]	3 (18) [6]	6 (31) [9]	3 (16) [6]	3 (16)
15	||	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	3 (18) [6]	3 (19) [6]	5 (21) [6]	3 (17)
16	||	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	4 (19) [7]	3 (20) [6]	5 (22)
17	||	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	1 (5) [2]	7 (36)
18	||	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	0 (0) [0]	4 (21)

The graphic above is a small sample of the printout that resulted from the program, wtih $4 \leq x \leq 18$ and $x+1 \leq y \leq 18$. The first column indicates the value assigned to x and the first row the values of y. The corresponding entry gives the $\kappa(D)$ values that were generated, with each triplet of numbers representing the time (free floating), circumference (in parenthesis) and distance (in brackets) of that particular $\kappa(D)$ value. This led to conjectures about the values of $\kappa(D)$ for a D-set of certain special forms. For instance, the first pattern noticed was that if $D = \{2, 3, x, y\}$ where $x = 2 \mod 5$ or $3 \mod 5$ and $y = 2 \mod 5$ or $3 \mod 5$, then $\kappa(D) = \frac{2}{5}$.

Theorem 2.1. *Let* $D = \{2, 3, x, y\}$ *where* $x = 2 \mod 5$ *or* $3 \mod 5$ *and* $y = 2 \mod 5$ *or* $3 \mod 5$*. Then* $\kappa(D) = \frac{2}{5}$*.*

Proof: Calculating the value of the maximum of minimum distances for $D = \{2, 3, x, y\}$ where x and y are allowed to equal $2 \mod 5$ or $3 \mod 5$ with the circumference of 5 (the addition of the elements 2 and 3 from our D-set) is easy, as $t = 1$ places all of the elements a distance of 2 away from the origin as seen in the graphic below.

Figure 2.2: *Movement of Runners for the First Time Iteration*



This can also be visualized with a table as previously seen:

Table 2.5: $c = 5,\ D = \{2, 3, x, y\}$

5				
t	2	3	2(mod5)	3(mod5)
2	2	2	2	2
:	:	:	:	:
:	:	:	:	:
:	:	:	:	:
:	:	:	:	:

It is obvious that if x and y are only allowed to either be $2 \mod 5$ or $3 \mod 5$ then at the first time iteration on the circle of circumference 5 all elements are at least $\frac{2}{5}$ away from the origin. Thus, we know that $\kappa(D) \geq \frac{2}{5}$, as $\kappa(D)$ is the supremum of the minimum distances, and the minimum distance for a circumference 5 is $\frac{2}{5}$. Furthermore, it is known that $\kappa(D') = \frac{2}{5}$ for $D' = \{2, 3\}$. As $D' \subseteq D$, it is thus proved that $\kappa(D) = \frac{2}{5}$, as $\kappa(D) \leq \kappa(D')$.

With the $2 \mod 5$, $3 \mod 5$ cases completely solved, we must look for another pattern in the generated $\kappa(D)$ values. The most obvious place to begin an investigation is to fix $y - x$. And indeed, a pattern was found and equalities proven.

CHAPTER 3

Finding $\kappa(D)$ for $D = \{2, 3, x, x+1\}$

Let $D = \{2, 3, x, x+1\}$. An algorithm in the previous chapter was used to find $\kappa(D)$ for the first 50 values of x, but now another equivalent definition of $\kappa(D)$ is used to prove conjectured values for all x-values. This continuous definition of $\kappa(D)$ is stated as follows.

Let $\alpha \in (0, \frac{1}{2})$, where $\alpha = \frac{d}{c}$ with $c, d \in \mathbb{Z}$. For each positive integer i, define

$$I_i(\alpha) = \{t \in (0,1) : \| \, ti \, \| \geq \alpha\}.$$

That is [11],

$$I_i(\alpha) = \{t : n + \alpha \leq ti \leq n + 1 - \alpha, 0 \leq n \leq i - 1\},$$

or alternatively

$$I_i(\alpha) = \{t : \tfrac{n+\alpha}{i} \leq t \leq \tfrac{n+1-\alpha}{i}, 0 \leq n \leq i - 1\},$$

where n represents the number of revolutions the element goes around a circle, d being the absolute minimum distance to the origin and c being the circumference of the circle.

The following relationship will be used extensively throughout this paper to establish a lower bound for $\kappa(D)$:

$$\kappa(D) \geq \sup\{\alpha \in (0, \tfrac{1}{2}) : \textstyle\bigcap_{i \in D} I_i(\alpha) \neq \emptyset\}.$$

It is sufficient to show that given $\alpha = \frac{d}{c}$, if

$$I_i(\alpha) = \textstyle\bigcup_{n=0}^{l} [\frac{d+(n)(c)}{i}, \frac{c-d+(n)(c)}{i}], \; l \in \mathbb{N} \text{ with the max } l \text{ being constrained by}$$

$$\frac{c-d+(l)(c)}{i} \leq c, \text{ and with } \bigcap_{i \in D} I_i(\alpha) \neq \emptyset,$$

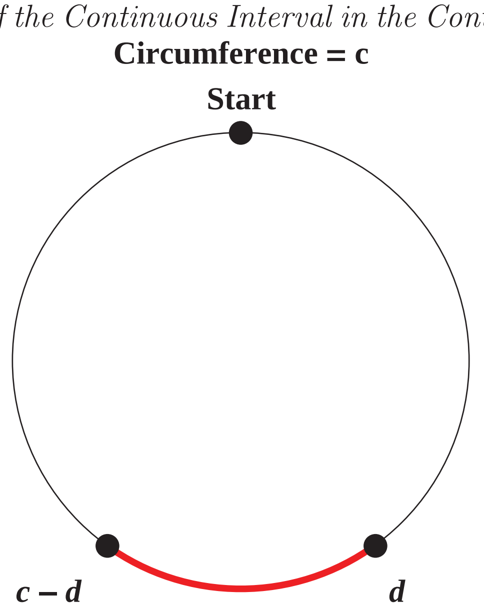
then

$$\kappa(D) \geq \tfrac{d}{c}.$$

Furthermore, if it can be shown that $\{\alpha \in (0, \frac{1}{2}) : \bigcap_{i \in D} I_i(\alpha)\}$ consists only of a set of isolated points, then this is sufficient to prove that the conjectured $\alpha = \kappa(D)$. To this end, we introduce the notation $I_{2,1}(\alpha)$ to indicate the first time interval from the set of intervals generated by the element 2, so that we may identify specific time intervals within the set of intervals an element generates over α for which it is at least an absolute distance d away from the origin.

The continuous intervals, I_i, noted above can be thought of as the intervals of time for which a runner is in $[d, c-d]$. The runners all start at the same time at the origin, and can be thought to stop the first instant they find themselves to all be back at the origin. Within that interval of time between the start and their first coincidence at the start after beginning, the number of times a runner goes through $[d, c-d]$ depends directly on the speed of the runner, with faster participants passing through more times than a slower, and thus having more intervals of time generated.

Figure 3.1: *Illustration of the Continuous Interval in the Context of the Track Analogy*



Fix the fourth element in the D-set to be a distance of 1 away from the third element. Thus we have $D = \{2, 3, x, x+1\}$. When investigating the $\kappa(D)$ for iterations of x, starting at $x = 4$, a pattern emerges where all $\kappa(D)$ for this D-set can be described as follows.

Theorem 3.1. *Suppose $D = \{2, 3, x, x+1\}$. Let $x + 3 = 5\gamma + r$ with $0 \leq r \leq 4$, then*

$$\kappa(D) = \begin{cases} \frac{2\gamma}{y+2} & \text{if } 0 \leq r \leq 3; \\ \\ \frac{2\gamma+1}{y+2} & \text{if } r = 4. \end{cases}$$

We will prove this conjecture using the continuous definition of $\kappa(D)$, meaning that we will investigate individual time intervals from each of the sets generated by the elements, and discover where these intervals intersect. As a reminder, if we can demonstrate that all intersections of intervals contain only singleton points, then we have in fact proven the conjectured equalities.

Case 1 $x = 5k+2$. Suppose $x = 5k+2$ for some $k \in \mathbb{N}$. Then $x \equiv 2 \mod 5$ and $y \equiv 3 \mod 5$. As both $x = 2 \mod 5$ and $y = 3 \mod 5$, by Theorem 2.1 we know $\kappa(D) = \frac{2}{5}$.

Case 2 $x = 5k+3$. Suppose $x = 5k+3$ for some $k \in \mathbb{N}$. Then $y + 2 = 5k + 6$, $\gamma = k + 1$ and $r = 1$. We claim

$$\kappa(D) = \frac{2k+2}{5k+6}.$$

Let $\alpha = \frac{2k+2}{5k+6}$. Using this value of α, the set of intervals is

$I_i(\alpha) = \bigcup_{n=0}^{l}[\frac{2k+2+n(5k+6)}{i}, \frac{3k+4+n(5k+6)}{i}]$ with $l \in \mathbf{N}$ with the maximum l being such that $\frac{3k+4+l(5k+6)}{i} \leq (5k+6)$.

Calculating the intersection of intervals for $I_{2,1}(\alpha)$ and $I_{3,1}(\alpha)$ gives the following result

$$\bigcap_{i=2,3} I_{i,1}(\alpha) = [\tfrac{2k+2}{2}, \tfrac{3k+4}{3}] \;=\; [k+1, \tfrac{3k+4}{3}].$$

As the intervals for 2 and 3 act symmetrically across the $t = \frac{c}{2}$ line, and further that there is a single interval $I_2(\alpha)$ before this line, we need only investigate the intersection $\bigcap_{i=2,3} I_{i,1}(\alpha)$, as $I_{3,2}(\alpha)$ does not intersect $I_{2,1}(\alpha)$ at all.

Continuing our investigation, we look to see if we can identify single intervals from the other elements in D that might intersect with $\bigcap_{i=2,3} I_{i,1}(\alpha)$. Letting $n = k$ for the I_{x+1} interval generates

$$I_{x+1,k}(\alpha) \;=\; [\tfrac{2k+2+(k)(5k+6)}{5k+4}, \tfrac{3k+4+(k)(5k+6)}{5k+4}] = [\tfrac{5k^2+8k+2}{5k+4}, k+1].$$

The interval of intersection for 2, 3, and $x+1$ is thus

$$I_{2,1}(\alpha) \bigcap I_{3,1}(\alpha) \bigcap I_{x+1,k}(\alpha) = \{k+1\}.$$

As noted previouosly, $I_{2,1}(\alpha) \bigcap I_{3,1}(\alpha) = [k+1, \frac{3k+4}{3}]$ is the only interval which must be investigated, excluding symmetrically identical intervals, which in turn restricts the values of n for $x+1$ that need to be investigated. As we started with I_{x+1}, note that letting $n = k-1$ for the I_{x+1} interval generates

$$I_{x+1,k-1}(\alpha) \;=\; [\tfrac{2k+2+(k-1)(5k+6)}{5k+4}, \tfrac{3k+4+(k-1)(5k+6)}{5k+4}] = [\tfrac{5k^2+3k-4}{5k+4}, \tfrac{5k^2+4k-2}{5k+4}].$$

Comparing this to $I_{2,1}(\alpha) \bigcap I_{3,1}(\alpha)$ immediately reveals that there is no value of k that will ever be sufficient to create a non-empty intersection, which further gives

$$I_{2,1}(\alpha) \bigcap I_{3,1}(\alpha) \bigcap I_{x+1,n \le k-1}(\alpha) = \emptyset, \text{ for all } n \le k-1.$$

Therefore we can eliminate $n \le k-1$ from the possible values that might generate an intersection.

