CONSTRUCTION OF OPTIMAL RADIO LABELINGS OF GRID GRAPHS AND A SURVEY OF RELEVANT RESULTS

A Thesis

Presented to

The Faculty of the Department of Mathematics

California State University, Los Angeles

In Partial Fulllment

of the Requirements for the Degree

Master of Science

in

Mathematics

By

Sam Zi-Ming Chyau

May 2017

The thesis of Sam Zi-Ming Chyau is approved.

Daphne Liu PhD, Committee Chair

Michael Krebs PhD

Anthony Shaheen PhD

Grant Fraser PhD, Department Chair

California State University, Los Angeles

May 2017

Abstract

A *multi-distance labeling* (or radio labeling) of a graph G is a function f that assigns each vertex a non-negative integer label such that the separation of labels between distinct vertices u, v is at least $diam(G) + 1 - d(u, v)$, where the *distance* $d(u, v)$ between u and v is the length of a shortest path from u to v and the *diameter* $diam(G)$ is the maximum distance between any two vertices in G. The *span* of a radio labeling f is the difference between the smallest and largest labels assigned by f, and the *radio number* of G is the smallest possible span for any radio labeling of G. We will prove a general formula for the radio number of all grid graphs dependent only on their horizontal and vertical lengths and their parity. We also survey other graphs and their radio numbers, some of which are completely determined. Of special interest within this survey is the radio number of different tree graphs.

Acknowledgements

This work is dedicated to my research supervisor, Dr. Daphne Liu, whose unwavering confidence and tireless support over the last 7.5 years were instrumental to my success, both in the classroom and in my research. Mere words would never fully describe the depth of my gratitude for her patience and guidance and for demonstrating in person what a truly dedicated instructor commits to his or her profession. I also extend many thanks to the members of my thesis committee, Dr. Michael Krebs and Dr. Anthony Shaheen, for their contributions to the completion of this thesis.

I thank my sisters Michelle and Fendi, my brother Stephen, my Uncle James, my Aunt Shu, and my cousin Truly for all their love, strength, and support. None of this work have been possible without their encouragement over the years.

I thank my many students, past and present, for being the light of my life and the reason I persist. They inspire me to work my very hardest in all tasks.

I thank my many friends, particularly Rosanna, Angeline, and Alba, for sharing this journey and the many priceless memories we have created.

Contents

1 Introduction to Radio Labelings and Meshes

1.1 The Channel Assignment Problem

Consider a set of radio stations or transmitters at fixed distances apart, each of which paired with a given frequency. It is of no surprise that proximity of stations induces interference; that is, the closer that two stations are, the stronger their interference may be. To circumvent this disruption to the stations, the separation of channels or frequencies of stations must be sufficiently large.

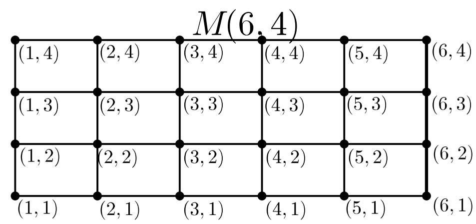
This leads researchers to design a graph model for the system of stations. We first represent the stations and their proximity as vertices and edges on a graph, respectively. We then define a function on the vertex set of this graph to assign frequencies, the goal being to find a function that precludes any interference but does not require unnecessarily high frequencies. This problem is known ubiquitously as the *channel assignment problem*, introduced by Hale in [2] and pursued by many prominent researchers in the field of graph theory.

To date, researchers have discovered many strong results for various classes of graphs in the channel assignment problem. The plethora of commonalities and differences in their methods and results is undeniable.

1.2 Definitions and Preliminary Observations

Definition 1.1. Let $m, n \in \mathbb{N}$. An m x n *grid graph* $M(m, n)$ (or *mesh*) with n rows and m columns is $P_m \square P_n = \{(u, v) : u \in V(P_m), v \in V(P_n)\}$, called the *Cartesian product* of P_m and P_n. P_m and P_n denote paths with m and n vertices, respectively. $(u, v) \sim (u', v')$ in $P_m \square P_n$ if and only if either $u \sim_{P_m} u'$ and $v = v'$ <u>or</u> $u = u'$ and $v \sim_{P_n} v'$.

Beginning from the bottom left corner of $M(m, n)$, we denote the vertices of $M(m, n)$ in a manner analogous to the Cartesian coordinate system in the first quadrant of $\mathbb{R}^2$. That is, $V(M(m, n)) = \{(a, b) \in \mathbb{N} \times \mathbb{N}\colon\ 1 \leq a \leq m\ 1 \leq b \leq n\}$. For additional consistency with terminology used in the Cartesian coordinate system, for any $v = (a, b) \in G$, we refer to a as the *x-coordinate* of v and b as the *y-coordinate* of v.



Above is the grid graph $M(6, 4)$. Each vertex is juxtaposed with an ordered pair of integers.

Definition 1.2. Let G be a connected graph.

1. The *distance* between elements $u, v \in V(G)$, denoted $d(u, v)$, is the length of a shortest path from u to v.
2. The *diameter* of G, denoted $diam(G)$, is $max\{d(u, v) : u, v \in V(G)\}$.

Observation 1.1. *Let* $G = M(m, n)$.

1. $diam(G) = m + n - 2$ *and* $|V(G)| = mn$.
2. *If* $u = (a, b)$ *and* $v = (c, d)$, *then* $d(u, v) = |c - a| + |d - b|$.
3. $M(m, n) \cong M(n, m)$. *So if* m *and* n *have different parity, then we assume that* m *is even and* n *is odd.*

Definition 1.3. Let $G = M(m, n)$.

1. A *center* of G is a middle vertex of G.
   (a) If $m = 2l + 1$ and $n = 2k + 1$, then G has a unique center $(l + 1, k + 1)$.
   (b) If $m = 2l$ and $n = 2k + 1$, then G has 2 centers, $(l, k + 1)$ and $(l + 1, k + 1)$.
   (c) If $m = 2l$ and $n = 2k$, then G has 4 centers, (l, k), $(l+1, k)$, $(l, k+1)$, and $(l+1, k+1)$.
2. The *left region* of G is the set of vertices on the left side of G.
   (a) If $m = 2l + 1$, then the left region of G is $\{(a, b) : 1 \leq a \leq l + 1\}$.
   (b) If $m = 2l$, then the left region of G is $\{(a, b) : 1 \leq a \leq l\}$.
3. The *right region* of G is the set of vertices on the right side of G.
   (a) If $m = 2l + 1$, then the right region of G is $\{(a, b) : l + 1 \leq a \leq m\}$.
   (b) If $m = 2l$, then the right region of G is $\{(a, b) : l + 1 \leq a \leq m\}$.

4. The *corners* of G are $(1,1)$, $(1,n)$, $(m,1)$, and (m,n).
5. The *upper section* of G is the set of upper vertices of G.
   (a) If $n = 2k+1$, then the upper section of G is $\{(a,b) : k+1 \leq b \leq n\}$
   (b) If $n = 2k$, then the upper section of G is $\{(a,b) : k+1 \leq b \leq n\}$
6. The *lower section* of G is the set of lower vertices of G.
   (a) If $n = 2k+1$, then the lower section of G is $\{(a,b) : 1 \leq b \leq k+1\}$
   (b) If $n = 2k$, then the lower section of G is $\{(a,b) : 1 \leq b \leq k\}$.

To simplify our calculations, we introduce an artificial *horizontal axis* and an artificial *vertical axis* to separate the sections and regions of G.

1. If $n = 2k+1$, then all vertices with y-coordinate $k+1$ lie on the horizontal axis. Vertices on or above the horizontal axis form the upper section of G. Vertices on or below the horizontal axis form the lower section of G.
2. If $n = 2k$, then the lower section of G is below the horizontal axis, and the upper section of G is above the horizontal axis. No vertices of G lie on the horizontal axis if n is even, but from a visual perspective, each vertex whose y-coordinate is k or $k+1$ appears to be "half a unit" away from the horizontal axis. The horizontal axis therefore partitions the vertices of G if n is even.
3. If $m = 2l+1$, then all vertices with x-coordinate $l+1$ lie on the vertical axis. Vertices on or to the left of the vertical axis form the left region of G. Vertices on or to the right of the vertical axis form the right region of G.
4. If $m = 2l$, then the left region of G is to the left of the vertical axis, and the right region of G is to the right of the vertical axis. No vertices of G lie on the vertical axis if m is even, but from a visual perspective, each vertex whose x-coordinate is l or $l+1$ appears to be "half a unit" away from the vertical axis. The vertical axis therefore partitions the vertices of G if m is even.

Definition 1.4. Let $v \in V(G)$.

1. The *level* of v, denoted $L(v)$, is the vertical separation of v from the horizontal axis of G. Let w be the vertex in the same column and section of G nearest the horizontal axis.
   (a) If $n = 2k+1$, then $L(v) = d(v,w)$
   (b) If $n = 2k$, then $L(v) = d(v,w) + \frac{1}{2}$.
2. The *displacement* of v, denoted $D(v)$, is the horizontal separation of v from the vertical axis of G. Let u be the vertex in the same row and region of G nearest the vertical axis.

(a) If $m = 2l + 1$, then $D(v) = d(v, u)$

(b) If $m = 2l$, then $D(v) = d(v, u) + \frac{1}{2}$.

Proposition 1.1. *Let $G = M(m, n)$. Let $u, v \in V(G)$.*

1. *If u, v are in the same region of G, then*

$$d(u,v) = \begin{cases} L(u) + L(v) + |D(u) - D(v)| & \textit{if } u, v \textit{ are in opposite sections;} \\ |L(u) - L(v)| + |D(u) - D(v)| & \textit{if } u, v \textit{ are in the same section.} \end{cases}$$

2. *If u, v are in opposite regions of G, then*

$$d(u,v) = \begin{cases} L(u) + L(v) + D(u) + D(v) & \textit{if } u, v \textit{ are in opposite sections;} \\ |L(u) - L(v)| + D(u) + D(v) & \textit{if } u, v \textit{ are in the same section.} \end{cases}$$

Remark 1. If $d(u, v) = L(u) + L(v) + D(u) + D(v)$, then we say the u, v jump is of the *best type*.

Definition 1.5. Let G be a connected graph.

1. A *radio labeling* of G (also known as a *multi-distance labeling*) of G is a function $f : V(G) \to \{0, 1, 2, ...\}$ such that $|f(u) - f(v)| \geq diam(G) + 1 - d(u, v)$ for all distinct $u, v \in V(G)$.
2. The *span* of a radio labeling f, denoted *span*(f), is $max\{|f(u) - f(v)| : u, v \in V(G)\}$
3. The *radio number* of G, denoted $rn(G)$, is the minimum span of all radio labelings of G.
4. If *span*$(f) = rn(G)$, then f is an *optimal radio labeling* of G.

Proposition 1.2. *Every radio labeling on a connected graph G is a one-to-one function on its vertex set $V(G)$, since $d(u, v) \leq diam(G)$ for all $u, v \in V(G)$ by definition. So for every radio labeling f on G, the elements of $V(G) = \{u_1, u_2, ..., u_n\}$ can be uniquely ordered such that*

$$f(u_1) < f(u_2) < ... < f(u_n).$$

If we fix $f(u_1) = 0$ for convenience, then $f(u_n) = span(f)$.

Notation: Let f be a radio labeling of $G = M(m, n)$ with the vertex ordering $(x_1, y_1), ..., (x_{mn}, y_{mn})$.

1. Let v_i denote (x_i, y_i), and let d_i denote $d(v_i, v_{i+1})$.
2. Let f_i denote $f(v_{i+1}) - f(v_i)$, so $f_i \geq n + 1 - d_i$ for every $1 \leq i \leq mn - 1$.
3. Let d_{x_i} denote $|x_{i+1} - x_i|$, the horizontal separation of v_i and v_{i+1}.
4. Let d_{y_i} denote $|y_{i+1} - y_i|$, the vertical separation of v_i and v_{i+1}.