Letting $n = k+1$ for the I_{x+1} interval generates

$$I_{x+1,k+1}(\alpha) \;=\; [\tfrac{2k+2+(k+1)(5k+6)}{5k+4}, \tfrac{3k+4+(k+1)(5k+6)}{5k+4}] = [\tfrac{5k^2+13k+8}{5k+4}, \tfrac{5k^2+14k+10}{5k+4}].$$

Once again we find that regardless of chosen k values, there will never be a non-empty intersection, let alone an interval. This gives

$$I_{2,1}(\alpha)\bigcap I_{3,1}(\alpha)\bigcap I_{x+1,n}(\alpha)=\emptyset, \text{ for all } n\geq k+1.$$

Thus we must choose $n=k$ for our I_{x+1} interval such that $I_2(\alpha)\bigcap I_3(\alpha)\bigcap I_{x+1}(\alpha)\neq\emptyset$, excluding symmetry.

Now all that remains is to show that $k+1\in I_x(\alpha)$. Choosing $n=k$ gives

$$I_{x,k}(\alpha) = [\tfrac{2k+2+(k)(5k+6)}{5k+3}, \tfrac{3k+4+(k)(5k+6)}{5k+3}] = [\tfrac{5k^2+8k+2}{5k+3}, \tfrac{5k^2+9k+4}{5k+3}].$$

Since $\{k+1\}\ \in I_{x,k}(\alpha)$, as is apparent when comparing $k+1$ to the interval $I_{x,k}(\alpha)$, the proof that

$$I_2(\alpha)\bigcap I_3(\alpha)\bigcap I_x(\alpha)\bigcap I_{x+1}(\alpha)\neq\emptyset$$

is complete, implying

$$\kappa(D)\geq\tfrac{2k+2}{5k+6}.$$

The calculations above show that $I_2(\alpha)\bigcap I_3(\alpha)\bigcap I_x(\alpha)\bigcap I_{x+1}(\alpha)$ contain only singleton points of intersection. Therefore we may then conclude that $\kappa(D)=\frac{2k+2}{5k+6}$, as desired.

Computationally, this is quite arduous, so to expedite the process, we articulate the steps that we did by hand in a Mathematica program, so that we may input values for n_x and n_y, the number of revolutions x and y complete before generating the desired interval, and investigate whether there are intersections without having to perform the computations. The code for this is attached in Appendix A.2. When writing this code, it is useful to note that we are comparing fractions, and thus may use a computer to compute a cross-multiplication that will have the same effect for evaluating intersection intervals. Additionally, we observe that $I_2 \bigcap I_3$ create two intervals of intersection, which are symmetric about the line $t = \frac{c}{2}$. As such we focus on the left of these intervals $\bigcap\limits_{i=2,3} I_{i,1} = [I_{2l}, I_{3r}]$, where I_{2l} symbolizes the left side of the interval $I_{2,1}$, and I_{3r} symbolizes the right side of $I_{3,1}$. Let us use the next case to show how these facts can be used to prove the claim of the next case.

Case 3 $x = 5k + 4$

One additional fact that we will use when employing the Mathematica program is that when comparing fractions, we do so by creating an LCD, which implies that we must only really investigate the overlapping or lack thereof of the numerators when multiplied by the denominator of the intervals we are comparing it against. For example, if we were comparing I_{2l} to some $I_{y,n}$, we must only investigate the numerator of I_{2l} multiplied by y against the numerators of $I_{y,n}$ multiplied by 2.

Let $x = 5k + 4$, setting $y = 5k + 5, x + 3 = 5\gamma + 2$, $\gamma = k + 1$ and $r = 2$. We claim

$$\kappa(D) = \tfrac{2k+2}{5k+7}.$$

Let $\alpha = \frac{2k+2}{5k+7}$. Using this value of α, the set of intervals is

$I_i(\alpha) = \bigcup_{n=0}^{l}[\frac{2k+2+n(5k+7)}{i}, \frac{3k+5+n(5k+7)}{i}]$ with $l \in \mathbb{N}$ such that $\frac{3k+5+l(5k+7)}{i} \leq (5k + 7)$.

Calculating the $I_{2,1}(\alpha) \bigcap I_{3,1}(\alpha)$ gives the following result

$$\bigcap_{i=2,3} I_{i,1}(\alpha) = [\tfrac{2k+2}{2}, \tfrac{3k+5}{3}] = [k + 1, \tfrac{3k+5}{3}].$$

Let $n = k$ for the $I_{y,k}(\alpha)$. Using Mathematica to compare this interval against $I_{2l}(\alpha)$ gives

$$[4 + 18k + 10k^2, 10 + 20k + 10k^2]$$

as the numerators of the interval $I_{y,k}(\alpha)$ are multiplied by 2. As $I_{2l}(\alpha)$ is multiplied by $5k+5$, with $y = 5k+5$, this gives the numerator

$$\{10 + 20k + 10k^2\},$$

it is obvious that the right-most endpoint of $I_{y,k}(\alpha)$ is exactly equal to $I_{2l}(\alpha)$, thus demonstrating that there is a singleton point of intersection, namely $k+1$. This is also convenient because it automatically shows that any $n < k$, $I_{y,n<k}(\alpha) \bigcap I_{2l}(\alpha) = \emptyset$, as any smaller k value would remove I_{2l} from intersecting with I_y. All that remains is to ensure $n > t$ also eliminates the intersection, and additionally that $I_x(\alpha)$ contains the same singleton point of intersection for some n. For $I_{y,k+1}(\alpha)$, we once again use Mathematica, but this time to compare the interval against $I_{3r}(\alpha)$ gives

$$[27 + 42k + 15k^2, 36 + 45k + 15k^2]$$

for the numerators of $I_{y,k+1}(\alpha)$ multiplied by 3. As $I_{3r}(\alpha)$ multiplied by y gives the numerator

$$\{25 + 40k + 15k^2\},$$

we can easily conclude that $I_{y,n>k}(\alpha) \bigcap I_{2l}(\alpha) = \emptyset$, proving that the intersection only occurs when $n = k$.

Additionally, calculating $I_{x,k}(\alpha)$ multiplied by 2 gives the numerators

$$[4 + 18k + 10k^2, 10 + 20k + 10k^2]$$

and the $I_{2l}(\alpha)$ multiplied by x gives the numerator

$$\{8 + 18k + 10k^2\}.$$

It is immediately apparent that any k chosen will have an intersection, which shows that $I_{2l}(\alpha) \bigcap I_{x,k}(\alpha) \neq \emptyset$, proving that $\{k+1\} \in I_{x,k}(\alpha)$. Thus the proof that $\kappa(D) = \frac{2k+2}{5k+7}$ is complete.

Case 4 $x = 5k + 5$

Case 5 $x = 5k + 6$

The remaining proofs are identical in structure, and result in the same singleton points of intersections. Thus $\kappa(D)$ is completely known with $D = \{2, 3, x, x+1\}$, as all values of $x \mod 5$ have been considered.

CHAPTER 4

Extending Patterns: Finding $\kappa(D)$ for $D = \{2, 3, x, x+i\}$ with $2 \leq i \leq 5$

As $\kappa(D)$ has been completely established for $D = \{2, 3, x, x+1\}$, in this chapter $\kappa(D)$ is found for $D = \{2, 3, x, y\}$ with $y = x+2$, $y = x+3$, $y = x+4$, and $y = x+5$.

Theorem 4.1. *Let $D = \{2, 3, x, x+2\}$. Let $x+4 = 6\gamma + r$ with $0 \leq r \leq 5$. Then*

$$\kappa(D) = \begin{cases} \frac{2\gamma}{y+2} & \text{if } 0 \leq r \leq 2; \\ \frac{4\gamma+r-4}{x+y} & \text{if } 3 \leq r \leq 5. \end{cases}$$

The proof is the same as in the $x+1$ case, and thus will be left to the reader. The concept is the same as in Chapter 3, with intersections only happening at singleton points of intersection for any given n and t.

Increasing the difference between the third and fourth elements of the D-set by amounts of 1 results in the following $\kappa(D)$ values.

Theorem 4.2. *Let $D = \{2, 3, x, x+3\}$. Let $2x+3 = 9\gamma + r$ with $0 \leq r \leq 8$. Then*

$$\kappa(D) = \begin{cases} \frac{4}{15} & \text{if } x = 10; \\ \frac{3\gamma}{2x+3} & \text{if } 0 \leq r \leq 5; \\ \frac{\lfloor \frac{x+5}{3} \rfloor}{x+6} & \text{if } 6 \leq r \leq 8. \end{cases}$$

Theorem 4.3. *Let* $D = \{2, 3, x, x+4\}$*. Let* $x + 4 = 5\gamma + r$ *with* $0 \le r \le 4$*. Then*

$$\kappa(D) = \begin{cases} \frac{2\gamma+r}{x+7} & \text{if } 0 \le r \le 1; \\ \frac{2\gamma}{x+2} & \text{if } 2 \le r \le 4. \end{cases}$$

Theorem 4.4. *Let* $D = \{2, 3, x, x+5\}$*. Let* $x + 3 = 5\gamma + r$ *with* $0 \le r \le 4$*. Then*

$$\kappa(D) = \begin{cases} \frac{2}{5} & \text{if } 0 \le r \le 1; \\ \frac{2\gamma}{x+2} & \text{if } 2 \le r \le 3; \\ \frac{2\gamma+1}{x+3} & \text{if } r = 4. \end{cases}$$

All of the proofs for these are similar to those performed in the $y = x + 1$ cases, and thus will be left to the reader.

CHAPTER 5

Extending Results for $y \geq x + 6$

In this chapter, similar methods to those in the previous two chapters are used to explore the values of $\kappa(D)$ for more generalized families of D-sets. From the previous chapter, $\kappa(D)$ is known for $|y - x| \leq 5$, where $D = \{2, 3, x, y\}$. The next step is to proceed to the $|y - x| = 6$ case. This is when it first becomes apparent that there is an even more general overarching behavior to the $\kappa(D)$ values. It becomes clear that *all* distances between x and y could be generalized into 5 main cases, labeled $x + 5\beta + i, 0 \leq i \leq 4$, respectively. The results of $\kappa(D)$ in these cases are presented in the following tables.

In the following theorems, notation is used to simplify the tables. The meanings for the symbols are as follows:

$\gamma =$ term defined by the modulo relationship, used in creating the numerator of $\kappa(D)$.

$d =$ the numerator of $\kappa(D)$.

$c =$ the denominator of $\kappa(D)$.

$r =$ the modulo difference being used to define the specific values of x for the subcases in the tables for $\kappa(D)$.

$n_x =$ the number of revolutions the element x goes around the circle before generating the interval which intersects all other intervals in $\kappa(D)$.

$n_y =$ the number of revolutions the last element in the D-set goes around the circle before generating the interval which intersects all other intervals in $\kappa(D)$.

$POI =$ the exact value of the singleton point of intersection generated by the two elements which are added together to obtain the circumference of $\kappa(D)$, when applicable.

$\kappa(D)$ = the value of $\kappa(D)$ within the given constraints.

5.1 $y = x + 5\beta,\ \beta \geq 1$

In the table, k, r and γ are determined by $x = 2 \mod 5 + r$, $x \geq 4$, with $0 \leq r \leq 4$, and $x + 3 = 5\gamma + r$.

Theorem 5.1. *Let $D = \{2, 3, x, x + 5\beta\},\ k \geq 2\beta$. Then the values of $\kappa(D)$ given in the tables are exact for $k \geq 2\beta$ and provide upper bounds when $k < 2\beta$.*

Table 5.1: $\kappa(D)$, $D = \{2, 3, x, x + 5\beta\}$

$y = x + 5\beta$, $\beta \geq 2k$

x	γ	d	c	r
$5k + 7$	$k+2$	2γ	$x+3$	0
$5k + 8$	$k+2$	2γ	$x+2$	1
$5k + 4$	$k+1$	2γ	$x+2$	2
$5k + 5$	$k+1$	2γ	$x+2$	3
$5k + 6$	$k+1$	$2\gamma+1$	$x+3$	4

Table 5.2: $\kappa(D)$, $D = \{2, 3, x, x + 5\beta\}$ Continued

$y = x + 5\beta$, $\beta \geq 2k$				
x	n_x	n_y	POI	$\kappa(D)$
$5k + 7$	$k + 1$	$k+\beta+1$	$2 + k$	$\frac{2}{5}$
$5k + 8$	$k + 1$	$k+\beta+1$	$2 + 2k$	$\frac{2}{5}$
$5k + 4$	k	$k + \beta$	$1 + k$	$\frac{2+2k}{6+5k}$
$5k + 5$	k	$k + \beta$	$1 + k$	$\frac{2+2k}{7+5k}$
$5k + 6$	$k + 1$	$k+\beta+1$	$2 + k$	$\frac{3+2k}{9+5k}$

5.2 $y = x + 5\beta + 1$, $\beta \geq 1$

In the table, k, r and γ are determined by $x = 2 \mod 5 + r$, $x \geq 4$, with $0 \leq r \leq 4$, and $x + 3 = 5\gamma + r$.

Theorem 5.2. *Let $D = \{2, 3, x, x+5\beta+1\}$, $k \geq 2\beta$. Then the values of $\kappa(D)$ given in the tables are exact for $k \geq 2\beta$ and provide upper bounds when $k < 2\beta$ for $x = i$* mod $5, i = 0, 2, 3, 4$. *For $x = 1 \mod 5$, the value of $\kappa(D)$ given in the table begins at $k \geq 3\beta$ and provides an upper bound when $k < 3\beta$.*

Table 5.3: $\kappa(D),\ D = \{2, 3, x, x + 5\beta + 1\}$

$y = x + 5\beta + 1$				
x	γ	d	c	r
$5k + 7$	$k + 2 + \beta$	2γ	$y+2$	0
$5k + 8$	$k + 2 + \beta$	2γ	$y+2$	1
$5k + 4$	$k + 1 + \beta$	2γ	$y+2$	2
$5k + 5$	$k + 1 + \beta$	2γ	$y+2$	3
$5k + 6$	$k + 1 + \beta$	2γ	$y+2$	4

Table 5.4: $\kappa(D)$, $D = \{2, 3, x, x + 5\beta + 1\}$ Continued

$y = x + 5\beta + 1$				
x	n_x	n_y	POI	$\kappa(D)$
$5k + 7$	$k + 1$	$k+\beta+1$	$2 + k+\beta$	$\frac{2}{5}$
$5k + 8$	$k + 1$	$k+\beta+1$	$2 + 2k+\beta$	$\frac{4+2k+2\beta}{11+5k+5\beta}$
$5k + 4$	k	$k+\beta$	$1 + k+\beta$	$\frac{2+2k+2\beta}{7+5k+5\beta}$
$5k + 5$	k	$k+\beta$	$1 + k+\beta$	$\frac{2+2k+2\beta}{8+5k+5\beta}$
$5k + 6$	k	$k+\beta+1$	$1 + k+\beta$	$\frac{2+2k+2\beta}{9+5k+5\beta}$

5.3 $y = x + 5\beta + 2,\ \beta \geq 1$

Let γ be defined by $x + 5\beta + 4 = 6\gamma + 5\beta\gamma + r$, $0 \leq r \leq 6 + 5m + 4$. Note that in this section, by our definition of γ, there are necessarily $6 + 5\beta$ values of x that must occur. Let us call the 5β cases the remainder cases. The amount of remainder cases occurring are then added to the original 6 subcases. Thus, we will use m with $m = 0, 1, 2, ..., \beta - 1$ to describe the additional cases added to the original 6.

Theorem 5.3. *Let $D = \{2, 3, x, x + 5\beta + 2\}$. Then the values of $\kappa(D)$ given in the tables are exact for $k \geq 2\beta$ and provide upper bounds when $k < 2\beta$ for all non-remainder cases. For the remainder cases, the values of $\kappa(D)$ provided in the table give a lower bound for $\kappa(D)$ when $k \geq 2\beta$.*

Table 5.5: $\kappa(D),\ D=\{2,3,x,x+5\beta+2\}$

$y = x + 5\beta + 2, \quad 0 \le m \le \beta - 1$				
x	γ	d	c	r
$6k+5\ \beta\ k{+}5\beta{+}8$	$k+2$	$(2\gamma)(\beta+1)$	$y+2$	0
$6k{+}5\beta k{+}5\beta{+}9$	$k+2$	$(2\gamma)(\beta+1)$	$y+2$	1
$6k+5\ \beta\ k{+}4$	$k{+}1$	$(2\gamma)(\beta+1)$	$y+2$	2
$6k+5\ \beta\ k{+}5$	$k+1$	$(2\gamma)(\beta+1)$	$y+2$	3
$6k{+}5\beta\ k{+}6$	$k+1$	$(4\gamma)(\beta+1)-3\beta$	$x+y$	4
$6k+5\ \beta\ k{+}7$	$k+1$	$(4\gamma)(\beta+1)+1-2\beta$	$x+y$	5
$6k{+}5\beta\ k+8+5m$	$k+1$	$(2\gamma)(\beta+1)+2m$	$y+2$	$6+5m$
$6k{+}5\beta\ k+9+5m$	$k+1$	$(2\gamma)(\beta+1)+2m$	$y+2$	$6{+}1{+}5m$
$6k{+}5\beta\ k+10+5m$	$k+1$	$(2\gamma)(\beta+1)+2m$	$y+2$	$6{+}2{+}5m$
$6k{+}5\beta\ k+11+5m$	$k+1$	$(2\gamma)(\beta+1)+2m$	$y+2$	$6{+}3{+}5m$
$6k{+}5\beta\ k+12+5m$	$k+1$	$(4\gamma)(\beta{+}1){+}1{-}2\beta{+}3m$	$x+y$	$6{+}4{+}5m$

Table 5.6: $\kappa(D)$, $D=\{2,3,x,x+5\beta+2\}$ Continued

$y = x + 5\beta + 2, \quad 0 \le m \le \beta - 1$				
x	n_x	n_y	POI	$\kappa(D)$
$6k{+}5\beta\ k{+}5\beta{+}8$	$(\beta+1)(k+1)$	$(\beta+1)(k+1)+\beta$	$(3+2k)(1+\beta)$	$\frac{2+2\beta}{6+5\beta}$
$6k{+}5\beta\ k{+}5\beta{+}9$	$(\beta+1)(k+1)$	$(\beta+1)(k+1)+\beta$	$(2+k)(1+\beta)$	$\frac{(2+2\beta)(2+k)}{13+10\beta+k(6+5\beta)}$
$6k{+}5\beta\ k{+}4$	$(\beta+1)(k)$	$(\beta+1)(k)+\beta$	$(1+k)(1+\beta)$	$\frac{(2+2\beta)(1+k)}{8+5\beta+k(6+5\beta)}$
$6k{+}5\beta\ k{+}5$	$(\beta+1)(k)$	$(\beta+1)(k)+\beta$	$(1+k)(1+\beta)$	$\frac{(2+2\beta)(1+k)}{9+5\beta+k(6+5\beta)}$
$6k{+}5\beta\ k{+}6$	$(\beta+1)(k)+1$	$(\beta)(k)+\beta+1$	$(1+k)(1+\beta)$	$\frac{4+\beta+4k(\beta+1)}{14+5\beta+2k(6+5\beta)}$
$6k{+}5\beta\ k{+}7$	$(\beta+1)(k)+1$	$(\beta+1)(k)+\beta+1$	$3+\beta+2k(1+\beta)$	$\frac{5+2\beta+4k(1+\beta)}{16+5\beta+2k(6+5\beta)}$
$6k{+}5\beta\ k+8+5m$	$(\beta+1)(k)+m$	$(\beta+1)(k)+\beta+m$	—-	$\frac{2(1+k+m+\beta+k\beta)}{12+5m+5\beta+k(6+5\beta)}$
$6k{+}5\beta k+9+5m$	$(\beta+1)(k)+m$	$(\beta+1)(k)+\beta+m$	—-	$\frac{2(1+k+m+\beta+k\beta)}{13+5m+5\beta+k(6+5\beta)}$
$6k{+}5\beta k+10+5m$	$(\beta+1)(k)+m$	$(\beta+1)(k)+\beta+m$	—-	$\frac{2(1+k+m+\beta+k\beta)}{14+5m+5\beta+k(6+5\beta)}$
$6k{+}5\beta k+11+5m$	$(\beta{+}1)(k){+}m{+}1$	$(\beta{+}1)(k){+}\beta{+}m{+}1$	—-	$\frac{2(1+k+m+\beta+k\beta)}{15+5m+5\beta+k(6+5\beta)}$
$6k{+}5\beta k+12+5m$	$(\beta{+}1)(k){+}m{+}2$	$(\beta{+}1)(k){+}\beta{+}m{+}2$	—-	$\frac{6+4m+2\beta+4k(1+\beta)}{26+10m+5\beta+2k(6+5\beta)}$

5.4 $y = x + 5\beta + 3,\ \beta \geq 1$

Let γ be defined by $x + 5\beta + 5 = 9\gamma + 5\beta\gamma + r$, $0 \leq r \leq 9 + 5m + 4$. Note that in this section, by our definition of γ, there are necessarily $9 + 5\beta$ values of x that must occur. Let us call the 5β cases the remainder cases. The amount of remainder cases occurring are then added to the original 9 subcases. Thus, we will use m with $m = 0, 1, 2, ..., \beta - 1$ to describe the additional cases added to the original 9.