Lemma 1.1. *Let $G = M(m,n)$. If f is a radio labeling of G such that $0 = f(v_1) < f(v_2) < \ldots < f(v_{mn}) = span(f)$, then $span(f)$ is bounded below by*

$$span(f) \geq (mn-1)(m+n-1) - \sum_{i=1}^{mn-1} d(v_i, v_{i+1}).$$

Proof. Let f be a radio labeling of G. Since $diam(G) = m+n-2$, we obtain the following inequalities.

$$\begin{cases} f(v_2) - f(v_1) & \geq m+n-1-d(v_2,v_1) \\ f(v_3) - f(v_2) & \geq m+n-1-d(v_3,v_2) \\ & \vdots \\ f(v_{mn-1}) - f(v_{mn-2}) & \geq m+n-1-d(v_{mn-1},v_{mn-2}) \\ f(v_{mn}) - f(v_{mn-1}) & \geq m+n-1-d(v_{mn},v_{mn-1}) \end{cases}$$

Summing up these $mn-1$ inequalities, we obtain

$$span(f) \geq (mn-1)(m+n-1) - \sum_{i=1}^{mn-1} d(v_i, v_{i+1}).$$

□

2 Radio Number of Ladder Graphs - Lower Bound

2.1 Lower Bound of $rn\left(M(2,n)\right)$ for n Odd

Definition 2.1. A *ladder graph* with $n \geq 3$ steps is a mesh $M(2,n)$.

Observation 2.1. *Let $G = M(2,n)$. Let $v \in V(G)$.*

1. *If $n = 2k+1$, then $L(v) = 0$ if and only if v is a center. The level of any vertex is non-negative.*
2. *If $n = 2k$, then $L(v) = \frac{1}{2}$ if and only if v is a center. The level of any vertex is at least $\frac{1}{2}$.*
3. *Since G has only two columns, every vertex has a displacement of $\frac{1}{2}$. Hence, $D(u) - D(v) = 0$ and $D(u) + D(v) = 1$ for any $u, v \in V(G)$.*
4. *The first column is the left region of G; the second column is the right region of G.*

Proposition 2.1. *Let $G = M(2,n)$. Let $u, v \in V(G)$.*

1. *If u, v are in the same column of G, then*

$$d(u,v) = \begin{cases} L(u) + L(v) & \text{if } u, v \text{ are on opposite sections;} \\ |L(u) - L(v)| & \text{if } u, v \text{ are in the same section.} \end{cases}$$

2. *If u, v are in opposite columns of G, then*

$$d(u,v) = \begin{cases} L(u) + L(v) + 1 & \text{if } u, v \text{ are on opposite sections;} \\ |L(u) - L(v)| + 1 & \text{if } u, v \text{ are in the same section.} \end{cases}$$

So if $d(u,v) = L(u) + L(v) + 1$, then we say that the u, v jump is of the best type.

Lemma 2.1. *Let $n = 2k+1$. Let f be a radio labeling of $G = M(2,n)$, where $0 = f(v_1) < f(v_2) < ... < f(v_{2n}) = span(f)$ gives the ordering of the vertices of G. Then*

$$\sum_{i=1}^{2n-1} d(v_i, v_{i+1}) \leq \sum_{i=1}^{2n-1} [L(v_i) + L(v_{i+1}) + 1] = n^2 + 2n - 2 - [L(v_1) + L(v_{2n})].$$

Proof. Suppose the assumption holds. It is clear from an earlier proposition defining the distance between vertices in terms of their levels that $d(v_i, v_j) \leq L(v_i) + L(v_j) + 1$ for all $v_i, v_j \in V(G)$, with equality holding only when v_i and v_j are on opposite sections and different columns, unless one of the vertices is a center. This gives us the following inequality.

$$\sum_{i=1}^{2n-1} d(v_i, v_{i+1}) \leq \sum_{i=1}^{2n-1} [L(v_i) + L(v_{i+1}) + 1]$$

Notice that each vertex level $L(v_i)$ appears exactly twice in the above inequality except $L(v_1)$ and $L(v_{2n})$, each of which appears only once. Also notice that for any integer $1 \leq t \leq k$, there exist exactly four $v_i \in V(G)$ such that $L(v_i) = t$. Hence, we have

$$\begin{aligned}
\sum_{i=1}^{2n-1} d(v_i, v_{i+1}) &\leq \sum_{i=1}^{2n-1} [L(v_i) + L(v_{i+1}) + 1] \\
&= (2n-1) - L(v_1) - L(v_{2n}) + 2\sum_{i=1}^{2n} L(v_i) \\
&= (2n-1) + 2\left[4\left(1 + 2 + ... + \frac{n-1}{2}\right)\right] - [L(v_1) + L(v_{2n})] \\
&= (2n-1) + 8\left[\frac{1}{2}\left(\frac{n+1}{2}\right)\left(\frac{n-1}{2}\right)\right] - [L(v_1) + L(v_{2n})] \\
&= n^2 + 2n - 2 - [L(v_1) + L(v_{2n})].
\end{aligned} \tag{1}$$

□

From Lemma 2.1, it becomes evident that the values of $L(v_1)$ and $L(v_{2n})$ are directly related to minimizing the sum of the distances between consecutively labeled vertices and the search for a lower bound of $rn(G)$.

Definition 2.2. Let f be a radio labeling of $G = M(m, n)$. A *secluded vertex* of G is a corner vertex v_j of G (where $2 \leq j \leq mn-1$) satisfying $d(v_j, v_{j+1}) = L(v_j)+L(v_{j+1})+D(v_j)+D(v_{j+1})$ and $d(v_j, v_{j-1}) = L(v_j) + L(v_{j-1}) + D(v_j) + D(v_{j-1})$.

Observation 2.2. *If $G = M(2, n)$, then a secluded vertex v_j (where $2 \leq j \leq 2n-1$) is a corner vertex satisfying $d(v_j, v_{j+1}) = L(v_j) + L(v_{j+1}) + 1$ and $d(v_j, v_{j-1}) = L(v_j) + L(v_{j-1}) + 1$.*

Lemma 2.2. *Let $n = 2k + 1$. Let f be a radio labeling of $G = M(2, n)$. If $f(v_{i+1}) - f(v_i) = n + 1 - d(v_{i+1}, v_i)$ for all $1 \leq i \leq 2n - 1$, then for any secluded vertex v_j, we have that either v_{j-1} or v_{j+1} is a center of G.*

Proof. Suppose the assumption holds. Let v_j be a secluded vertex of G. By the assumption, we have

$$\begin{cases} f(v_{j+1}) - f(v_j) = n+1-d(v_{j+1},v_j) = n+1-[L(v_{j+1})+L(v_j)+1] \\ f(v_j) - f(v_{j-1}) = n+1-d(v_j,v_{j-1}) = n+1-[L(v_j)+L(v_{j-1})+1] \end{cases} \tag{2}$$

$$\begin{aligned} \implies f(v_{j+1}) - f(v_{j-1}) &= 2n - 2L(v_j) - L(v_{j+1}) - L(v_{j-1}) \\ &= 2n - 2\left(\frac{n-1}{2}\right) - [L(v_{j+1}) + L(v_{j-1})] \\ &= n+1-[L(v_{j+1})+L(v_{j-1})]\,. \end{aligned} \tag{3}$$

By our assumption, we know that v_{j+1} and v_{j-1} are both on the column and section opposite of v_j. Hence we know

$$\begin{aligned} f(v_{j+1}) - f(v_{j-1}) &\geq n+1-d(v_{j+1},v_{j-1}) \\ &= n+1-|L(v_{j+1})-L(v_{j-1})|. \end{aligned} \tag{4}$$

By (3), we have $n+1-[L(v_{j+1})+L(v_{j-1})] \geq n+1-|L(v_{j+1})-L(v_{j-1})|$

$$\begin{aligned} &\implies L(v_{j+1})+L(v_{j-1}) \leq |L(v_{j+1})-L(v_{j-1})| \\ &\implies L(v_{j+1}) = 0 \text{ or } L(v_{j-1}) = 0. \end{aligned} \tag{5}$$

So either v_{j-1} or v_{j+1} is a center. ☐

Lemma 2.3. *Let $n = 2k+1$. Then $rn(G) \geq n^2 - n + 3$.*

Proof. Let f be any radio labeling of $G = M(2,n)$, with $n = 2k+1$. Since $L(v_1) + L(v_{2n}) \geq 0$, we observe three cases to show that $span(f) \geq n^2 - n + 3$.

Case 1: $L(v_1) + L(v_{2n}) = 0$, so v_1 and v_{2n} are the centers of G.

Subcase 1a: For every $1 \leq i \leq 2n-1$, v_i and v_{i+1} are in different columns of G.
Since G has exactly two centers v_1 and v_{2n} in this case (the first and last vertices in the labeling sequence of f), we know that $v_2, v_3, ..., v_{2n-1}$ are not centers and therefore cannot concurrently be in both the upper and lower sections of G. Hence, at most n consecutively labeled vertices can be labeled by switching sections and columns in each jump, so there must exist an index $2 \leq j \leq 2n-2$ such that v_j and v_{j+1} are in the same section of G, since we assumed they are not in the same column in this subcase.

Without loss of generality, suppose $L(v_{j+1}) \leq L(v_j)$. Notice that v_{j+1} is not a center of G, so $L(v_{j+1}) \geq 1$. Therefore, we have

$$\begin{aligned} d(v_j, v_{j+1}) &= |L(v_j) - L(v_{j+1})| + 1 \\ &= L(v_j) - L(v_{j+1}) + 1 \\ &= L(v_j) + [L(v_{j+1}) - 2L(v_{j+1})] + 1 \\ &\leq L(v_j) + [L(v_{j+1}) - 2] + 1. \end{aligned} \tag{6}$$

$$\begin{aligned} \text{Hence, } \sum_{i=1}^{2n-1} d(v_i, v_{i+1}) &\leq \sum_{i=1}^{2n-1} [L(v_i) + L(v_{j+1}) + 1] - 2 \\ &\leq n^2 + 2n - 4. \end{aligned} \tag{7}$$

Therefore, $span(f) \geq (2n-1)(n+1) - (n^2 + 2n - 4) = n^2 - n + 3$.

Subcase 1b: There exists an index $1 \leq i \leq 2n-1$ such that v_i and v_{i+1} are in the same column of G.

First note that v_1 and v_{2n} are on opposite columns, since they are both centers. Since v_i and v_{i+1} are in the same column and the two columns of G have equal cardinality, there must exist a pair of consecutively labeled vertices v_j and v_{j+1} in the column opposite v_i and v_{i+1}. Therefore, we have

$$\begin{cases} d(v_i, v_{i+1}) \leq L(v_i) + L(v_{i+1}) \\ d(v_j, v_{j+1}) \leq L(v_j) + L(v_{j+1}). \end{cases} \tag{8}$$

$$\begin{aligned} \Longrightarrow \sum_{i=1}^{2n-1} d(v_i, v_{i+1}) &\leq \sum_{i=1}^{2n-1} [L(v_i) + L(v_{i+1}) + 1] - 2 \\ &\leq n^2 + 2n - 4. \end{aligned} \tag{9}$$

So $span(f) \geq (2n-1)(n+1) - (n^2 + 2n - 4) = n^2 - n + 3$.