Theorem 5.4. *Let $D = \{2, 3, x, x + 5\beta + 3\}$, $k \geq 2\beta$. Then the values of $\kappa(D)$ given in the tables are exact for $k \geq 2\beta$ and provide upper bounds when $k < 2\beta$ for all non-remainder cases. For the remainder cases, the values of $\kappa(D)$ provided in the table give a lower bound for $\kappa(D)$ when $k \geq 2\beta$*

Table 5.7: $\kappa(D),\ D=\{2,3,x,x+5\beta+3\}$

$y=x+5\beta+3,\quad 0\le m\le\beta-1,\ \beta\ge 2k$				
x	γ	d	c	r
$9k+5\beta k+4$	$k+1$	$(2\gamma)(3+2\beta)-3(\beta+1)$	$x+y$	0
$9k+5\beta k+5$	$k+1$	$(\gamma)(3+2\beta)$ -1	$y+3$	1
$9k+5\beta k+6$	$k+1$	$(\gamma)(\beta+1)$	$y+3$	2
$9k+5\beta k+7$	$k+1$	$(\gamma)(\beta+1)$ +1	$y+3$	3
$9k+5\beta\ k+8$	$k+1$	$(2\gamma)(3+2\beta)-2\beta$	$x+y$	4
$9k+5\beta\ k+9$	$k+1$	$(2\gamma)(3+2\beta)-3\beta$	$x+y$	5
$9k+5\beta\ k+10$	$k+1$	$(\gamma)(3+2\beta)+1$	$y+3$	6
$9k+5\beta\ k+11$	$k+1$	$(\gamma)(3+2\beta)+2$	$y+3$	7
$9k+5\beta\ k+12$	$k+1$	$(\gamma)(3+2\beta)+3$	$y+3$	8
$9k+5\beta\ k+13+5m$	$k+1$	$(2\gamma)(3+2\beta)-2\beta+3m+3$	$x+y$	$9+5m$
$9k+5\beta\ k+14+5m$	$k+1$	$(\gamma)(3+2\beta)+2m+2$	$y+3$	$9+5m+1$
$9k+5\beta\ k+15+5m$	$k+1$	$(\gamma)(3+2\beta)+2m+3$	$y+3$	$9+5m+2$
$9k+5\beta\ k+16+5m$	$k+1$	$(\gamma)(3+2\beta)+2m+4$	$y+3$	$9+5m+3$
$9k+5\beta\ k+17+5m$	$k+1$	$(\gamma)(3+2\beta)+2m+5$	$y+3$	$9+5m+4$

Table 5.8: $\kappa(D)$, $D=\{2,3,x,x+5\beta+3\}$ Continued

$y = x + 5\beta + 3, \quad 0 \le m \le \beta - 1,\ \beta \ge 2k$				
x	n_x	n_y	POI	$\kappa(D)$
$9k+5\beta\ k+4$	$2k+\beta k$	$2k+\beta k+\beta+1$	$(1+2k)(2+\beta)$	$\frac{3+\beta+k(6+4\beta)}{11+5\beta+2k(9+5\beta)}$
$9k+5\beta\ k+5$	$2k+\beta k+1$	$2k+2\beta k+\beta+2$	$(1+2k)(2+\beta)$	$\frac{2+3k+2\beta+2\beta k}{11+9k+5\beta+5\beta k}$
$9k+5\beta\ k+6$	$2k+\beta k+1$	$2k+2\beta k+\beta+2$	$(1+2k)(2+\beta)$	$\frac{(1+k)(3+2\beta)}{12+5\beta+k(9+5\beta)}$
$9k+5\beta\ k+7$	$2k+\beta k+1$	$2k+2\beta k+\beta+2$	$(1+2k)(2+\beta)$	$\frac{1+(1+k)(3+2\beta)}{13+5\beta+k(9+5\beta)}$
$9k+5\beta\ k+8$	$2k+\beta k+1$	$2k+2\beta k+\beta+2$	$(1+2k)(2+\beta)$	$\frac{2(3+\beta+k(3+2\beta))}{19+5\beta+2k(9+5\beta)}$
$9k+5\beta\ k+9$	$2k+\beta k+1$	$2k+2\beta k+\beta+2$	$(1+2k)(2+\beta)$	$\frac{6+\beta+k(6+4\beta))}{21+5\beta+2k(9+5\beta)}$
$9k+5\beta\ k+10$	$2k+\beta k+2$	$2k+2\beta k+\beta+3$	$(1+2k)(2+\beta)$	$\frac{1+(1+k)(3+2\beta)}{16+5\beta+k(9+5\beta)}$
$9k+5\beta\ k+11$	$2k+\beta k+2$	$2k+2\beta k+\beta+3$	$(1+2k)(2+\beta)$	$\frac{2+(1+k)(3+2\beta)}{17+5\beta+k(9+5\beta)}$
$9k+5\beta\ k+12$	$2k+\beta k+2$	$2k+2\beta k+\beta+3$	$(1+2k)(2+\beta)$	$\frac{3+(1+k)(3+2\beta)}{18+5\beta+k(9+5\beta)}$
$9k+5\beta\ k+13+5m$	$2k+\beta k+m+2$	$2k+\beta k+m+\beta+3$	—-	$\frac{9+3m+2\beta+k(6+4\beta)}{29+10m+5\beta+2k(9+5\beta)}$
$9k+5\beta\ k+14+5m$	$2k+\beta k+m+3$	$2k+\beta k+m+\beta+4$	—-	$\frac{5+2m+2\beta+k(3+2\beta)}{5(4+m+\beta)+k(9+5\beta)}$
$9k+5\beta\ k+15+5m$	$2k+\beta k+m+3$	$2k+\beta k+m+\beta+4$	—-	$\frac{5+2m+2\beta+k(3+2\beta)}{5(4+m+\beta)+k(9+5\beta)}$
$9k+5\beta\ k+16+5m$	$2k+\beta k+m+3$	$2k+\beta k+m+\beta+4$	—-	$\frac{5+2m+2\beta+k(3+2\beta)}{5(4+m+\beta)+k(9+5\beta)}$
$9k+5\beta\ k+17+5m$	$2k+\beta k+m+3$	$2k+\beta k+m+\beta+4$	—-	$\frac{5+2m+2\beta+k(3+2\beta)}{5(4+m+\beta)+k(9+5\beta)}$

5.5 $y = x + 5\beta + 4,\ \beta \geq 1$

In the table, k, r and γ are determined by $x = 2 \mod 5 + r$, $x \geq 4$, with $0 \leq r \leq 4$, and $x + 7 = 5\gamma + r$.

Theorem 5.5. *Let $D = \{2, 3, x, x + 5\beta + 4\}$, $k \geq 2\beta$. Then the values of $\kappa(D)$ provided in the tables are exact for $k \geq 2\beta$ and provide upper bounds when $k < 2\beta$.*

Table 5.9: $\kappa(D)$, $D = \{2, 3, x, x + 5\beta + 4\}$

$y = x + 5\beta + 4$, $\beta \geq 2k$				
x	γ	d	c	r
$5k+8$	$k+3$	2γ-2	$x+2$	0
$5k+4$	$k+2$	4γ+β-8	$x+y$	1
$5k+5$	$k+2$	2γ+2β-3	$y+3$	2
$5k+6$	$k+2$	2γ+2β-2	$y+3$	3
$5k+7$	k+2	2γ+2β-11	$y+3$	4

Table 5.10: $\kappa(D)$, $D = \{2, 3, x, x + 5\beta + 4\}$ Continued

$y = x + 5\beta + 4$, $\beta \geq 2k$				
x	n_x	n_y	POI	$\kappa(D)$
$5k+3$	k	$k+\beta+1$	$2+k$	$\frac{2}{5}$
$5k+4$	k	$k+\beta+1$	$2+2k+\beta$	$\frac{4+4k+\beta}{12+10k+5\beta}$
$5k+5$	$k+1$	$k+\beta+2$	$3+k+\beta$	$\frac{3+2k+2\beta}{12+5k+5\beta}$
$5k+6$	$k+1$	$k+\beta+2$	$3+k+\beta$	$\frac{4+2k+2\beta}{13+5k+5\beta}$
$5k+7$	$k+1$	$k+\beta+2$	$3+k+\beta$	$\frac{5+2k+2\beta}{14+5k+5\beta}$

5.6 Conclusions

In all of the results presented in this chapter, the same methods of proof were used as in earlier chapters. β must be designated first in order to establish the distance between x and y. Once this distance is established, k is allowed to go from 0 to ∞. Because of this choice ordering, there is a secondary pattern that emerges. In particular, when β is large and k is small, β becomes the dominant term in $\kappa(D)$, forcing smaller values for $\kappa(D)$, thus making the obtained values an upper bounds for $\kappa(D)$. Thus we must restrict the equalities for the proven values of $\kappa(D)$. Within that restriction again we find only singleton points of intersection and thus $\kappa(D)$ is known.

CHAPTER 6

Connections Between $\mu(D)$ and $\kappa(D)$

It is known that $\mu(D) \geq \kappa(D)$ [19]. In this chapter known values and bounds of $\mu(D)$ are stated, as well as techniques for finding values of $\mu(D)$.

For two-element sets $D = \{a, b\}$, Cantor and Gordon [5] proved that

$$\kappa(D) = \mu(D) = \frac{\lfloor \frac{a+b}{2} \rfloor}{a+b}$$

. For 3-element sets D, if $D = \{a, b, a+b\}$ it was proved that $\kappa(D) = \mu(D)$ and the exact values were determined.

Theorem 6.1. [15] *Suppose $M = \{a, b, a+b\}$ for some positive integers a and b with $gcd(a, b) = 1$, where $0 < a < b$. Then*

$$\kappa(M) = \mu(M) = \begin{cases} \frac{1}{3} & \text{if } b-a = 3k; \\ \frac{a+k}{3a+3k+1} & \text{if } b-a = 3k+1; \\ \frac{a+2k+1}{3a+6k+4} & \text{if } b-a = 3k+2 \end{cases}$$

For the general case $D = \{i, j, k\}$, various lower bounds of $\kappa(D)$ were given by Gupta, in which $\mu(D)$ was also studied.

Theorem 6.2. [11] *If $D = \{i, j, k\}$ with $i \geq j \geq k$ and $\gcd(i, j) = d$, then*

$$\mu(D) \geq \begin{cases} \frac{k}{2(j+k)} & \text{if } \frac{i}{d} \equiv \frac{j}{d} \mod 2; \\ \frac{k(i+j-d)}{2(2ij+(i+j)k)} & \text{if } \frac{i}{d} \not\equiv \frac{j}{d} \mod 2 \text{ and } k > \frac{j(j-i-d)}{d}; \\ \frac{k}{2(j+k)} & \text{if } \frac{i}{d} \not\equiv \frac{j}{d} \mod 2 \text{ and } k < \frac{j(j-i-d)}{d}. \end{cases}$$

In addition, among other results it was shown in [11] that if D is an arithmetic sequence then $\kappa(D) = \mu(D)$ and the value was determined.

Theorem 6.3. [11] *If* $D = \{n, n+d, n+2d, ..., n+(k-1)d\}$ *with* $\gcd(n,d) = 1$ *and* $k \geq 1$*, then*

$$\mu(D) = \begin{cases} \frac{2n+(k-1)(d-1)}{2(2n+(k-1)d} & \text{if } d \text{ is even;} \\ \\ \frac{1}{2} & \text{if } d \text{ is odd.} \end{cases}$$

It should be noted that throughout the literature thus far, equalities have been established primarily by utilizing arithmetic relationships in the D-set, or by using $\kappa(D)$ values, such as in [11], or by finding specific patterns within the relationships of the D-set elements. For instance, Liu and Zhu proved the following.

Theorem 6.4. [15] *If* $D = \{x, y, y-x, y+x\}$*, where* $y > x$*,* $x = 2k+1$*,* $y = 2m+1$ *and* $\gcd(x,y) = 1$*, then*

$$\mu(D) \geq \frac{(k+1)m}{4(k+1)m+1},$$

although, they conjecture that this is in fact an equality.

Many of the above results used the following Lemma by Haralambdis. For a D-sequence S, denote $S[n] = |\{0, 1, 2, \ldots, n\} \cap S|$.

Lemma 6.5. [19] *Let* D *be a set of positive integers, and let* $\alpha \in (0,1]$*. If for every* D*-sequence* S *with* $0 \in S$ *there exists a positive integer* n *such that* $S[n]/(n+1) \leqslant \alpha$*, then* $\mu(D) \leqslant \alpha$*. Alternatively, if* $\mu(D) > \alpha$*, then there is a* D*-sequence* S *such that* $S[n] > \alpha(n+1)$*, for all non-negative integers* n*.*

For a given D-sequence S, write the elements of S in increasing order, $S = (s_0, s_1, s_2, \ldots)$ with $s_0 < s_1 < s_2 < \ldots$, and denote its *difference sequence* by $\Delta(S) = (\delta_0, \delta_1, \delta_2, \ldots)$ where $\delta_i = s_{i+1} - s_i$. A subsequence of consecutive terms in $\Delta(S)$, $(\delta_a, \delta_{a+1}, \ldots, \delta_{a+b-1})$, generate a periodic interval of k copies, $k \geq 1$, if $\delta_{j(a+b)+i} = \delta_{a+i}$

for all $0 \le i \le b-1$, $1 \le j \le k-1$. We denote such a periodic subsequence of $\Delta(S)$ by $(\delta_a, \delta_{a+1}, \ldots, \delta_{a+b-1})^k$. If the periodic interval repeats infinitely we simply denote it by $(\delta_a, \delta_{a+1}, \ldots, \delta_{a+b-1})$. If $\Delta(S)$ is infinite periodic, except for a finite number of terms, with the periodic interval $(t_1, t_2, \ldots, t_k)$, then the density of S is $(\sum_{i=1}^{k} t_i)/k$.