CASE 2: $L(v_1) + L(v_{2n}) \geq 2$.

$$\begin{aligned} \implies \sum_{i=1}^{2n-1} d(v_i, v_{i+1}) &\leq \sum_{i=1}^{2n-1} [L(v_i) + L(v_{i+1}) + 1] \\ &= n^2 + 2n - 2 - [L(v_1) + L(v_{2n})] \\ &\leq n^2 + 2n - 4. \end{aligned} \tag{10}$$

So $span(f) \geq (2n-1)(n+1) - (n^2 + 2n - 4) = n^2 - n + 3$.

CASE 3: $L(v_1)+L(v_{2n}) = 1$. Without loss of generality, assume that $L(v_1) = 0$ and $L(v_{2n}) = 1$.

Subcase 3a: There exists an index $1 \leq j \leq 2n-1$ such that v_j and v_{j+1} are not on opposite columns and sections. Then

$$d(v_j, v_{j+1}) < L(v_j) + L(v_{j+1}) + 1. \tag{11}$$

$$\begin{aligned} \implies \sum_{i=1}^{2n-1} d(v_i, v_{i+1}) &\leq \sum_{i=1}^{2n-1} [L(v_i) + L(v_{i+1}) + 1] - 1 \\ &= n^2 + 2n - 2 - [L(v_1) + L(v_{2n})] - 1 \\ &= n^2 + 2n - 4. \end{aligned} \tag{12}$$

So $span(f) \geq (2n-1)(n+1) - (n^2 + 2n - 4) = n^2 - n + 3$.

Subcase 3b: For every index $1 \leq i \leq 2n-1$, v_i and v_{i+1} are on opposite columns and sections. Since $L(v_1) = 0$ and $L(v_{2n}) = 1$, we know v_1 is a center of G and that the only unlabeled center is adjacent to v_1 in the opposite column. Since the column opposite v_1 has 2 corner vertices and no vertices precede v_1 in the labeling sequence of f, it is impossible for both corner vertices in the column opposite v_1 to immediately precede or follow a center of G, since we assumed in this subcase that consecutively labeled vertices alternate columns and sections.

Say that v_t (where $3 \leq t \leq 2n-2$) is a corner vertex in the column opposite v_1 that does not immediately follow v_1. Since v_{t+1} and v_{t-1} are both in the column and section opposite v_t, we know that v_{t+1} and v_{t-1} both reside in the same column and section. Therefore, we have

$$\begin{cases} d(v_t, v_{t+1}) = L(v_t) + L(v_{t+1}) + 1 \\ d(v_t, v_{t-1}) = L(v_t) + L(v_{t-1}) + 1 \end{cases} \tag{13}$$

Hence, the conclusion of Lemma 2.2 fails, and so it follows that there exists an index $1 \leq i \leq 2n-1$ such that $f(v_{i+1}) - f(v_i) > n+1-d(v_i, v_{i+1})$, with strict inequality.

$$\begin{aligned} \implies span(f) &> (2n-1)(n+1) - (n^2+2n-3) \\ &= n^2 - n + 2 \end{aligned} \tag{14}$$

So $span(f) \geq n^2 - n + 3$.

Hence, in all cases, we have $span(f) \geq n^2 - n + 3$, thus proving our lower bound for $rn(G)$ when n is odd. □

2.2 Lower Bound of $rn\left(M(2,n)\right)$ for n Even

Lemma 2.4. *Let $n = 2k$. Let f be a radio labeling of $G = M(2,n)$, where $0 = f(v_1) < f(v_2) < ... < f(v_{2n}) = span(f)$ gives the ordering of the vertices of G. Then*

$$\sum_{i=1}^{2n-1} d(v_i, v_{i+1}) \leq \sum_{i=1}^{2n-1} \left[L(v_i) + L(v_{i+1}) + 1\right] \leq n^2 + 2n - 2.$$

Proof. Suppose the assumption holds. We know that $d(v_i, v_j) \leq L(v_i) + L(v_j) + 1$ for all $v_i, v_j \in V(G)$, with equality holding only when v_i and v_j are on opposite sections and different columns. This gives us the following inequality.

$$\sum_{i=1}^{2n-1} d(v_i, v_{i+1}) \leq \sum_{i=1}^{2n-1} \left[L(v_i) + L(v_{i+1}) + 1\right].$$

Notice that each vertex level $L(v_i)$ appears exactly twice in the above inequality except $L(v_1)$ and $L(v_{2n})$, each of which appears only once. Also notice that for any $t \in \left\{\frac{1}{2}, \frac{3}{2}, ..., \left(\frac{n-1}{2}\right)\right\}$, there exist exactly four $v_i \in V(G)$ such that $L(v_i) = t$. Hence, we have

$$\begin{aligned}
\sum_{i=1}^{2n-1} d(v_i, v_{i+1}) &\leq \sum_{i=1}^{2n-1} [L(v_i) + L(v_{i+1}) + 1] \\
&= (2n-1) - L(v_1) - L(v_{2n}) + 2\sum_{i=1}^{2n} L(v_i) \\
&= (2n-1) + 2\left[4\left(\frac{1}{2} + \frac{3}{2} + ... + \frac{n-1}{2}\right)\right] - [L(v_1) + L(v_{2n})] \qquad (15) \\
&= (2n-1) + 4[1 + 3 + ... + (n-1)] - [L(v_1) + L(v_{2n})] \\
&= (2n-1) + 4\left[\frac{1}{2}\left(\frac{n}{2}\right)(n)\right] - [L(v_1) + L(v_{2n})] \\
&= n^2 + 2n - 1 - [L(v_1) + L(v_{2n})].
\end{aligned}$$

Since n is even, we have $L(v_1) + L(v_{2n}) \geq \frac{1}{2} + \frac{1}{2} = 1$. Therefore, we have

$$\sum_{i=1}^{2n-1} [L(v_i) + L(v_{i+1}) + 1] \leq n^2 + 2n - 2.$$

□

Lemma 2.5. *Let $n = 2k$. If v_j (where $2 \leq j \leq 2n-1$) is a secluded vertex, then either $f(v_{j+1}) - f(v_j) > n + 1 - d(v_{j+1}, v_j)$ or $f(v_j) - f(v_{j-1}) > n + 1 - d(v_j, v_{j-1})$.*

Proof. Let $G = M(2, n)$, where $n = 2k$. Let $v_j \in V(G)$ be a secluded vertex, where $2 \leq j \leq 2n-1$. Suppose for contradiction that $f(v_{j+1}) - f(v_j) = n+1-d(v_{j+1}, v_j)$ and $f(v_j) - f(v_{j-1}) = n + 1 - d(v_j, v_{j-1})$. Then

$$\begin{aligned}
f(v_{j+1}) - f(v_{j-1}) &= 2(n+1) - d(v_{j+1}, v_j) - d(v_j, v_{j-1}) \\
&= 2(n+1) - [L(v_{j+1}) + L(v_j) + 1] - [L(v_j) + L(v_{j-1}) + 1] \\
&= 2n - L(v_{j+1}) - L(v_{j-1}) - 2L(v_j) \qquad (16) \\
&= 2n - [L(v_{j+1}) + L(v_{j-1})] - 2\left(\frac{n-1}{2}\right) \\
&= n + 1 - [L(v_{j+1}) + L(v_{j-1})].
\end{aligned}$$

By our assumption, we know that v_{j+1} and v_{j-1} are both on the column and section opposite of v_j. Hence we know

$$\begin{aligned} f(v_{j+1}) - f(v_{j-1}) &\geq n + 1 - d(v_{j+1}, v_{j-1}) \\ &= n + 1 - |L(v_{j+1}) - L(v_{j-1})|. \end{aligned} \tag{17}$$

By (16), we have $n + 1 - [L(v_{j+1}) + L(v_{j-1})] \geq n + 1 - |L(v_{j+1}) - L(v_{j-1})|$

$$\begin{aligned} &\implies L(v_{j+1}) + L(v_{j-1}) \leq |L(v_{j+1}) - L(v_{j-1})| \\ &\implies L(v_{j+1}) = 0 \text{ or } L(v_{j-1}) = 0. \end{aligned} \tag{18}$$

But this is contradictory, since $L(v_{j+1}) \geq \frac{1}{2}$ and $L(v_{j-1}) \geq \frac{1}{2}$. So our assumption is false; therefore, either $f(v_{j+1}) - f(v_j) > n+1-d(v_{j+1}, v_j)$ or $f(v_j) - f(v_{j-1}) > n+1-d(v_j, v_{j-1})$. $\square$

Definition 2.3. Let $G = M(m, n)$. Let f be a radio labeling of $G = M(m, n)$, where $0 = f(v_1) < f(v_2) < ... < f(v_{mn}) = \mathit{span}(f)$ gives the ordering of the vertices of G. The *α-number* of f, denoted $\alpha(f)$, is the number of indices $1 \leq i \leq mn - 1$ such that $f_i > m + n - 1 - d_i$.

From this definition, we obtain the following:

$$\mathit{span}(f) \geq (mn - 1)(m + n - 1) + \alpha(f) - \sum_{i=1}^{mn-1} d(v_i, v_{i+1})$$

Observation: If $m = 2$ (i.e. G is a ladder graph with n steps), then $\alpha(f)$ is the number of indices $1 \leq i \leq 2n - 1$ such that $f_i > n + 1 - d_i$.

Lemma 2.6. *Let $n = 2k$. Then $rn(G) \geq n^2 - n + 4$.*

Proof. Let f be any radio labeling of $G = M(2, n)$, with $n = 2k$. Notice that G has exactly 4 corners and therefore at most 4 secluded vertices. We examine different cases based on the number of secluded vertices in G to show that $\mathit{span}(f) \geq n^2 - n + 4$.

CASE 1: G has four secluded vertices $v_{t_1}, v_{t_2}, v_{t_3}$, and v_{t_4}, where $1 < t_1 < t_2 < t_3 < t_4 < 2n$. Then, we have

$$\begin{cases} f_{t_1} > n+1-d_{t_1} \text{ or } f_{t_1-1} > n+1-d_{t_1-1} \\ f_{t_2} > n+1-d_{t_2} \text{ or } f_{t_2-1} > n+1-d_{t_2-1} \\ f_{t_3} > n+1-d_{t_3} \text{ or } f_{t_3-1} > n+1-d_{t_3-1} \\ f_{t_4} > n+1-d_{t_4} \text{ or } f_{t_4-1} > n+1-d_{t_4-1}. \end{cases} \tag{19}$$

If two secluded vertices are consecutively labeled, then two of the conditions can be concurrently satisfied in a single jump (the jump between the two said secluded vertices). However, no three secluded vertices may be consecutive, since any corner vertex whose predecessor and successor in the labeling pattern of f are both corners would necessarily induce a jump not of the best type. Since there are two pairs of secluded vertices, we can satisfy the first and second conditions above with a single jump, and likewise the third and fourth conditions with a single jump.

Hence, $\alpha(f) \geq 2$, with equality possible only if $t_1 + 1 = t_2$ and $t_3 + 1 = t_4$.

Observe that when n is even, no vertex concurrently exists in both the upper and lower sections of G. Hence, at most n consecutively labeled vertices can be labeled by alternating columns and sections in each jump; in other words, at most $n-1$ consecutive jumps can be of the best type. Therefore, there exists an index $1 \leq j \leq 2n-1$ such that v_j and v_{j+1} are either in the same column or the same section. Hence, we have $d(v_j, v_{j+1}) \leq L(v_j) + L(v_{j+1})$. Consequently,

$$\sum_{i=1}^{2n-1} d(v_i, v_{i+1}) \leq (n^2 + 2n - 2) - 1 = n^2 + 2n - 3.$$

So $span(f) \geq (2n-1)(n+1) + 2 - (n^2 + 2n - 3) = n^2 - n + 4$.