Observation 1. *A sequence of non-negative integers S is a D-sequence if and only if $\sum_{i=a}^{b} \delta_i \notin D$ for any $a \le b$.*

Observation 2. *Assume $2, 3 \in D$. If S is a D-sequence, then $\delta_i + \delta_{i+1} \ge 5$ for all i. Equality holds only when $\{\delta_i, \delta_{i+1}\} = \{1, 4\}$. Consequently, $\mu(D) \le 2/5$. More relevant to the connection between $\mu(D)$ and $\kappa(D)$ is the fact that this results in $S[5t] \le 2t+1$, $S[5t+4] \le 2t+2$ for any non-negative t, and $S[5t+5] \le 2t+3$.*

Lemma 6.6. [7] *Let $D = \{2, 3\} \cup A$ for some $A \subseteq \mathbb{Z}$. Then $\kappa(D) = 2/5$ if and only if $A \subseteq \{x : x \equiv 2, 3 \mod 5\}$. Furthermore, if $\kappa(D) = 2/5$ then $\mu(D) = 2/5$.*

Proof. Let $D = \{2, 3\} \cup A$. Assume $A \subseteq \{x : x \equiv 2, 3 \mod 5\}$. Let $t = 1/5$. Then $||td|| \ge 2/5$ for all $d \in D$. Hence $\kappa(D) \ge 2/5$. On the other hand, the density of the infinite periodic D-sequence S with $\Delta(S) = (1, 4)$ is $2/5$. By Observation 2, this is an optimal D-sequence. Hence, $\mu(D) = 2/5$, implying $\kappa(D) = 2/5$.

Conversely, assume $\kappa(D) = 2/5$. Then $\mu(D) \ge 2/5$. By Observation 2, $\mu(D) = 2/5$. By Observation 1, this implies that if $d \in D$, then $d \not\equiv 0, 1, 4 \pmod 5$. Thus the result follows. □

Note, in $D = \{2, 3, x, y\}$, if $x = 1$, then it is known [15] and easy to see that $\mu(D) = \kappa(D) = 1/4$ if y is not a multiple of 4 (with $\Delta(S) = (4)$); otherwise $y = 4k$ and $\mu(D) = \kappa(D) = (k)/(4k+1)$ (with $\Delta(S) = ((4)^{k-1}5)$). Thus throughout the remainder of this thesis it is assumed that $x \ne 1$.

CHAPTER 7

Values of $\mu(D)$, $D = \{2, 3, x, x+i\}, 1 \leq i \leq 5$

As that $\kappa(D)$ is known for $D = \{2, 3, x, x+i\}, 1 \leq i \leq 5$, and techniques have been established to find values of $\mu(D)$, it is possible to connect the two parameters. This Chapter is dedicated to stating the values of $\mu(D$ for $D = \{2, 3, x, x+i\}, 1 \leq i \leq 5$, where possible.

Theorem 7.1. [7]

Let $D = \{2, 3, x, x+1\}$ with $x \geq 4$, then $\kappa(D) = \mu(D)$.

Proof. Consider the following cases.

Case 1. $x = 5k+2$**.** The result follows by Lemma 6.6.

Case 2. $x = 5k+3$**.** By Theorem 3.1, we have established $\kappa(D) = (2k+2)/(5k+6)$.

Now we claim $\mu(D) \leq (2k+2)/(5k+6)$. Assume to the contrary that $\mu(D) > (2k+2)/(5k+6)$. By Lemma 6.5, there exists a D-sequence S with $S[n]/(n+1) > (2k+2)/(5k+6)$ for all non-negative n. This implies, for instance, $S[0] \geq 1$, so $s_0 = 0$; $S[2] \geq 2$, so $s_1 = 1$; $S[5] \geq 3$, so $s_3 = 5$ (as $2, 3 \in D$). Moreover, $S[5k+5] \geq 2k+3$. By Observation 2, it must be $(\delta_0, \delta_1, \delta_2, \ldots, \delta_{2k+1}) = (1, 4, 1, 4, \ldots, 1, 4)$. This implies $5k+5 \in S$, which is impossible since $1 \in S$ and $5k+4 \in D$. Therefore, $\mu(D) = \kappa(D)$.

Case 3. $x = 5k+4$**.** By Theorem 3.1, we have established $\kappa(D) = (2k+2)/(5k+7)$. Now we claim $\mu(D) \leq (2k+2)/(5k+7)$. Assume to the contrary that $\mu(D) > (2k+2)/(5k+7)$. By Lemma 6.5, there exists a D-sequence S with $S[n]/(n+1) > (2k+2)/(5k+7)$ for all non-negative n. This implies, for instance, $S[0] \geq 1$, so $s_0 = 0$; $S[3] \geq 2$, so $s_1 = 1$ (as $2, 3 \in D$); and $S[5k+6] \geq 2k+3$. By Observation 2, either $5k+5$ or $5k+6$ is an element in S. This is impossible since $0, 1 \in S$ and

$5k + 4, 5k + 5 \in D$. Thus $\mu(D) = \kappa(D)$.

Case 4. $x = 5k+5$**.** By Theorem 3.1, we have established $\kappa(D) = (2k+2)/(5k+8)$. Now we claim $\mu(D) \leq (2k + 2)/(5k + 8)$. Assume to the contrary that $\mu(D) > (2k + 2)/(5k + 8)$. By Lemma 6.5, there exists a D-sequence S with $S[n]/(n + 1) > (2k + 2)/(5k + 8)$ for all non-negative n. Similar to the proof from Case 3, one has $0, 1 \in S$ and $S[5k + 7] \geq 2k + 3$. This implies that one of $5k + 5$, $5k + 6$, or $5k + 7$ is an element in S, which is again impossible, as $0, 1 \in S$ and $5k + 5, 5k + 6 \in D$. Therefore, $\mu(D) = \kappa(D)$.

Case 5. $x = 5k+1$**.** By Theorem 3.1, we have established $\kappa(D) = (2k+1)/(5k+4)$.

Now we claim $\mu(D) \leq (2k+1)/(5k+4)$. Assume to the contrary that $\mu(D) > (2k + 1)/(5k + 4)$. By Lemma 6.5, $(s_0, s_1) = (0, 1)$, and $S[5k + 3] \geq 2k + 2$. Because $S[5k] \leq 2k + 1$, so $S \cap \{5k + 1, 5k + 2, 5k + 3\} \neq \emptyset$, which is impossible, as $0, 1 \in S$ and $5k + 1, 5k + 2 \in D$. Therefore, $\mu(D) = \kappa(D)$. □

Notice that the denominators in the values of $\kappa(D)$ for $D = \{2, 3, x, x + 1\}$ are always $x + 3$. By the above proofs, one can extend the results to other families of sets which contain D as follows:

Corollary 7.2. *Let* $D = \{2, 3, x, x + 1\} \cup D'$*, where* $D' \subseteq \{y : y \equiv \pm 2, \pm 3 \mod x + 3\}$*. Then* $\mu(D) = \kappa(D) = \mu(\{2, 3, x, x + 1\})$*.*

Theorem 7.3. [7]

Let $D = \{2, 3, x, x + 2\}$ *with* $x \geq 4$*. Then* $\mu(D) = \kappa(D)$ *for all* $x \not\equiv 6k + 5 \mod 6$*.*

Proof. We prove the following cases.

Case 1. $x = 6k + 2$. By Theorem 4.1 we have established $\kappa(D) = 1/3$. Now we claim $\mu(D) \leq 1/3$. Let $M' = \{2, x, x+2\} = \{2, 6k+2, 6k+4\}$. By Theorem 6.1 with $M = \{1, 3k+1, 3k+2\}$, we obtain $\mu(M') = \mu(M) = 1/3$. Because $M' \subseteq D$, so $\kappa(D) = \mu(D) \leq \mu(M') = 1/3$.

Case 2. $x = 6k+3$. By Theorem 4.1 we have established $\kappa(D) = (2k+2)/(6k+7)$.

By Theorem 6.1 with $M = \{2, x, x+2\} = \{2, 6k+3, 6k+5\}$, we get $\mu(M) = (2k+2)/(6k+7)$. Because $M \subseteq D$, so $\mu(D) \leq \mu(M) = (2k+2)/(6k+7)$. Thus, the result follows that $\mu(D) = \kappa(D)$.

Case 3. $x = 6k+4$. By Theorem 4.1 we have established $\kappa(D) = (2k+2)/(6k+8)$.

By Theorem 6.1 with $M = \{2, x, x+2\} = \{2, 6k+4, 6k+6\}$ which can be reduced to $M' = \{1, 3k+2, 3k+3\}$, we obtain $\mu(M) = (k+1)/(3k+4)$. Therefore, $\mu(D) \leq \mu(M) = (2k+2)/(6k+8)$. So the result follows that $\mu(D) = \kappa(D)$.

Case 4. $x = 6k+6$. By Theorem 4.1 we have established $\kappa(D) = (4k+4)/(12k+14)$.

By Theorem 6.1 with $M = \{2, x, x+2\} = \{2, 6k+6, 6k+8\}$ which can be reduced to $M' = \{1, 3k+3, 3k+4\}$, we get $\mu(M) = \kappa(M) = (2k+2)/(6k+7)$. Hence, $\mu(D) \leq \mu(M) = (2k+2)/(6k+7)$ and $\mu(D) = \kappa(D)$.

Case 5. $x = 6k+7$. By Theorem 4.1 we have established $\kappa(D) = (4k+5)/(12k+16)$.

By Theorem 6.1 with $M = \{2, x, x+2\} = \{2, 6k+7, 6k+9\}$, we obtain $\mu(M) = \kappa(M) = (4k+5)/(12k+16)$.

Therefore, $\mu(D) \leq (4k+5)/(12k+16)$ and $\mu(D) = \kappa(D)$. □

Theorem 7.4. [7] *Let* $D = \{2, 3, x, x+3\}$ *with* $x \geq 4$. *Then* $\mu(D) = \kappa(D)$, $x \not\equiv 1, 2, 4, 5 \mod 9$.

Proof. Consider the following cases:

Case 1. $x = 9k + 3$. By Theorem 4.2 we have $\kappa(D) = (6k + 3)/(18k + 9) = 1/3$. By Theorem 6.1, $M = \{3, x, x + 3\} = \{3, 9k + 3, 9k + 6\}$, which can be reduced to $M' = \{1, 3k + 1, 3k + 2\}$. This results in $\mu(M) = \kappa(M) = 1/3$. Because $M \subseteq D$, so $\kappa(D) = \mu(D) = \mu(M) = 1/3$.

Case 2. $x = 9k + 6$. By Theorem 4.2 we have $\kappa(D) = (k + 1)/(3k + 4)$. By Theorem 6.1, $M = 3, x, x + 3 = \{3, 9t + 6, 9t + 9\}$, which can be reduced to $M' = \{1, 3t+2, 3t+3\}$. Because $M \subseteq D$, $\kappa(D) \le \mu(D) \le \mu(M) \le \kappa(M) = (k+1)/(3k+4)$, which results in $\mu(D) = \kappa(D) = (k + 1)/(3k + 4)$.

Case 3. $x = 9k+7$. By Theorem 4.2 we have $\kappa(D) = (3k+4)/(9k+13)$. By Theorem 6.1 with $M = \{3, x, x + 3\} = \{3, 9t + 7, 9t + 10\}$, we get $\kappa(M) = (3k + 4)/(9k + 13)$. Because $M \subseteq D$, so $\kappa(D) \le \mu(D) \le \mu(M) = \kappa(M) = (3k + 4)/(9k + 13)$.

Case 4. $x = 9k + 8$. By Theorem 4.2 we have $\kappa(D) = (6k + 6)/(18k + 19)$. By Theorem 6.1 with $M = \{3, x, x + 3\}$, we get $\kappa(M) = (6k + 6)/(18k + 19)$. Hence, $\mu(D) = \kappa(D) = (6k + 6)/(18k + 19)$.

Case 5. $x = 9k$. By Theorem 4.2 we have $\kappa(D) = (2k)/(6k + 1)$. By Theorem 6.1 with $M = \{3, x, x + 3\} = \{3, 9k, 9k + 3\}$, $\mu(M) = \kappa(M) = (2k)/(6k + 1)$. Hence, the result follows that $\mu(D) = \kappa(D)$. □

Theorem 7.5. *Let $D = \{2, 3, x, x + 4\}$ with $x \ge 4$. Then $\mu(D) = \kappa(D)$.*

Proof. Consider the following cases:

Case 1. $x = 5k + 8$. By Lemma 6.6 we know that $\kappa(D) = \mu(D) = \frac{2}{5}$.

Case 2. $x = 5k + 4$. By Theorem 4.3 we have established $\kappa(D) = \frac{2k+2}{5k+6}$.

Now we claim $\mu(D) \le \frac{2k+2}{5k+6}$. Assume to the contrary that $\mu(D) > \frac{2k+2}{5k+6}$. Then by Lemma 6.5, there exists a D-sequence S such that $\frac{S[n]}{n+1} > \frac{2k+2}{5k+6}$, for all non-negative integers n. Thus $S[2] \geq 2$ which shows that $1 \in S$, as $0 \in S$ and $2 \in D$. This also results in $S[5k+3] \geq 2k+2$. By Observation 2, $S[5k+3] \leq S[5k+4] \leq 2k+2$, which in turn gives the equality $S[5k+3] = 2k+2$. Continuing this investigation, we find that $S[5k+5] \geq 2k+3$, implying that $S \cap \{5k+4, 5k+5\} \neq \emptyset$, which is impossible as $0, 1 \in S$, and $5k+4 \in D$. Thus $\mu(D) \leq \frac{2k+2}{5k+6}$, which further results in $\mu(D) = \kappa(D)$, as desired.

Case 3. $x = 5k+5$**.** By Theorem 4.3 we have established $\kappa(D) = \frac{2k+2}{5k+7}$.

Now we claim $\mu(D) \leq \frac{2k+2}{5k+7}$. Assume to the contrary that $\mu(D) > \frac{2k+2}{5k+7}$. By Lemma 6.5, there exists a D-sequence S such that $\frac{S[n]}{n+1} > \frac{2k+2}{5k+7}$, for all non-negative integers n. Thus $S[3] \geq 2$ which shows that $1 \in S$, as $0 \in S$ and $2, 3 \in D$. This also results in $S[5k+4] \geq 2k+2$. By Observation 2, $S[5k+4] \leq 2k+2$, which in turn gives the equality $S[5k+4] = 2k+2$. Continuing this investigation, we find that $S[5k+6] \geq 2k+3$, implying that $S \cap \{5k+5, 5k+6\} \neq \emptyset$, which is impossible as $0, 1 \in S$, and $5k+5 \in D$. Thus $\mu(D) \leq \frac{2k+2}{5k+7}$, which further results in $\mu(D) = \kappa(D)$, as desired.

Case 4. $x = 5k+6$**.** By Theorem 4.3 we have established $\kappa(D) = \frac{2k+4}{5k+13}$.

Now we claim $\mu(D) \leq \frac{2k+4}{5k+13}$. Assume to the contrary that $\mu(D) > \frac{2k+4}{5k+13}$. Then by Lemma 6.5 there exists a D-sequence S such that $\frac{S[n]}{n+1} > \frac{2k+4}{5k+13}$, for all non-negative integers n. Thus $S[3] \geq 2$ which shows that $1 \in S$, as $0 \in S$ and $2, 3 \in D$. This also results in $S[5k+7] \geq 2k+3$. By Observation 2, $S[5k+5] \leq 2k+3$, this gives the equality $S[5k+5] = 2k+3$. Indeed, as we know from the same Observation that $S[5k+4] \leq 2k+2$, we obtain $5k+5 \in S$. Continuing this investigation, we find that $S[5k+10] \geq 2k+4$, implying that $5k+9 \in S$, as $S \cap \{5k+8, 5k+9, 5k+10\} \neq \emptyset$, and both $0, 5k+5 \in S$ and $3, 5k+10 \in D$. Finally, $S[5k+12] \geq 2k+5$ implying that $S \cap \{5k+11, 5k+12\} \neq \emptyset$, which is impossible as $5k+9 \in S$, and $2, 3 \in D$. Thus $\mu(D) \leq \frac{2k+4}{5k+13}$, which further results in $\mu(D) = \kappa(D)$, as desired.

Case 5. $x = 5k+7$**.** By Theorem 4.3 we have established $\kappa(D) = \frac{2k+5}{5k+14}$.

Now we claim $\mu(D) \leq \frac{2k+5}{5k+14}$. Assume to the contrary that $\mu(D) > \frac{2k+5}{5k+14}$. Then by Lemma 6.5 there exists a D-sequence S such that $\frac{S[n]}{n+1} > \frac{2k+5}{5k+14}$, for all non-negative integers n. Thus $S[2] \geq 2$ and $S[5] \geq 3$, which shows that $1, 5 \in S$, as $0 \in S$ and $2, 3 \in D$. This also results in $S[5k+4] \geq 2k+2$. By Observation 2, $S[5k+4] \leq 2k+2$, which in turn gives the equality $S[5k+4] = 2k+2$. Continuing this investigation, we find that $S[5k+6] \geq 2k+3$, implying that $\{5k+5, 5k+6\} \cap S \neq \emptyset$. This option does not present a problem. As $S[5k+8] \geq 2k+4$, and as $0, 1 \in S$ and $5k+7 \in D$, this means that both $5k+5, 5k+6 \in S$. Additionally, $S[5k+11] \geq 2k+5$, giving $5k+10 \in S$, as $3, 5k+11 \in D$ and $0, 5k+6 \in S$. Finally, $S[5k+13] \geq 2k+6$ implying that $S \cap \{5k+12, 5k+13\} \neq \emptyset$, which is impossible as $2, 3 \in D$, and $5k+10 \in S$. Thus $\mu(D) \leq \frac{2k+5}{5k+14}$, which further results in $\mu(D) = \kappa(D)$, as desired. □

Theorem 7.6. *Let* $D = \{2, 3, x, x+5\}$*. Then* $\mu(D) = \kappa(D)$*.*

Proof. Consider the following cases:

Case 1. $x = 5k+2$**.** By Lemma 6.6, we know that $\kappa(D) = \mu(D) = \frac{2}{5}$.

Case 2. $x = 5k+3$**.** By Lemma 6.6, we know that $\kappa(D) = \mu(D) = \frac{2}{5}$.

Case 3. $x = 5k+4$**.** By Theorem 4.4 we have established $\kappa(D) = \frac{2k+2}{5k+6}$.

Now we claim $\mu(D) \leq \frac{2k+2}{5k+6}$. Assume to the contrary that $\mu(D) > \frac{2k+2}{5k+6}$. Then by Lemma 6.5 there exists a D-sequence S such that $\frac{S[n]}{n+1} > \frac{2k+2}{5k+6}$, for all non-negative integers n. Thus $S[2] \geq 2$, which shows that $1 \in S$, as $0 \in S$ and $2 \in D$. By our assumption, $S[5k+3] \geq 2k+2$. By Observation 2, $S[5k+3] \leq S[5k+4] \leq 2k+2$, which in turn gives the equality $S[5k+3] = 2k+2$. Continuing this investigation, we find that $S[5k+5] \geq 2k+3$, implying that $S \cap \{5k+4, 5k+5\} \neq \emptyset$, which is impossible as $0, 1 \in S$, and $5k+4 \in D$. Thus $\mu(D) \leq \frac{2k+2}{5k+6}$ as desired, which further results in $\mu(D) = \kappa(D)$.

Case 4. $x = 5k+5$**.** By Theorem 4.4 we have established $\kappa(D) = \frac{2k+2}{5k+7}$.

Now we claim $\mu(D) \leq \frac{2k+2}{5k+7}$. Assume to the contrary that $\mu(D) > \frac{2k+2}{5k+7}$. Then by Lemma 6.5 there exists a D-sequence S such that $\frac{S[n]}{n+1} > \frac{2k+2}{5k+7}$, for all non-negative integers n. Thus $S[2] \geq 2$ which shows that $1 \in S$, as $0 \in S$ and $2 \in D$. By our assumption, $S[5k+4] \geq 2k+2$. By Observation 2, $S[5k+4] \leq 2k+2$, which in turn gives the equality $S[5k+4] = 2k+2$. Continuing this investigation, we find that $S[5k+6] \geq 2k+3$, implying that $S \cap \{5k+5, 5k+6\} \neq \emptyset$, which is impossible as $0, 1 \in S$, and $5k+5 \in D$. Thus $\mu(D) \leq \frac{2k+2}{5k+7}$, which further results in $\mu(D) = \kappa(D)$, as desired.

Case 5. $x = 5k+6$**.** By Theorem 4.4 we have established $\kappa(D) = \frac{2k+3}{5k+9}$.

Now we claim $\mu(D) \leq \frac{2k+3}{5k+9}$. Assume to the contrary that $\mu(D) > \frac{2k+3}{5k+9}$. Then by Lemma 6.5 there exists a D-sequence S such that $\frac{S[n]}{n+1} > \frac{2k+3}{5k+9}$, for all non-negative integers n. Thus $S[2] \geq 2$ which shows that $1 \in S$, as $0 \in S$ and $2 \in D$. This also results in $S[5k+4] \geq 2k+2$. By Observation 2, $S[5k+4] \leq 2k+2$, which in turn gives the equality $S[5k+4] = 2k+2$. Continuing this investigation, we find that $S[5k+6] \geq 2k+3$, implying that $5k+5 \in S$, as $0 \in S$ and $5k+6 \in D$. Furthermore,

$S[5k+8] \geq 2k+4$, implying that $S \cap \{5k+7, 5k+8\} \neq \emptyset$, which is impossible as $5k+5 \in S$, and $2, 3 \in D$. Thus $\mu(D) \leq \frac{2k+3}{5k+9}$, which further results in $\mu(D) = \kappa(D)$, as desired. $\square$

CHAPTER 8

Generalized Results on $\mu(D)$ and $\kappa(D)$

After examining patterns between $\mu(D)$ and $\kappa(D)$ for the first few fixed distances between the x and y elements of the D-set, namely the D-sets $\{2,3,x,x+i\}, 1 \leq i \leq 5$, it remains to be seen if these results may be extended to the generalized $\kappa(D)$ values from Chapter 5. In this Chapter we show that these results can currently be extended to $y = x + 5\beta$ and $y = x + 5\beta + 1$.