CASE 2: G has exactly three secluded vertices v_{t_1}, v_{t_2}, and v_{t_3}, where $1 < t_1 < t_2 < t_3 < 2n$, and one non-secluded corner vertex v_c. Then we have

$$\begin{cases} f_{t_1} > n+1-d_{t_1} \text{ or } f_{t_1-1} > n+1-d_{t_1-1} \\ f_{t_2} > n+1-d_{t_2} \text{ or } f_{t_2-1} > n+1-d_{t_2-1} \\ f_{t_3} > n+1-d_{t_3} \text{ or } f_{t_3-1} > n+1-d_{t_3-1} \\ d_c \leq L(v_c) + L(v_{c+1}) \text{ or } d_{c-1} \leq L(v_{c-1}) + L(v_c) \text{ unless } v_c \in \{v_1, v_{2n}\}. \end{cases} \tag{20}$$

Hence, by the same argument from Case 1, we have $\alpha(f) \geq 2$ with equality possible only if

$t_1 + 1 = t_2$ or $t_2 + 1 = t_3$.

Also, since v_c is a non-secluded corner vertex, we know that either v_c is in $\{v_1, v_{2n}\}$ (in which case $L(v_1) + L(v_{2n}) \geq 2$) or at least one jump to or from v_c is not of the best type. In either situation, from Lemma 2.4 we have

$$\sum_{i=1}^{2n-1} d(v_i, v_{i+1}) \leq (n^2 + 2n - 2) - 1 = n^2 + 2n - 3.$$

So $span(f) \geq (2n-1)(n+1) + 2 - (n^2 + 2n - 3) = n^2 - n + 4$.

CASE 3: G has exactly two secluded vertices v_{t_1} and v_{t_2}, where $1 < t_1 < t_2 < 2n$, and two non-secluded corner vertices v_{c_1} and v_{c_2}, where $c_1 < c_2$. Then, we have

$$\begin{cases} f_{t_1} > n + 1 - d_{t_1} \text{ or } f_{t_1 - 1} > n + 1 - d_{t_1 - 1} \\ f_{t_2} > n + 1 - d_{t_2} \text{ or } f_{t_2 - 1} > n + 1 - d_{t_2 - 1} \\ d_{c_1} \leq L(v_{c_1}) + L(v_{c_1+1}) \text{ or } d_{c_1 - 1} \leq L(v_{c_1 - 1}) + L(v_{c_1}) \text{ unless } v_{c_1} \in \{v_1, v_{2n}\} \\ d_{c_2} \leq L(v_{c_2}) + L(v_{c_2+1}) \text{ or } d_{c_2 - 1} \leq L(v_{c_2 - 1}) + L(v_{c_2}) \text{ unless } v_{c_2} \in \{v_1, v_{2n}\}. \end{cases} \tag{21}$$

Hence, $\alpha(f) \geq 1$ with equality possible only if $t_1 + 1 = t_2$.

Also, we know that each non-secluded corner vertex v_{c_1} and v_{c_2} either is in $\{v_1, v_{2n}\}$ or necessarily induces a jump not of the best type. Any corner vertex in $\{v_1, v_{2n}\}$ would automatically increase $L(v_1) + L(v_{2n})$ by at least 1 and therefore sufficiently minimize the sum of the distances between consecutive vertices (as in Case 2), so we examine the different possibilities if neither of these corner vertices is in $\{v_1, v_{2n}\}$.

1. If v_{c_1} and v_{c_2} are not consecutive, then they each induce a distinct jump not of the best type, since they are non-secluded.
2. If v_{c_1} and v_{c_2} are consecutive and antipodal (i.e. in different columns and sections), then the jump from v_{c_1} to v_{c_2} is of the best type, so both these vertices must induce a distinct jump not of the best type, since they are non-secluded.
3. If v_{c_1} and v_{c_2} are consecutive but not antipodal, then the jump from v_{c_1} to v_{c_2} concurrently satisfies the third and fourth conditions of this case; however, that would force t_2 to be strictly larger than $t_1 + 1$ (since secluded vertices v_{t_1} and v_{t_2} cannot be in the same column

or section if they are consecutive), which would make the preceding inequality strict.

From our observations, in Case 3 one of the two following statements must hold.

$$\sum_{i=1}^{2n-1} d(v_i, v_{i+1}) \leq (n^2+2n-2)-2 = n^2+2n-4 \quad \text{and} \quad \alpha(f) \geq 1$$

OR

$$\sum_{i=1}^{2n-1} d(v_i, v_{i+1}) \leq (n^2+2n-2)-1 = n^2+2n-3 \quad \text{and} \quad \alpha(f) \geq 2.$$

In either case, $span(f) \geq n^2 - n + 4$.

CASE 4: G has exactly one secluded vertex v_t, where $1 < t < 2n$, and three non-secluded corner vertices v_{c_1}, v_{c_2}, and v_{c_3}, where $c_1 < c_2 < c_3$. Then, we have

$$\begin{cases} f_t > n+1-d_t \text{ or } f_{t-1} > n+1-d_{t-1} \\ d_{c_1} \leq L(v_{c_1}) + L(v_{c_1+1}) \text{ or } d_{c_1-1} \leq L(v_{c_1-1}) + L(v_{c_1}) \text{ unless } v_{c_1} \in \{v_1, v_{2n}\} \\ d_{c_2} \leq L(v_{c_2}) + L(v_{c_2+1}) \text{ or } d_{c_2-1} \leq L(v_{c_2-1}) + L(v_{c_2}) \text{ unless } v_{c_2} \in \{v_1, v_{2n}\} \\ d_{c_3} \leq L(v_{c_3}) + L(v_{c_3+1}) \text{ or } d_{c_3-1} \leq L(v_{c_3-1}) + L(v_{c_3}) \text{ unless } v_{c_3} \in \{v_1, v_{2n}\}. \end{cases} \tag{22}$$

Hence, by the first condition, we know $\alpha(f) \geq 1$.

Also, unless one of the three non-secluded vertices v_{c_1}, v_{c_2}, and v_{c_3} is in $\{v_1, v_{2n}\}$ (in which case $L(v_1) + L(v_{2n}) \geq 2$), it is impossible for v_{c_1}, v_{c_2}, and v_{c_3} to collectively induce only one jump not of the best type, since a single jump can only satisfy at most two of the above conditions. Therefore, we have

$$\sum_{i=1}^{2n-1} d(v_i, v_{i+1}) \leq (n^2+2n-2)-2 = n^2+2n-4.$$

So $span(f) \geq (2n-1)(n+1)+1-(n^2+2n-4) = n^2-n+4$.

CASE 5: G has four non-secluded corner vertices v_{c_1}, v_{c_2}, v_{c_3}, and v_{c_4} with $c_1 < c_2 < c_3 < c_4$. Then

$$\begin{cases} d_{c_1} \leq L(v_{c_1}) + L(v_{c_1+1}) \text{ or } d_{c_1-1} \leq L(v_{c_1-1}) + L(v_{c_1}) \text{ unless } v_{c_1} \in \{v_1, v_{2n}\} \\ d_{c_2} \leq L(v_{c_2}) + L(v_{c_2+1}) \text{ or } d_{c_2-1} \leq L(v_{c_2-1}) + L(v_{c_2}) \text{ unless } v_{c_2} \in \{v_1, v_{2n}\} \\ d_{c_3} \leq L(v_{c_3}) + L(v_{c_3+1}) \text{ or } d_{c_3-1} \leq L(v_{c_3-1}) + L(v_{c_3}) \text{ unless } v_{c_3} \in \{v_1, v_{2n}\} \\ d_{c_4} \leq L(v_{c_4}) + L(v_{c_4+1}) \text{ or } d_{c_4-1} \leq L(v_{c_4-1}) + L(v_{c_4}) \text{ unless } v_{c_4} \in \{v_1, v_{2n}\}. \end{cases} \tag{23}$$

We examine the different possibilities for the non-secluded corner vertices v_{c_1}, v_{c_2}, v_{c_3}, and v_{c_4}.

1. If two of the four corner vertices are in $\{v_1, v_{2n}\}$, then $L(v_1)+L(v_{2n}) \geq 3$ and the remaining two corner vertices would induce a jump not of the best type.
2. If only one of the four corner vertices is in $\{v_1, v_{2n}\}$, then $L(v_1) + L(v_{2n}) \geq 2$, and the remaining three corner vertices would induce at least two distinct jumps not of the best type, as was determined in Case 4.
3. If none of the corner vertices are in $\{v_1, v_{2n}\}$, then the four corner vertices would induce at least two distinct jumps not of the best type. However, if $L(v_1) + L(v_{2n}) = 1$ and the four corners induce *exactly* 2 jumps not of the best type (in other words, if $c_4 = c_3 + 1$ and $c_2 = c_1 + 1$), then we consider all of the following observations:
   (a) The two jumps not of the best type must be between corner vertices within the same section or column. However, if the jumps are between corner vertices of the same section, then each jump has a distance 1, which is the worst possible type and further reduces the sum of the distances between consecutive vertices. So we set the two jumps that are not of the best type to be between corners in the same column.
   (b) If $c_3 = c_2 + 1$ (so the jump from v_{c_2} to v_{c_3} is of the best type) where v_{c_1} and v_{c_2} are in one column and v_{c_3} and v_{c_4} are in the other, then v_{c_1} and v_{c_4} will be in opposite columns and sections, which would would force an additional jump not of the best type in order to label all vertices. So $c_3 \neq c_2 + 1$, which prevents labeling v_{c_3} immediately after v_{c_2}.
   (c) The labeling pattern must begin and end at central vertices. This forces an additional jump not of the best type unless v_1 and v_{2n} are on opposite sections and columns (since no vertices are in more than one section or column). But f already includes two jumps between vertices in the same region, since $c_2 = c_1 + 1$ and $c_4 = c_3 + 1$. So v_1 and v_{2n} must be in opposite columns and sections to avoid another jump not of the best type.
   (d) If the two jumps that are not of the best type are between corners in the same column

as indicated in (a), then v_{c_4+1} and v_{c_3} are in the same section but opposite columns.

CLAIM: If v_{c_4+1} and v_{c_3} are in the same section but opposite columns, then we have $f_{c_3} > n+1-d_{c_3}$ or $f_{c_4} > n+1-d_{c_4}$.

Proof. Assume that $f_{c_3} = n+1-d_{c_3}$ and $f_{c_4} = n+1-d_{c_4}$. Then

$$\begin{aligned} f_{c_4} = n+1-d_{c_4} &= n+1-L(v_{c_4+1})-L(v_{c_4})-1 \\ &= n-\frac{n-1}{2}-L(v_{c_4+1}) \\ &= \frac{n}{2}-L(v_{c_4+1})+\frac{1}{2}. \end{aligned} \tag{24}$$

Therefore, since $f_{c_3} = n+1-d_{c_3} = (n+1)-(n-1) = 2$, we have $f(v_{c_4+1})-f(v_{c_3}) = \frac{n}{2}+\frac{5}{2}-L(v_{c_4+1})$. However,

$$\begin{aligned} f(v_{c_4+1})-f(v_{c_3}) &\geq n+1-d(v_{c_4+1},v_{c_3}) \\ &= n+1-L(v_{c_3})+L(v_{c_4+1})-1 \\ &= n-\frac{n-1}{2}+L(v_{c_4+1}) \\ &= \frac{n}{2}+L(v_{c_4+1})+\frac{1}{2} \\ &> \frac{n}{2}+\frac{5}{2}-L(v_{c_4+1}),\ \text{since } L(v_{c_4+1}) > 1. \end{aligned} \tag{25}$$

This contradiction shows that $f_{c_3} > n+1-d_{c_3}$ or $f_{c_4} > n+1-d_{c_4}$, which indicates that $\alpha(f) \geq 1$. □

From our observations, in Case 5 one of the two following statements must hold.

$$\sum_{i=1}^{2n-1} d(v_i,v_{i+1}) \leq (n^2+2n-2)-3 = n^2+2n-5$$

OR

$$\sum_{i=1}^{2n-1} d(v_i,v_{i+1}) \leq (n^2+2n-2)-2 = n^2+2n-4 \quad \text{and} \quad \alpha(f) \geq 1.$$

In either case, $span(f) \geq n^2-n+4$.

Hence, in all 5 cases, we have $span(f) \geq n^2-n+4$, thus proving our lower bound for $rn(G)$ when n is even. □

3 Radio Number of Ladder Graphs - Upper Bound

3.1 Upper Bound of $rn\,(M(2,n))$ for n Odd

Lemma 3.1. *Let $n = 2k+1$. Then $rn(G) \leq n^2 - n + 3$.*

Observation 3.1. *This proposed upper bound is precisely the lower bound previously proven for $rn(M(2,n))$.*

Proof. Let $n = 2k+1$. It suffices to find one radio labeling of $G = M(2,n)$ with a span of $n^2 - n + 3$. Let $\{w_1, w_2, ..., w_{2n}\}$ be a permutation of $V(G)$ given by the following pattern.

$$w_1 = (1,k+1) \xrightarrow{k+1} (2,1) \xrightarrow{k+2} (1,k+2) \xrightarrow{k+1} (2,2) \xrightarrow{k+2} (1,k+3) \xrightarrow{k+1} (2,3) \xrightarrow{k+2} ... \xrightarrow{k+1}$$
$$w_{2k} = (2,k) \xrightarrow{k+2} (1,n) \xrightarrow{2k} (1,1) \xrightarrow{k+2} (2,k+2) \xrightarrow{k+1} (1,2) \xrightarrow{k+2} (2,k+3) \xrightarrow{k+1} ... \xrightarrow{k+1} (1,k)$$
$$\xrightarrow{k+2} (2,n) \xrightarrow{k} (2,k+1) = w_{2n}$$

Let the quantity above each arrow indicate the distance between the two consecutive vertices. Let $f : V(G) \to \{0,1,2,...\}$ be a function such that $f(w_1) = 0$ and $f(w_{i+1}) - f(w_i) = n + 1 - d(w_{i+1}, w_i)$ for all $1 \leq i \leq 2n-1$.

CLAIM: f is a radio labeling of G.

To prove this, we show that for all $1 \leq i \leq 2n-2$, we have

$$f(w_j) - f(w_i) \geq n + 1 - d(w_j, w_i) \text{ for any } j \geq i+2.$$

By assumption,

$$\begin{aligned} f(w_{i+1}) - f(w_i) &= n + 1 - d(w_{i+1}, w_i) \\ f(w_{i+2}) - f(w_{i+1}) &= n + 1 - d(w_{i+2}, w_{i+1}) \\ &\vdots \\ f(w_j) - f(w_{j-1}) &= n + 1 - d(w_j, w_{j-1}). \end{aligned}$$

Summing up these $j - i$ equations, we obtain

$$f(w_j) - f(w_i) = (j-i)(n+1) - \sum_{t=i}^{j-1} d(w_t, w_{t+1}).$$

Let $j = i + p$, where $p \geq 2$. We observe 2 cases, using distances indicated in the labeling pattern for f for reference.

CASE 1: $p = 2$.

Subcase 1a: $i = 2k$ or $2k+1$ (i.e. $i = n-1$ or n). Then according to the labeling pattern, $d(w_j, w_i) = k$.

$$\begin{aligned} \implies f(w_j) - f(w_i) &= 2(n+1) - d(w_i, w_{i+1}) - d(w_{i+1}, w_{i+2}) \\ &= n+1+(2k+2) - [(k+2)+2k] \\ &= n+1-k = n+1-d(w_j, w_i). \end{aligned} \tag{26}$$

Subcase 1b: $i = 4k$ (i.e. $i = 2n-2$).

$$\begin{aligned} \implies f(w_j) - f(w_i) &= 2(n+1) - d(w_i, w_{i+1}) - d(w_{i+1}, w_{i+2}) \\ &= n+1+(2k+2) - [(k+2)+k] \\ &= n+1 > n+1-d(w_j, w_i) \quad \text{since } d(w_j, w_i) \geq 1. \end{aligned} \tag{27}$$

Subcase 1c: $i \neq 2k$, $2k+1$, or $4k$. Then according to the labeling pattern, $d(w_j, w_i) = 1$.

$$\begin{aligned} \implies f(w_j) - f(w_i) &= 2(n+1) - d(w_i, w_{i+1}) - d(w_{i+1}, w_{i+2}) \\ &= n+1+(2k+2) - [(k+1)+(k+2)] \\ &= n+1-1 = n+1-d(w_j, w_i). \end{aligned} \tag{28}$$

CASE 2: $p \geq 3$.

First note that the distances in the labeling pattern alternate between $k+1$ and $k+2$ except on two occasions.

1. The distance from $w_n = (1, n)$ to $w_{n+1} = (1, 1)$ is $2k$ instead of $k+1$.
2. The distance from $w_{2n-1} = (2, n)$ to $w_{2n} = (2, k+1)$ is k instead of $k+1$.

Also, for $p \geq 3$, at most $\lceil \frac{p}{2} \rceil$ jumps between w_i and w_j in the labeling pattern are of distance $k+1$.

$$\begin{aligned} \text{Hence, } \sum_{t=i}^{j-1} d(w_t, w_{t+1}) &\leq p(k+2) - \left\lceil \frac{p}{2} \right\rceil + [2k - (k+1)] \\ &= p(k+2) - \left\lceil \frac{p}{2} \right\rceil + k - 1. \end{aligned} \tag{29}$$

$$
\begin{aligned}
\implies f(w_j) - f(w_i) &= p(n+1) - \sum_{t=i}^{j-1} d(w_t, w_{t+1}) \\
&= n + 1 + (p-1)(2k+2) - \sum_{t=i}^{j-1} d(w_t, w_{t+1}) \\
&\geq n + 1 + (p-1)(2k+2) - \left[p(k+2) - \left\lceil \frac{p}{2} \right\rceil + k - 1\right] \qquad (30) \\
&= n + 1 + (p-3)k + \left\lceil \frac{p}{2} \right\rceil - 1 \\
&\geq n + 1 \qquad \text{since } p \geq 3 \text{ and } \left\lceil \frac{p}{2} \right\rceil \geq 1 \\
&> n + 1 - d(w_j, w_i).
\end{aligned}
$$

By Cases 1 and 2, we know that f is a radio labeling of G, proving our claim.

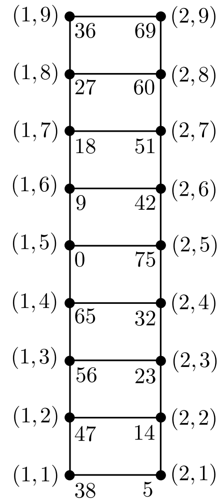
Notice from the labeling pattern that there are four possible distances between consecutively labeled vertices, namely k, $k+1$, $2k$, and $k+2$, with the number of occurrences 1, $2k-1$, 1, and $2k$, respectively.

$$
\begin{aligned}
\implies span(f) &= (2n-1)(n+1) - \sum_{i=1}^{2n-1} d(w_i, w_{i+1}) \\
&= (2n-1)(n+1) - [(k)(1) + (k+1)(2k-1) + (2k)(1) + (k+2)(2k)] \\
&= (2n-1)(n+1) - (4k^2 + 8k - 1) \\
&= (2n-1)(n+1) - \left[4\left(\frac{n-1}{2}\right)^2 + 8\left(\frac{n-1}{2}\right) - 1\right] \qquad (31) \\
&= (2n-1)(n+1) - \left[(n-1)^2 + 4(n-1) - 1\right] \\
&= n^2 - n + 3.
\end{aligned}
$$

Therefore, $rn(G) \leq n^2 - n + 3$, since f is a radio labeling of G with this span.

□

$rn(M(2,9)) = 75$



Above is an optimal radio labeling of $M(2,9)$ following the labeling pattern described in the proof of Lemma 3.1. The radio number of $M(2,9)$ is $9^2 - 9 + 3 = 75$.

3.2 Upper Bound of $rn\left(M(2,n)\right)$ for n Even

Lemma 3.2. *Let $n = 2k$. Then $rn(G) \leq n^2 - n + 4$.*

Proof. Let $n = 2k$. It suffices to find one radio labeling of $G = M(2,n)$ with a span of $n^2 - n + 4$. Let $\{w_1, w_2, ..., w_{2n}\}$ be a permutation of $V(G)$ given by the following pattern.

$$w_1 = (1, k+1) \xrightarrow{k+1} (2,1) \xrightarrow{2k} (1,2k) \xrightarrow{k+1} (2,k) \xrightarrow{k} (1,2k-1) \xrightarrow{k+1} (2,k-1) \xrightarrow{k} (1,2k-2) \xrightarrow{k+1} (2,k-2) \xrightarrow{k} (1,2k-3) \xrightarrow{k+1} (2,k-3) \xrightarrow{k} ... \xrightarrow{k} (1,k+2) \xrightarrow{k+1} w_n = (2,2) \xrightarrow{2k-3} (2,2k-1) \xrightarrow{k+1} (1,k-1) \xrightarrow{k} (2,2k-2) \xrightarrow{k+1} (1,k-2) \xrightarrow{k} (2,2k-3) \xrightarrow{k+1} (1,k-3) \xrightarrow{k} ... \xrightarrow{k} (2,k+2) \xrightarrow{k+1} (1,2) \xrightarrow{k} (2,k+1) \xrightarrow{k+1} (1,1) \xrightarrow{2k} (2,2k) \xrightarrow{k+1} (1,k) = w_{2n}$$

Let the quantity above each arrow indicate the distance between the two consecutive vertices. Let $f : V(G) \to \{0, 1, 2, ...\}$ be a function such that $f(w_1) = 0$ and

$$f(w_{i+1}) - f(w_i) = \begin{cases} n + 1 - d(w_{i+1}, w_i) & \text{if } 1 \leq i \leq 2n-1 \text{ unless } i = 2 \text{ or } 2n-2; \\ n + 2 - d(w_{i+1}, w_i) & \text{if } i = 2 \text{ or } 2n-2. \end{cases}$$

CLAIM: f is a radio labeling of G.

To prove this, we show that for all $1 \leq i \leq 2n-2$, we have

$$f(w_j) - f(w_i) \geq n + 1 - d(w_j, w_i) \text{ for any } j \geq i + 2.$$

By the same justification in the previous section, we have

$$f(w_j) - f(w_i) = f_{j-1} + f_{j-2} + ... + f_{i+1} + f_i.$$

Let $j = i + p$, where $p \geq 2$. We observe 3 cases, using distances indicated in the labeling pattern for f for reference.

CASE 1: $p = 2$.

Subcase 1a: $i = 1, 2, 2n-3$, or $2n-2$. Then according to the labeling pattern, $d(w_j, w_i) = k-1$.

$$\begin{aligned}\implies f(w_j) - f(w_i) &= 2(n+1) + 1 - d(w_i, w_{i+1}) - d(w_{i+1}, w_{i+2}) \\ &= 2(n+1) + 1 - [(k+1) + 2k] \\ &= n + 1 - (k-1) = n + 1 - d(w_j, w_i).\end{aligned} \tag{32}$$

Subcase 1b: $i = n-1$ or n. Then according to the labeling pattern, $d(w_j, w_i) = k-2$.

$$\begin{aligned}\implies f(w_j) - f(w_i) &= 2(n+1) - d(w_i, w_{i+1}) - d(w_{i+1}, w_{i+2}) \\ &= 2(n+1) - [(k+1) + (2k-3)] \\ &= n + 1 - (k-3) \\ &> n + 1 - (k-2) = n + 1 - d(w_j, w_i).\end{aligned} \tag{33}$$

Subcase 1c: $i \neq 1, 2, n-1, n, 2n-3$, or $2n-2$. Then according to the labeling pattern, $d(w_j, w_i) = 1$.

$$\begin{aligned}\implies f(w_j) - f(w_i) &= 2(n+1) - d(w_i, w_{i+1}) - d(w_{i+1}, w_{i+2}) \\ &= 2(n+1) - [(k+1) + k] \\ &= n + 1 > n + 1 - d(w_j, w_i).\end{aligned} \tag{34}$$

CASE 2: $p = 3$.