Theorem 8.1. *Let* $D = \{2, 3, x, x + 5\beta\}, \beta \geq 1$. *Then*

$$\mu(D) \;\leq \begin{cases} \frac{2}{5} & \textit{if } x = 5k+7 \textit{ or } x = 5k+8; \\ \frac{2k+2}{5k+6} & \textit{if } x = 5k+4; \\ \frac{2k+2}{5k+7} & \textit{if } x = 5k+5; \\ \frac{2k+3}{5k+9} & \textit{if } x = 5k+6. \end{cases}$$

Furthermore, $\mu(D) = \kappa(D)$ *when* $k \geq 2\beta$.

Proof. Consider the following cases:

Case 1. $x = 5k + 7$**.** By Lemma 6.6, we know that $\kappa(D) = \mu(D) = \frac{2}{5}$.

Case 2. $x = 5k + 8$**.** By Lemma 6.6, we know that $\kappa(D) = \mu(D) = \frac{2}{5}$.

Case 3. $x = 5k + 4$**.** By Theorem 5.1, we have established $\kappa(D) = \frac{2k+2}{5k+6}$ when $k \geq 2\beta$.

Now we claim $\mu(D) \leq \frac{2k+2}{5k+6}$. Assume to the contrary $\mu(D) > \frac{2k+2}{5k+6}$. Then by Lemma 6.5 there exists a D-sequence S such that $\frac{S[n]}{n+1} > \frac{2k+2}{5k+6}$, for all non-negative integers n. Thus $S[0] \geq 1$ and $S[2] \geq 2$, making $0, 1 \in S$. Additionally, $S[5k+4] \geq 2k+2$. By Observation 2, $S[5k+4] \leq 2k+2$, so combined with $S[5k+4] \geq 2k+2$, we know that $S[5k+4] = 2k+2$. $S[5k+5] \geq 2k+3$, forcing $5k+5 \in S$, which creates a contradiction, as $1 \in S$ and $5k+4 \in D$. Thus $\mu(D)$ is bounded above by $\frac{2k+2}{5k+6}$ and furthermore, $\kappa(D) = \mu(D)$, when $k \geq 2\beta$ as desired.

Case 4. $x = 5k+5$**.** By Theorem 5.1, we have established $\kappa(D) = \frac{2k+2}{5k+7}$ when $k \geq 2\beta$.

We claim $\mu(D) \leq \frac{2k+2}{5k+7}$. Assume to the contrary $\mu(D) > \frac{2k+2}{5k+7}$. Then by Lemma 6.5 there exists a D-sequence S such that $\frac{S[n]}{n+1} > \frac{2k+2}{5k+7}$, for all non-negative integers n. Thus $S[0] \geq 1$ and $S[2] \geq 2$, making $0, 1 \in S$, as $2 \in D$. Additionally, $S[5k+4] \geq 2k+2$. By Observation 2, $S[5k+4] \leq 2k+2$, so combined with $S[5k+4] \geq 2k+2$, we know that $S[5k+4] = 2k+2$. $S[5k+6] \geq 2k+3$, implying that $S \cap \{5k+5, 5k+6\} \neq \emptyset$ which creates a contradiction, as $0, 1 \in S$ and $5k+5 \in D$. Thus $\mu(D)$ is bounded above by $\frac{2k+2}{5k+7}$ and furthermore, $\mu(D) = \kappa(D)$, when $k \geq 2\beta$, as desired.

Case 5. $x = 5k+6$**.** By Theorem 5.1, we have established $\kappa(D) = \frac{2k+3}{5k+9}$ when $k \geq 2\beta$.

In general, we claim $\mu(D) \leq \frac{2k+3}{5k+9}$. Assume to the contrary $\mu(D) > \frac{2k+3}{5k+9}$. Then by Lemma 6.5 there exists a D-sequence S such that $\frac{S[n]}{n+1} > \frac{2k+3}{5k+9}$, for all non-negative integers n. Thus $S[0] \geq 1$ and $S[2] \geq 2$, making $0, 1 \in S$, as $2 \in D$. Additionally, $S[5k+6] \geq 2k+3$. By Observation 2, $S[5k+4] \leq 2k+2$, forcing $5k+5 \in S$, as $0 \in S$ and $5k+6 \in D$. Finally, $S[5k+8] \geq 2k+4$, implying that $S \cap \{5k+7, 5k+8\} \neq \emptyset$, which is impossible as $5k+5 \in S$ and $2, 3 \in D$. Thus $\mu(D)$ is bounded above by $\frac{2k+3}{5k+9}$ and furthermore $\mu(D) = \kappa(D) = \frac{2k+3}{5k+9}$, when $k \geq 2\beta$, as desired. □

Observation 3. *Note that for Case 3 we have shown that there is an upper bound* *of* $\mu(A) \leq \frac{2k+2}{5k+6}$ *when* $A = \{2, 3, 5k+4\}$*, and thus* $\mu(D) \leq \mu(A) = \frac{2k+2}{5k+6}$ *for any* D *where* $A \subset D$*. Similarly, upper bounds of* $\mu(A) \leq \frac{2k+2}{5k+7}$ *when* $A = \{2, 3, 5k+5\}$ *and* $\mu(A) \leq \frac{2k+3}{5k+9}$ *when* $A = \{2, 3, 5k+6\}$ *have been established through Cases 4 and 5, and by extension,* $\mu(D) \leq \mu(A) = \frac{2k+3}{5k+9}$*, for any* D*-set with* $A \subset D$*.*

Theorem 8.2. *Let* $D = \{2, 3, x, x+5\beta+1\}$*. Then*

$$\mu(D) \leq \begin{cases} \frac{2}{5} & \text{if } x = 5k+7; \\ \frac{2k+2\beta+4}{5k+5\beta+11} & \text{if } x = 5k+8; \\ \frac{2k+2\beta+2}{5k+5\beta+7} & \text{if } x = 5k+4; \\ \frac{2k+2}{5k+7} & \text{if } x = 5k+5; \\ \frac{2k+3}{5k+9} & \text{if } x = 5k+6. \end{cases}$$

Furthermore, when $k \geq 2\beta$*, then the first three cases of the above theorem become equalities, and* $\mu(D) = \kappa(D)$*.*

Proof. Consider the following cases:

Case 1. $x = 5k+7$**.** By Lemma 6.6, we know that $\kappa(D) = \mu(D) = \frac{2}{5}$.

Case 2. $x = 5k+8$**.** By Theorem 5.2, $\kappa(D) = \frac{2k+2\beta+4}{5k+5\beta+11}$ when $k \geq 2\beta$.

Now we claim $\mu(D) \leq \frac{2k+2\beta+4}{5k+5\beta+11}$. Assume to the contrary that $\mu(D) > \frac{2k+2\beta+4}{5k+5\beta+11}$. Then by Lemma 6.5 there exists a D-sequence S such that $\frac{S[n]}{n+1} > \frac{2k+2\beta+4}{5k+5\beta+11}$, for all non-negative integers n. Thus $S[2] \geq 2$ which shows that $1 \in S$, as $0 \in S$ and $2 \in D$. This also results in $S[5k+5\beta+8] \geq 2k+2\beta+4$. By Observation 2, $S[5k+5\beta+8] \leq S[5k+5\beta+9] \leq 2k+2\beta+4$, as $2, 3 \in D$, which in turn gives the equality $S[5k+5\beta+8] = 2k+2\beta+4$. Continuing this investigation, we find that

$S[5k+5\beta+10] \geq 2k+2\beta+5$, implying that $S \cap \{5k+5\beta+9, 5k+5\beta+10\} \neq \emptyset$, which is impossible as $0,1 \in S$, and $5k+5\beta+9 \in D$. Thus $\mu(D) \leq \frac{2k+2\beta+4}{5k+5\beta+11}$ as desired, which further results in $\mu(D) = \kappa(D)$ when $k \geq 2\beta$.

Case 3. $x = 5k+4$**.** By Theorem 5.2, we have established $\kappa(D) = \frac{2k+2\beta+2}{5k+5\beta+7}$ when $k \geq 2\beta$.

Now we claim $\mu(D) \leq \frac{2k+2\beta+2}{5k+5\beta+7}$. Assume to the contrary $\mu(D) > \frac{2k+2\beta+2}{5k+5\beta+7}$. Then by Lemma 6.5 there exists a D-sequence S such that $\frac{S[n]}{n+1} > \frac{2k+2\beta+2}{5k+5\beta+7}$, for all non-negative integers n. Thus $S[0] \geq 1$ and $S[2] \geq 2$, making $0,1 \in S$. Also, we observe that $S[5k+5\beta+4] \geq 2k+2\beta+2$. By Observation 2, $S[5k+5\beta+4] \leq 2k+2\beta+2$, so combined with $S[5k+5\beta+4] \leq 2k+2\beta+2$, we know that $S[5k+5\beta+4] = 2k+2\beta+2$. Finally, $S[5k+5\beta+6] \geq 2k+2\beta+3$, implying that $S \cap \{5k+5\beta+5, 5k+5\beta+6\} \neq \emptyset$, which creates a contradiction, as $0,1 \in S$ and $5k+5\beta+5 \in D$. Thus $\mu(D)$ is bounded above by $\frac{2k+2\beta+2}{5k+5\beta+7}$ and $\mu(D) = \kappa(D)$, when $k \geq 2\beta$, as desired.

Case 4. $x = 5k+5$**.** By Observation 3, $\mu(D) \leq \mu(\{2,3,5k+5\}) = \frac{2k+2}{5k+7}$, as desired.

Case 5. $x = 5k+6$**.** By Observation 3, $\mu(D) \leq \mu(\{2,3,5k+6\}) = \frac{2k+3}{5k+9}$, as desired.

$\square$

CHAPTER 9

Interesting Properties and Questions for Future Study

When looking at the tables from Chapter 5 for the overarching generalized behaviors, we can see that $\lim_{\beta \to \infty} \kappa(D) = \frac{2}{5}$ or $\lim_{\beta \to \infty} \kappa(D) = \frac{1}{3}$. One of the next questions to be answered is what are the interactions dominating the first of the generalized cases, where $k < 2\beta$. The value of $\kappa(D)$ seems to initially drop dramatically towards a lower fractional value that is closer to $\frac{1}{4}$, but then resumes its asymptotic rise towards $\frac{2}{5}$. We conjecture that there is another modulo pattern at work in these initial cases, dominant only so long as $k < 2\beta$, and further ask whether or not there are perhaps interesting dynamics within the initial $\kappa(D)$ values that may shed light on how to extend these results to more inclusive D-sets, such as ones that do not require the first two elements to be 2 and 3. Finally, it remains to be seen if asymptotic upper bounds may be found for the $\mu(D)$ of other D-sets, and whether this may, in turn, be extended to generalize the complete behavior for all 4-element D-sets.

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APPENDIX A

Java Program Code for Generating $\kappa(D)$ Values

```
package colortheory;

public class TimeCircumDistanceSingle {

	public static void main(String[] args) {
//		//System.out.print("   y: ");
//		for (int columnCounter = 5; columnCounter < 99; columnCounter++) {
//			if (columnCounter < 10) {
//				System.out.print(columnCounter + "           ");
//			} else {
//				System.out.print(columnCounter + "          ");
//			}
////		}
		final int a = 2, b = 3;
		final int distanceBetweenXandY = 51;
		int C = 171, D = C + distanceBetweenXandY;
		int[][] timeMatrix = new int[C+1][C+1+distanceBetweenXandY];
		int[][] circumMatrix2 = new int[C+1][C+1+distanceBetweenXandY];
		int[][] distMatrix = new int[C+1][C+1+distanceBetweenXandY];
		for (int cVariable = 4; cVariable <= C; cVariable++) {
			int c = cVariable;
			for (int variable = c + 1; variable <= D; variable++) {
				int d = variable;
				int[] circumf = new int[6];
				double[] maxKappa = new double[6];
				int[] timeKeeper = new int[6];
				int[] circumMatrix = new int[6];
				int[] dist = new int[6];
				circumf[0] = a + b;
				circumf[1] = a + c;
				circumf[2] = a + d;
				circumf[3] = b + c;
				circumf[4] = b + d;
				circumf[5] = c + d;
				for (int counter = 0; counter < 6; counter++) {
					int circum = circumf[counter];
					int[][] distance = new int[4][circum];
					for (int time = 1; time <= circum; time++) {
						int min;
						min = (time * a) % circum;
						if (min <= circum / 2) {
							// System.out.println("min is " + min);
							distance[0][time - 1] = min;
							continue;
						} else {
							min = (circum - min) % circum;
							distance[0][time - 1] = min;
						}
						// System.out.println("min is " + min);
						continue;
					}
					for (int time = 1; time <= circum; time++) {
						int min;
						min = (time * b) % circum;
						if (min <= circum / 2) {
							// System.out.println("min for b is " + min);
							distance[1][time - 1] = min;
							continue;
						} else {
							min = (circum - min) % circum;
							distance[1][time - 1] = min;
						}
```

```
			// System.out.println("min for b is " + min);
		}
		for (int time = 1; time <= circum; time++) {
			int min;
			min = (time * c) % circum;
			if (min <= circum / 2) {
				// System.out.println("min for c is " + min);
				distance[2][time - 1] = min;
				continue;
			} else {
				min = (circum - min) % circum;
				distance[2][time - 1] = min;
			}
			// System.out.println("min for c is " + min);
		}
		for (int time = 1; time <= circum; time++) {
			int min;
			min = (time * d) % circum;
			if (min <= circum / 2) {
				// System.out.println("min for d is " + min);
				distance[3][time - 1] = min;
				continue;
			} else {
				min = (circum - min) % circum;
				distance[3][time - 1] = min;
			}
			// System.out.println("min for d is " + min);
		}
		for (int bob = 0; bob < 4; bob++) {
			for (int min = 0; min < circum; min++) {
				if (distance[bob][min] > 99) {
					// System.out.print(distance[bob][min] + " ");
				} else if (distance[bob][min] < 100
						&& 9 < distance[bob][min]) {
					// System.out.print(" " + distance[bob][min] +
					// "  ");
				} else {
					// System.out.print(" " + distance[bob][min] +
					// "   ");
				}
			}
			// System.out.println();
		}

		int[] minDist = new int[circum];
		for (int column = 0; column < circum; column++) {
			minDist[column] = distance[0][column];
			if (distance[1][column] < distance[0][column]) {
				minDist[column] = distance[1][column];
				if (distance[2][column] < distance[1][column]) {
					minDist[column] = distance[2][column];
					if (distance[3][column] < distance[2][column]) {
						minDist[column] = distance[3][column];
						continue;
					} else
						continue;
				} else if (distance[3][column] < distance[1][column]) {
					minDist[column] = distance[3][column];
					continue;
				} else
					continue;
			} else if (distance[2][column] < distance[0][column]) {
```

```java
                        minDist[column] = distance[2][column];
                        if (distance[3][column] < distance[2][column]) {
                            minDist[column] = distance[3][column];
                            continue;
                        } else
                            continue;
                    } else if (distance[3][column] < distance[0][column]) {
                        minDist[column] = distance[3][column];
                        continue;
                    }
                }

                /*
                 * System.out.println(); for (int minPlaceHolder = 0;
                 * minPlaceHolder < circum; minPlaceHolder++) { if
                 * (minDist[minPlaceHolder] < 10) { System.out.print(" " +
                 * minDist[minPlaceHolder] + "  "); } else {
                 * System.out.print(" " + minDist[minPlaceHolder] + " "); }
                 * }
                 */

                // System.out.println();
                int maxMinValue = minDist[0], timeCounter = 1;
                for (int findingMaxMinValue = 0; findingMaxMinValue < circum; findingMaxMinValue++) {
                    if (minDist[findingMaxMinValue] > maxMinValue) {
                        maxMinValue = minDist[findingMaxMinValue];
                        timeCounter = findingMaxMinValue + 1;
                        continue;
                    } else
                        continue;
                }
                // System.out.println();
                double kappa = (double) maxMinValue / circum;
                /*System.out.println("The maximum minimum distance is " +
                maxMinValue
                + " for circum " + circum);
                System.out.println("This gives the kappa value of " +
                kappa);*/
                // System.out.println("The first time giving this distance is "
                // + timeCounter);
                // System.out.println();
                maxKappa[counter] = kappa;
                timeKeeper[counter] = timeCounter;
                circumMatrix[counter] = circum;
                dist[counter] = (int) (kappa*circum);
            }
            int dist2 = 0;
            int time = 0;
            double kappa = 0;
            int circum = 0;
            for (int counter = 0; counter < 6; counter++) {
                if (maxKappa[counter] <= kappa) {
                    continue;
                } else {
                    kappa = maxKappa[counter];
                    time = timeKeeper[counter];
                    circum = circumMatrix[counter];
                    dist2 = dist[counter];
                    continue;
                }
            }
            //System.out.println(circumReference + " is the circum");
```

```
/*System.out.println("The circum that results in kappa is " +
circum + "  The final Kappa value  for {" + a + ", " + b + ", "
+ c + ", " + d + "} is " + kappa + " .");
System.out.println(time + " {"+ a + "," + b + "," + c + "," +
d + "}");*/
timeMatrix[c][d] = time;
circumMatrix2[c][d] = circum;
distMatrix[c][d] = dist2;
}
}
for (int place = 5; place < D; place++) {

}
System.out.println("x,  y");
for (int row = 4; row < C; row++) {
    if (row < 10) {
        System.out.print(row + ", " + (row+distanceBetweenXandY) + "  || ");
    } else if (row < 100 && 10 <= row) {
        System.out.print(row + ", " + (row+distanceBetweenXandY) + " || ");
    } else {
        System.out.print(row + ", " + (row+distanceBetweenXandY) + "|| ");
    }

    for (int column = row+distanceBetweenXandY; column == row+distanceBetweenXandY; column++) {
        if (timeMatrix[row][column] < 10) {
            if (circumMatrix2[row][column] <10){
                if (distMatrix[row][column]<10){
                    System.out.print(timeMatrix[row][column] + " (" +
circumMatrix2[row][column] + ")  [" + distMatrix[row][column] + "]    ");
                    continue;}
                else if (distMatrix[row][column]>=10 && distMatrix[row][column]<100){
                    System.out.print(timeMatrix[row][column] + " (" +
circumMatrix2[row][column] + ") [" + distMatrix[row][column] + "]   ");
                    continue;}
                else {
                    System.out.print(timeMatrix[row][column] + " (" +
circumMatrix2[row][column] + ") [" + distMatrix[row][column] + "]  ");
                    continue;}}
            else if (10 <= circumMatrix2[row][column] && circumMatrix2[row][column]<100){
                if (distMatrix[row][column]<10){
                    System.out.print(timeMatrix[row][column] + " (" +
circumMatrix2[row][column] + ") [" + distMatrix[row][column] + "]    ");
                    continue;}
                else if (distMatrix[row][column]>=10 && distMatrix[row][column]<100){
                    System.out.print(timeMatrix[row][column] + " (" +
circumMatrix2[row][column] + ") [" + distMatrix[row][column] + "]   ");
                    continue;}
                else {
                    System.out.print(timeMatrix[row][column] + " (" +
circumMatrix2[row][column] + ") [" + distMatrix[row][column] + "]  ");
                    continue;}}
            else {
                System.out.print(timeMatrix[row][column] + " (" + circumMatrix2[row][column] + ")["
+ distMatrix[row][column] + "]  ");
            }
        } else if (10 <= timeMatrix[row][column]&& timeMatrix[row][column] < 100) {
            if (circumMatrix2[row][column] <10){
                if (distMatrix[row][column]<10){
                    System.out.print(timeMatrix[row][column] + " (" +
circumMatrix2[row][column] + ") [" + distMatrix[row][column] + "]  ");
                    continue;}
```

```
						else if (distMatrix[row][column]>=10 && distMatrix[row][column]<100){
							System.out.print(timeMatrix[row][column] + "(" +
circumMatrix2[row][column] + ") [" + distMatrix[row][column] + "]   ");
							continue;}
						else {
							System.out.print(timeMatrix[row][column] + " (" +
circumMatrix2[row][column] + ") [" + distMatrix[row][column] + "] ");
							continue;}}
					else if (10 <= circumMatrix2[row][column] && circumMatrix2[row][column]<100){
							if (distMatrix[row][column]<10){
								System.out.print(timeMatrix[row][column] + " (" +
circumMatrix2[row][column] + ") [" + distMatrix[row][column] + "]  ");
								continue;}
							else if (distMatrix[row][column]>=10 && distMatrix[row][column]<100){
								System.out.print(timeMatrix[row][column] + "(" +
circumMatrix2[row][column] + ") [" + distMatrix[row][column] + "]   ");
								continue;}
							else {
								System.out.print(timeMatrix[row][column] + " (" +
circumMatrix2[row][column] + ") [" + distMatrix[row][column] + "]");
								continue;}}
					else {
							System.out.print(timeMatrix[row][column] + "(" +
circumMatrix2[row][column] + ")[" + distMatrix[row][column] + "]   ");
						}
						continue;
				} else {
					if (circumMatrix2[row][column] <10){
						System.out.print(timeMatrix[row][column] + " (" + circumMatrix2[row][column] + ") ["
+ distMatrix[row][column] + "]  ");
						continue;}
						else if (10 <= circumMatrix2[row][column] && circumMatrix2[row][column]<100){
							System.out.print(timeMatrix[row][column] + "(" +
circumMatrix2[row][column] + ") [" + distMatrix[row][column] + "]   ");
						}
						else {
							System.out.print(timeMatrix[row][column] + " (" +
circumMatrix2[row][column] + ") [" + distMatrix[row][column] + "]");
						}
					continue;
				}
			}
			System.out.println();
		}
	}
}
```

APPENDIX B

Mathematica Program Code for Checking Intersection Intervals

```
k=β;
"Clear[k]";
Clear[a];
x =5k+6;
y = x+5β+1;
γ=k+1+β;
r=0;
c=y+x
d =4γ-5
Kappa = Simplify[d/c]
c;
d;
Simplify[d];

Simplify[Expand[twoLeft=d/2]];
Simplify[Expand[threeRight = 1/3(c-d)]];
set = Simplify[{twoLeft,threeRight}];
xn =k+0;
yn=xn +β;
xl=Simplify[  1/x(d + xn* c)];
xr= 1/x(c-d+xn * c);
yl=Simplify[Expand[ 1/y(d + yn *c)]];
yr= 1/y(c-d+yn * c);
Expand[{Simplify[xl*2*x],Simplify[xr*2*x]}]
Expand[{twoLeft*2*x,threeRight*3*x}]
Expand[{Simplify[xl*3*x],Simplify[xr*3*x]}]
"-------------------"
Expand[{Simplify[yl*2*y],yr*2*y}]
Expand[{twoLeft*2*y,threeRight*3*y}]
Expand[{Simplify[yl*3*y],yr*3*y}]
"-------------------"
"   Left X Interval          Right X Interval"
Expand[{Simplify[xl*x*y],Simplify[xr*x*y]}]
Expand[{Simplify[yl*x*y],Simplify[yr*x*y]}]
"   Left Y Interval          Right Y Interval"
```

Simplify[{twoLeft, threeRight}]

```
Simplify[xl]
Simplify[yr]

Simplify[xr]
Simplify[yl]
```