Subcase 2a: $i = 1$ or $2n-3$.

$$\begin{aligned}\implies f(w_j) - f(w_i) &= 3(n+1) + 1 - \sum_{t=i}^{i+2} d(w_t, w_{t+1}) \\ &= 3(n+1) + 1 - [(k+1) + 2k + (k+1)] \\ &= n + 1 + 2(2k+1) + 1 - [(k+1) + 2k + (k+1)] \\ &= n + 1 + 1 > n + 1 - d(w_j, w_i).\end{aligned} \tag{35}$$

Subcase 2b: $i = 2$ or $2n-4$.

$$\begin{aligned}\implies f(w_j) - f(w_i) &= 3(n+1) + 1 - \sum_{t=i}^{i+2} d(w_t, w_{t+1}) \\ &= 3(n+1) + 1 - [2k + (k+1) + k] \\ &= n + 1 + 2(2k+1) + 1 - [2k + (k+1) + k] \\ &= n + 1 + 2 > n + 1 - d(w_j, w_i).\end{aligned} \tag{36}$$

Subcase 2c: $i = n-1$.

$$\begin{aligned}\implies f(w_j) - f(w_i) &= 3(n+1)+1-\sum_{t=i}^{i+2} d(w_t, w_{t+1}) \\ &= 3(n+1) - [(k+1)+(2k-3)+(k+1)] \\ &= n+1+2(2k+1) - [(k+1)+(2k-3)+(k+1)] \\ &= n+1+3 > n+1-d(w_j, w_i).\end{aligned} \tag{37}$$

Subcase 2d: $i = n-2$ or n.

$$\begin{aligned}\implies f(w_j) - f(w_i) &= 3(n+1)+1-\sum_{t=i}^{i+2} d(w_t, w_{t+1}) \\ &= 3(n+1) - [k+(k+1)+(2k-3)] \\ &= n+1+2(2k+1) - [k+(k+1)+(2k-3)] \\ &= n+1+4 > n+1-d(w_j, w_i).\end{aligned} \tag{38}$$

Subcase 2e: $i \in \{3, 5, 7, ..., n-3, n+1, n+3, ..., 2n-5\}$.

$$\begin{aligned}\implies f(w_j) - f(w_i) &= 3(n+1)+1-\sum_{t=i}^{i+2} d(w_t, w_{t+1}) \\ &= 3(n+1) - [(k+1)+k+(k+1)] \\ &= n+1+2(2k+1) - [(k+1)+k+(k+1)] \\ &= n+1+k-1 > n+1-d(w_j, w_i) \quad \text{since } k > 1.\end{aligned} \tag{39}$$

Subcase 2f: $i \in \{4, 6, 8, ..., n-4, n+2, n+4, ...2n-6\}$.

$$\begin{aligned}\implies f(w_j) - f(w_i) &= 3(n+1)+1-\sum_{t=i}^{i+2} d(w_t, w_{t+1}) \\ &= 3(n+1) - [k+(k+1)+k] \\ &= n+1+2(2k+1) - [k+(k+1)+k] \\ &= n+1+k+1 > n+1-d(w_j, w_i).\end{aligned} \tag{40}$$

CASE 3: $p \geq 4$.

First note that the distances in the labeling pattern alternate between k and $k+1$ except on three occasions.

1. The distance from $w_2 = (2, 1)$ to $w_3 = (1, 2k)$ is $2k$ instead of k.

2. The distance from $w_n = (2,2)$ to $w_{n+1} = (2, 2k-1)$ is $2k-3$ instead of k.
3. The distance from $w_{2n-2} = (1,1)$ to $w_{2n-1} = (2, 2k)$ is $2k$ instead of k.

Also notice that for any $p \geq 4$, at most $\lceil \frac{p}{2} \rceil$ jumps between w_i and w_j in the labeling pattern are of distance k.

Subcase 3a: $i \leq 2$ and $j \geq 2n-1$. So $p \geq 4k-3$.

$$\begin{aligned}
\text{In this subcase, } \sum_{t=i}^{j-1} d(w_t, w_{t+1}) &\leq p(k+1) - \left\lceil \frac{p}{2} \right\rceil + [(2k-3)-k] + 2(2k-k)] \\
&= p(k+1) - \left\lceil \frac{p}{2} \right\rceil + 3k - 3. \\
&\leq p(k+1) - \left\lceil \frac{4k-3}{2} \right\rceil + 3k - 3 \\
&\leq p(k+1) - (2k-1) + (3k-3) \\
&= p(k+1) + k - 2.
\end{aligned} \tag{41}$$

$$\begin{aligned}
\implies f(w_j) - f(w_i) &= p(n+1) + 2 - \sum_{t=i}^{j-1} d(w_t, w_{t+1}) \\
&= n + 1 + (p-1)(2k+1) + 2 - \sum_{t=i}^{j-1} d(w_t, w_{t+1}) \\
&\geq n + 1 + (p-1)(2k+1) + 2 - [p(k+1) + k - 2] \\
&= n + 1 + pk - 3k + 3 \\
&= n + 1 + k(p-3) + 3 \\
&> n + 1 + 3 \quad \text{since } p > 3 \\
&> n + 1 - d(w_j, w_i).
\end{aligned} \tag{42}$$

Subcase 3b: $i > 2$ or $j < 2n-1$.

$$\begin{aligned}
\text{In this subcase, } \sum_{t=i}^{j-1} d(w_t, w_{t+1}) &\leq p(k+1) - \left\lceil \frac{p}{2} \right\rceil + [(2k-3)-k] + (2k-k) \\
&= p(k+1) - \left\lceil \frac{p}{2} \right\rceil + 2k - 3. \\
&\leq p(k+1) - 2 + 2k - 3 \quad \text{since } p \geq 4 \\
&= p(k+1) + 2k - 5.
\end{aligned} \tag{43}$$

$$\begin{aligned}
\implies f(w_j) - f(w_i) &= p(n+1) - \sum_{t=i}^{j-1} d(w_t, w_{t+1}) \\
&= n+1+(p-1)(2k+1) - \sum_{t=i}^{j-1} d(w_t, w_{t+1}) \\
&\geq n+1+(p-1)(2k+1) - [p(k+1)+2k-5] \\
&= n+1+pk-4k+5 \\
&= n+1+k(p-4)+5 \\
&\geq n+1+5 \quad \text{since } p \geq 4 \\
&> n+1-d(w_j, w_i).
\end{aligned} \tag{44}$$

By Cases 1, 2, and 3, we know that f is a radio labeling of G, proving our claim.

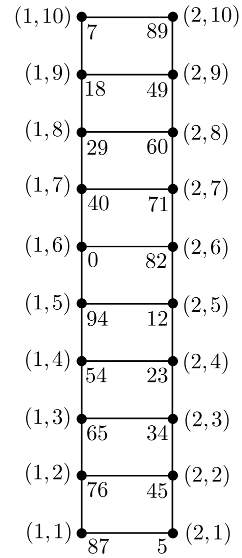
Notice from the labeling pattern that there are four possible distances between consecutively labeled vertices, namely $k+1$, $2k$, $2k-3$, and k, with the number of occurrences $2k$, 2, 1, and $2k-4$, respectively.

$$\begin{aligned}
\text{Hence, } span(f) &= (2n-1)(n+1)+2 - \sum_{i=1}^{2n-1} d(w_i, w_{i+1}) \\
&= (2n-1)(n+1)+2 - [(k+1)(2k)+(2k)(2)+(2k-3)(1)+(k)(2k-4)] \\
&= (2n-1)(n+1)+2-(4k^2+4k-3) \\
&= (2n-1)(n+1)+2-\left[4\left(\frac{n}{2}\right)^2+4\left(\frac{n}{2}\right)-3\right] \\
&= (2n-1)(n+1)+2-(n^2+2n-3) \\
&= n^2-n+4.
\end{aligned} \tag{45}$$

Therefore, $rn(G) \leq n^2-n+4$, since f is a radio labeling of G with this span.

□

$rn\,(M(2,10)) = 94$



Above is an optimal radio labeling of $M(2,10)$ following the labeling pattern described in the proof of Lemma 3.2. The radio number of $M(2,10)$ is $10^2 - 10 + 4 = 94$.

Theorem 3.3. *Let $G = M(2,n)$, where $n \geq 3$. Then*

$$rn(G) = \begin{cases} n^2 - n + 3 & \textit{if } n \textit{ is odd;} \\ n^2 - n + 4 & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. This result follows immediately from Lemmas 2.3, 2.6, 3.1 and 3.2. $\square$

4 Radio Number of Meshes $M(m,n)$ for $m,n \geq 3$ - Lower Bound

4.1 Lower Bound of $rn(M(m,n))$ for m Even and n Odd

Lemma 4.1. *Let $m = 2l$ and $n = 2k+1$, where $l \geq 2$ and $k \geq 1$. Let f be a radio labeling of $G = M(m,n)$, where $0 = f(v_1) < f(v_2) < ... < f(v_{mn}) = span(f)$ gives the ordering of the vertices of G. Then*

$$\sum_{i=1}^{mn-1} d(v_i, v_{i+1}) \leq \frac{1}{2}m(n^2-1) + \frac{1}{2}m^2 n - 1.$$

Proof. Suppose the assumption holds. We know that $d(v_i, v_j) \leq L(v_i) + L(v_j) + D(v_i) + D(v_{i+1})$ for all $v_i, v_j \in V(G)$, with equality holding only when v_i and v_j are on opposite sections and regions. This gives us the following inequality.

$$\sum_{i=1}^{mn-1} d(v_i, v_{i+1}) \leq \sum_{i=1}^{mn-1} [L(v_i) + L(v_{i+1}) + D(v_i) + D(v_{i+1})]$$

We make the following observations.

1. Each vertex level $L(v_i)$ and displacement $D(v_i)$ appears exactly twice in the above inequality except $L(v_1), L(v_{mn}), D(v_1)$, and $D(v_{mn})$, each of which appears only once.
2. For any $t \in \left\{1, 2, ..., \left(\frac{n-1}{2}\right)\right\}$, there exist exactly $2m$ vertices $v_i \in V(G)$ such that $L(v_i) = t$.
3. For any $s \in \left\{\frac{1}{2}, \frac{3}{2}, ..., \left(\frac{m-1}{2}\right)\right\}$, there exist exactly $2n$ vertices $v_i \in V(G)$ such that $D(v_i) = s$.

Hence, we have

$$
\begin{aligned}
\sum_{i=1}^{mn-1} d(v_i, v_{i+1}) &\le \sum_{i=1}^{mn-1} [L(v_i) + L(v_{i+1}) + D(v_i) + D(v_{i+1})] \\
&= 2\left[\sum_{i=1}^{mn} L(v_i) + \sum_{i=1}^{mn} D(v_i)\right] - [L(v_1) + L(v_{mn}) + D(v_1) + D(v_{mn})] \\
&= 2\left[2m\left(1 + 2 + ... + \frac{n-1}{2}\right)\right] + 2\left[2n\left(\frac{1}{2} + \frac{3}{2} + ... + \frac{m-1}{2}\right)\right] \\
&\quad - [L(v_1) + L(v_{mn}) + D(v_1) + D(v_{mn})] \\
&= 4m\left[1 + 2 + ... + \left(\frac{n-1}{2}\right)\right] + 2n[1 + 3 + ... + (m-1)] \\
&\quad - [L(v_1) + L(v_{mn}) + D(v_1) + D(v_{mn})] \\
&= 4m\left[\frac{1}{2}\left(\frac{n+1}{2}\right)\left(\frac{n-1}{2}\right)\right] + 2n\left[\frac{1}{2}\left(\frac{m}{2}\right)(m)\right] \\
&\quad - [L(v_1) + L(v_{mn}) + D(v_1) + D(v_{mn})] \\
&= \frac{1}{2}m(n^2-1) + \frac{1}{2}m^2 n - [L(v_1) + L(v_{mn}) + D(v_1) + D(v_{mn})].
\end{aligned}
\tag{46}
$$

Since m is even and n is odd, we have $L(v_1) + L(v_{mn}) + D(v_1) + D(v_{mn}) \ge 0 + 0 + \frac{1}{2} + \frac{1}{2} = 1$. Therefore, we have

$$\sum_{i=1}^{mn-1} [L(v_i) + L(v_{i+1}) + D(v_i) + D(v_{i+1})] \le \frac{1}{2}m(n^2-1) + \frac{1}{2}m^2 n - 1.$$

□

Lemma 4.2. *Let $m = 2l$ and $n = 2k+1$, where $l \ge 2$. Let $G = M(m,n)$. Then*

$$rn(G) \ge \frac{1}{2}(m^2 n + mn^2 - 2mn - m - 2n) + 2.$$

Proof. Let f be any radio labeling of $G = M(m,n)$, with $m = 2l$ and $n = 2k+1$ where $l \ge 2$. From Lemma 4.1, we have

$$\sum_{i=1}^{mn-1} d(v_i, v_{i+1}) \le \frac{1}{2}m(n^2-1) + \frac{1}{2}m^2 n - 1.$$

Hence, we have

$$
\begin{aligned}
span(f) &\geq (mn-1)(m+n-1) - \sum_{i=1}^{mn-1} d(v_i, v_{i+1}) \\
&\geq (mn-1)(m+n-1) - \left[\frac{1}{2}m(n^2-1) + \frac{1}{2}m^2n - 1\right] \\
&= m^2n + mn^2 - mn - m - n + 1 - \left[\frac{1}{2}m(n^2-1) + \frac{1}{2}m^2n - 1\right] \\
&= \frac{1}{2}(m^2n + mn^2 - 2mn - m - 2n) + 2.
\end{aligned} \tag{47}
$$

□

4.2 Lower Bound of $rn(M(m,n))$ for m,n Odd

Lemma 4.3. *Let $m = 2l+1$ and $n = 2k+1$. Let f be a radio labeling of $G = M(m,n)$, where $0 = f(v_1) < f(v_2) < ... < f(v_{mn}) = span(f)$ gives the ordering of the vertices of G. Then*

$$\sum_{i=1}^{mn-1} d(v_i, v_{i+1}) \leq \frac{1}{2}m(n^2-1) + \frac{1}{2}n(m^2-1) - 1.$$

Proof. Suppose the assumption holds. We know that $d(v_i, v_j) \leq L(v_i) + L(v_j) + D(v_i) + D(v_{i+1})$ for all $v_i, v_j \in V(G)$, with equality holding only when v_i and v_j are on opposite sections and regions. This gives us the following inequality.

$$\sum_{i=1}^{mn-1} d(v_i, v_{i+1}) \leq \sum_{i=1}^{mn-1} [L(v_i) + L(v_{i+1}) + D(v_i) + D(v_{i+1})]$$

We make the following observations.

1. Each vertex level $L(v_i)$ and displacement $D(v_i)$ appears exactly twice in the above inequality except $L(v_1), L(v_{mn}), D(v_1)$, and $D(v_{mn})$, each of which appears only once.
2. For any $t \in \left\{1, 2, ..., \left(\frac{n-1}{2}\right)\right\}$, there exist exactly $2m$ vertices $v_i \in V(G)$ such that $L(v_i) = t$.
3. For any $s \in \left\{1, 2, ..., \left(\frac{m-1}{2}\right)\right\}$, there exist exactly $2n$ vertices $v_i \in V(G)$ such that $L(v_i) = s$.

Hence, we have

$$\begin{aligned}
\sum_{i=1}^{mn-1} d(v_i, v_{i+1}) &\leq \sum_{i=1}^{mn-1} [L(v_i) + L(v_{i+1}) + D(v_i) + D(v_{i+1})] \\
&= 2\left[\sum_{i=1}^{mn} L(v_i) + \sum_{i=1}^{mn} D(v_i)\right] - [L(v_1) + L(v_{mn}) + D(v_1) + D(v_{mn})] \\
&= 2\left[2m\left(1 + 2 + ... + \frac{n-1}{2}\right)\right] + 2\left[2n\left(1 + 2 + ... + \frac{m-1}{2}\right)\right] \\
&\quad - [L(v_1) + L(v_{mn}) + D(v_1) + D(v_{mn})] \\
&= 4m\left[\frac{1}{2}\left(\frac{n+1}{2}\right)\left(\frac{n-1}{2}\right)\right] + 4n\left[\frac{1}{2}\left(\frac{m+1}{2}\right)\left(\frac{m-1}{2}\right)\right] \\
&\quad - [L(v_1) + L(v_{mn}) + D(v_1) + D(v_{mn})] \\
&= \frac{1}{2}m(n^2-1) + \frac{1}{2}n(m^2-1) - [L(v_1) + L(v_{mn}) + D(v_1) + D(v_{mn})].
\end{aligned} \tag{48}$$

Since m and n are both odd, we know that G has a unique center. Hence, only one vertex has a level and displacement both equal to zero. Since levels and displacements are both integral values when m and n are odd, we have $L(v_1) + L(v_{mn}) + D(v_1) + D(v_{mn}) \geq 1$. Therefore, we have

$$\sum_{i=1}^{mn-1} [L(v_i) + L(v_{i+1}) + D(v_i) + D(v_{i+1})] \leq \frac{1}{2}m(n^2-1) + \frac{1}{2}n(m^2-1) - 1.$$

□

Lemma 4.4. *Let $m = 2l + 1$ and $n = 2k + 1$. Let $G = M(m, n)$. Then*

$$rn(G) \geq \frac{1}{2}(m^2n + mn^2 - 2mn - m - n) + 2.$$

Proof. Let f be any radio labeling of $G = M(m, n)$, with $m = 2l + 1$ and $n = 2k + 1$. From Lemma 4.3, we have

$$\sum_{i=1}^{mn-1} d(v_i, v_{i+1}) \leq \frac{1}{2}m(n^2-1) + \frac{1}{2}n(m^2-1) - 1.$$

Hence, we have

$$
\begin{aligned}
span(f) &\geq (mn-1)(m+n-1) - \sum_{i=1}^{mn-1} d(v_i, v_{i+1}) \\
&\geq (mn-1)(m+n-1) - \left[\frac{1}{2}m(n^2-1) + \frac{1}{2}n(m^2-1) - 1\right] \\
&= m^2n + mn^2 - mn - m - n + 1 - \left[\frac{1}{2}m(n^2-1) + \frac{1}{2}n(m^2-1) - 1\right] \\
&= \frac{1}{2}(m^2n + mn^2 - 2mn - m - n) + 2.
\end{aligned} \tag{49}
$$

□

4.3 Lower Bound of $rn(M(m, n))$ for m, n Even

Lemma 4.5. *Let $m = 2l$ and $n = 2k$. Let f be a radio labeling of $G = M(m, n)$, where $0 = f(v_1) < f(v_2) < ... < f(v_{mn}) = span(f)$ gives the ordering of the vertices of G. Then*

$$\sum_{i=1}^{mn-1} d(v_i, v_{i+1}) \leq \frac{1}{2}mn^2 + \frac{1}{2}m^2n - 2.$$

Proof. Suppose the assumption holds. We know that $d(v_i, v_j) \leq L(v_i) + L(v_j) + D(v_i) + D(v_{i+1})$ for all $v_i, v_j \in V(G)$, with equality holding only when v_i and v_j are on opposite sections and regions. This gives us the following inequality.

$$\sum_{i=1}^{mn-1} d(v_i, v_{i+1}) \leq \sum_{i=1}^{mn-1} [L(v_i) + L(v_{i+1}) + D(v_i) + D(v_{i+1})]$$

We make the following observations.

1. Each vertex level $L(v_i)$ and displacement $D(v_i)$ appears exactly twice in the above inequality except $L(v_1), L(v_{mn}), D(v_1)$, and $D(v_{mn})$, each of which appears only once.
2. For any $t \in \left\{\frac{1}{2}, \frac{3}{2}, ..., \left(\frac{n-1}{2}\right)\right\}$, there exist exactly $2m$ vertices $v_i \in V(G)$ such that $L(v_i) = t$.
3. For any $s \in \left\{\frac{1}{2}, \frac{3}{2}, ..., \left(\frac{m-1}{2}\right)\right\}$, there exist exactly $2n$ vertices $v_i \in V(G)$ such that $D(v_i) = s$.

Hence, we have

$$\begin{aligned}
\sum_{i=1}^{mn-1} d(v_i, v_{i+1}) &\leq \sum_{i=1}^{mn-1} [L(v_i) + L(v_{i+1}) + D(v_i) + D(v_{i+1})] \\
&= 2\left[\sum_{i=1}^{mn} L(v_i) + \sum_{i=1}^{mn} D(v_i)\right] - [L(v_1) + L(v_{mn}) + D(v_1) + D(v_{mn})] \\
&= 2\left[2m\left(\frac{1}{2} + \frac{3}{2} + ... + \frac{n-1}{2}\right)\right] + 2\left[2n\left(\frac{1}{2} + \frac{3}{2} + ... + \frac{m-1}{2}\right)\right] \\
&\quad - [L(v_1) + L(v_{mn}) + D(v_1) + D(v_{mn})] \\
&= 2m[1 + 3 + ... + (n-1)] + 2n[1 + 3 + ... + (m-1)] \\
&\quad - [L(v_1) + L(v_{mn}) + D(v_1) + D(v_{mn})] \\
&= 2m\left[\frac{1}{2}\left(\frac{n}{2}\right)(n)\right] + 2n\left[\frac{1}{2}\left(\frac{m}{2}\right)(m)\right] - [L(v_1) + L(v_{mn}) + D(v_1) + D(v_{mn})] \\
&= \frac{1}{2}mn^2 + \frac{1}{2}m^2n - [L(v_1) + L(v_{mn}) + D(v_1) + D(v_{mn})].
\end{aligned} \tag{50}$$

Since m and n are both even, we have $L(v_1) + L(v_{mn}) + D(v_1) + D(v_{mn}) \geq \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 2$. Therefore, we have

$$\sum_{i=1}^{mn-1} [L(v_i) + L(v_{i+1}) + D(v_i) + D(v_{i+1})] \leq \frac{1}{2}mn^2 + \frac{1}{2}m^2n - 2.$$

□

Lemma 4.6. *Let $m = 2l$ and $n = 2k$. If v_j (where $2 \leq j \leq mn-1$) is a secluded vertex, then either $f(v_{j+1}) - f(v_j) > m + n - 1 - d(v_{j+1}, v_j)$ or $f(v_j) - f(v_{j-1}) > m + n - 1 - d(v_j, v_{j-1})$.*

Proof. Let $G = M(m, n)$, where $m = 2l$ and $n = 2k$. Let $v_j \in V(G)$ be a secluded vertex, where $2 \leq j \leq mn - 1$. Suppose to the contrary that $f(v_{j+1}) - f(v_j) = m + n - 1 - d(v_{j+1}, v_j)$ and

$f(v_j) - f(v_{j-1}) = m + n - 1 - d(v_j, v_{j-1})$. Then

$$\begin{aligned}
f(v_{j+1}) - f(v_{j-1}) &= 2(m+n-1) - [d(v_{j+1}, v_j) + d(v_j, v_{j-1})] \\
&= 2(m+n-1) - [L(v_{j+1}) + 2L(v_j) + L(v_{j-1}) + D(v_{j+1}) + 2D(v_j) + D(v_{j-1})] \\
&= 2(m+n-1) - \left[L(v_{j+1}) + 2\left(\frac{n-1}{2}\right) + L(v_{j-1}) + D(v_{j+1}) + 2\left(\frac{m-1}{2}\right) + D(v_{j-1})\right] \\
&= m + n - [L(v_{j+1}) + L(v_{j-1}) + D(v_{j+1}) + D(v_{j-1})] \\
&\leq m + n - [|L(v_{j+1}) - L(v_{j-1})| + |D(v_{j+1}) - D(v_{j-1})| + 2] \\
&\qquad \text{since } L(v_{j+1}) + L(v_{j-1}) \geq |L(v_{j+1}) - L(v_{j-1})| + 1 \\
&\qquad \text{and } D(v_{j+1}) + D(v_{j-1}) \geq |D(v_{j+1}) - D(v_{j-1})| + 1.
\end{aligned} \tag{51}$$

By our assumption, we know that v_{j+1} and v_{j-1} are both on the section and region opposite of v_j. Hence we know

$$\begin{aligned}
f(v_{j+1}) - f(v_{j-1}) &\geq m + n - 1 - d(v_{j+1}, v_{j-1}) \\
&= m + n - 1 - [|L(v_{j+1}) - L(v_{j-1})| + |D(v_{j+1}) - D(v_{j-1})|].
\end{aligned} \tag{52}$$

Hence, $m+n-2-[|L(v_{j+1}) - L(v_{j-1})| + |D(v_{j+1}) - D(v_{j-1})|] \geq m+n-1-[|L(v_{j+1}) - L(v_{j-1})| + |D(v_{j+1}) - D(v_{j-1})|]$, a contradiction. Therefore, either $f(v_{j+1}) - f(v_j) > m+n-1-d(v_{j+1}, v_j)$ or $f(v_j) - f(v_{j-1}) > m + n - 1 - d(v_j, v_{j-1})$. □

Recall from the previous section that for any radio labeling f on $G = M(m, n)$, $\alpha(f)$ is the number of indices $1 \leq i \leq mn - 1$ such that $f_i > (m + n - 1) - d_i$, so we have

$$span(f) \geq (mn-1)(m+n-1) + \alpha(f) - \sum_{i=1}^{mn-1} d(v_i, v_{i+1}).$$

Lemma 4.7. *Let $m = 2l$ and $n = 2k$, where $l \geq 2$ and $k \geq 2$. Let $G = M(m, n)$. Then*

$$rn(G) \geq \frac{1}{2}(m^2n + mn^2 - 2mn - 2m - 2n) + 6.$$

Proof. Let f be any radio labeling of $G = M(m, n)$, with $m = 2l$ and $n = 2k$ where $l \geq 2$ and $k \geq 2$. Without loss of generality, assume that $m \geq n$. From Lemma 4.5, we have

$$\sum_{i=1}^{mn-1} d(v_i, v_{i+1}) \leq \frac{1}{2}mn^2 + \frac{1}{2}m^2n - 2.$$

Notice that G has exactly 4 corners and therefore at most 4 secluded vertices. We examine different cases based on the number of secluded vertices in G to show that $span(f) \geq \frac{1}{2}(m^2n + mn^2 - 2mn - 2m - 2n) + 6$.

CASE 1: G has four secluded vertices $v_{t_1}, v_{t_2}, v_{t_3}$, and v_{t_4}, where $1 < t_1 < t_2 < t_3 < t_4 < mn$. Then, we have

$$\begin{cases} f_{t_1} > m+n-1-d_{t_1} \text{ or } f_{t_1-1} > m+n-1-d_{t_1-1} \\ f_{t_2} > m+n-1-d_{t_2} \text{ or } f_{t_2-1} > m+n-1-d_{t_2-1} \\ f_{t_3} > m+n-1-d_{t_3} \text{ or } f_{t_3-1} > m+n-1-d_{t_3-1} \\ f_{t_4} > m+n-1-d_{t_4} \text{ or } f_{t_4-1} > m+n-1-d_{t_4-1}. \end{cases} \tag{53}$$

If two secluded vertices are consecutively labeled, then two of the conditions can be concurrently satisfied in a single jump (the jump between the two said secluded vertices). However, no three secluded vertices may be consecutive, since any corner vertex whose predecessor and successor in the labeling pattern of f are both corners would necessarily induce a jump not of the best type. Since there are two pairs of secluded vertices, we can satisfy the first and second conditions above with a single jump, and likewise the third and fourth conditions with a single jump.

Hence, $\alpha(f) \geq 2$, with equality possible only if $t_1 + 1 = t_2$ and $t_3 + 1 = t_4$.

Observe that when m and n are both even, no vertex concurrently exists in both sections or both regions of G. Hence, at most $\frac{1}{2}mn$ consecutively labeled vertices can be labeled by alternating regions and sections in each jump; in other words, at most $\frac{1}{2}mn - 1$ consecutive jumps can be of the best type. Therefore, there exists an index $1 \leq j \leq mn - 1$ such that v_j and v_{j+1} are either in the same region or the same section, so the jump from v_j to v_{j+1} is not of the best type. Consequently,

$$\sum_{i=1}^{mn-1} d(v_i, v_{i+1}) \leq \left(\frac{1}{2}mn^2 + \frac{1}{2}m^2n - 2\right) - 1 = \frac{1}{2}mn^2 + \frac{1}{2}m^2n - 3.$$

So $span(f) \geq (mn-1)(m+n-1) + 2 - \left(\frac{1}{2}mn^2 + \frac{1}{2}m^2n - 3\right) = \frac{1}{2}(m^2n + mn^2 - 2mn - 2m - 2n) + 6$.

CASE 2: G has exactly three secluded vertices v_{t_1}, v_{t_2}, and v_{t_3}, where $1 < t_1 < t_2 < t_3 < mn$, and one non-secluded corner vertex v_c. Then we have

$$\begin{cases} f_{t_1} > m+n-1-d_{t_1} \text{ or } f_{t_1-1} > m+n-1-d_{t_1-1} \\ f_{t_2} > m+n-1-d_{t_2} \text{ or } f_{t_2-1} > m+n-1-d_{t_2-1} \\ f_{t_3} > m+n-1-d_{t_3} \text{ or } f_{t_3-1} > m+n-1-d_{t_3-1} \\ \text{Either the jump to } v_c \text{ or from } v_c \text{ is not of the best type, unless } v_c \in \{v_1, v_{mn}\}. \end{cases} \tag{54}$$

Hence, by the same argument from Case 1, $\alpha(f) \geq 2$ with equality possible only if $t_1 + 1 = t_2$ or $t_2 + 1 = t_3$.

Also, since v_c is a non-secluded corner vertex, we know that either v_c is in $\{v_1, v_{mn}\}$ (in which case $L(v_1) + L(v_{mn}) + D(v_1) + D(v_{mn}) \geq 4$) or at least one jump to or from v_c is not of the best type. In either situation, from Lemma 4.6 we have

$$\sum_{i=1}^{mn-1} d(v_i, v_{i+1}) \leq \left(\frac{1}{2}mn^2 + \frac{1}{2}m^2n - 2\right) - 1 = \frac{1}{2}mn^2 + \frac{1}{2}m^2n - 3.$$

So $span(f) \geq (mn-1)(m+n-1) + 2 - \left(\frac{1}{2}mn^2 + \frac{1}{2}m^2n - 3\right) = \frac{1}{2}(m^2n + mn^2 - 2mn - 2m - 2n) + 6$.

CASE 3: G has exactly two secluded vertices v_{t_1} and v_{t_2}, where $1 < t_1 < t_2 < mn$, and two non-secluded corner vertices v_{c_1} and v_{c_2}, where $c_1 < c_2$. Then, we have

$$\begin{cases} f_{t_1} > m+n-1-d_{t_1} \text{ or } f_{t_1-1} > m+n-1-d_{t_1-1} \\ f_{t_2} > m+n-1-d_{t_2} \text{ or } f_{t_2-1} > m+n-1-d_{t_2-1} \\ \text{Either the jump to } v_{c_1} \text{ or from } v_{c_1} \text{ is not of the best type, unless } v_{c_1} \in \{v_1, v_{mn}\}. \\ \text{Either the jump to } v_{c_2} \text{ or from } v_{c_2} \text{ is not of the best type, unless } v_{c_2} \in \{v_1, v_{mn}\}. \end{cases} \tag{55}$$

Hence, $\alpha(f) \geq 1$ with equality possible only if $t_1 + 1 = t_2$.

Also, we know that each non-secluded corner vertex v_{c_1} and v_{c_2} either is in $\{v_1, v_{mn}\}$ or necessarily induces a jump not of the best type. Note that since $m, n \geq 4$, we have that any corner vertex has a level and displacement of at least $\frac{3}{2}$. Therefore, any corner vertex in $\{v_1, v_{mn}\}$ increases $L(v_1) + L(v_{mn}) + D(v_1) + D(v_{mn})$ by at least 2 and therefore sufficiently minimizes the sum of the distances between consecutive vertices (as in Case 2), so we examine the different possibilities if neither of these corner vertices is in $\{v_1, v_{mn}\}$.

1. If v_{c_1} and v_{c_2} are not consecutive, then they each induce a distinct jump not of the best type, since they are non-secluded.
2. If v_{c_1} and v_{c_2} are consecutive and antipodal (i.e. in different regions and sections), then the jump from v_{c_1} to v_{c_2} is of the best type. But both these vertices are non-secluded, so both these vertices induce a distinct jump that is not of the best type.
3. If v_{c_1} and v_{c_2} are consecutive but not antipodal, then the jump from v_{c_1} to v_{c_2} concurrently satisfies the third and fourth conditions of this case; however, that would force t_2 to be strictly larger than $t_1 + 1$ (since secluded vertices v_{t_1} and v_{t_2} cannot be in the same region or section if they are consecutive), which would make the preceding inequality strict.

Our observations above lead to the conclusion that in Case 3, one of the two following statements must hold.

$$\sum_{i=1}^{mn-1} d(v_i, v_{i+1}) \leq \left(\frac{1}{2}mn^2 + \frac{1}{2}m^2n - 2\right) - 2 = \frac{1}{2}mn^2 + \frac{1}{2}m^2n - 4 \quad \text{and} \quad \alpha(f) \geq 1$$

OR

$$\sum_{i=1}^{mn-1} d(v_i, v_{i+1}) \leq \left(\frac{1}{2}mn^2 + \frac{1}{2}m^2n - 2\right) - 1 = \frac{1}{2}mn^2 + \frac{1}{2}m^2n - 3 \quad \text{and} \quad \alpha(f) \geq 2.$$

In either case, $span(f) \geq \frac{1}{2}(m^2n + mn^2 - 2mn - 2m - 2n) + 6$.

CASE 4: G has exactly one secluded vertex v_t, where $1 < t < mn$, and three non-secluded corner vertices v_{c_1}, v_{c_2}, and v_{c_3}, where $c_1 < c_2 < c_3$. Then, we have

$$\begin{cases} f_t > m + n - 1 - d_t \text{ or } f_{t-1} > m + n - 1 - d_{t-1} \\ \text{Either the jump to } v_{c_1} \text{ or from } v_{c_1} \text{ is not of the best type, unless } v_{c_1} \in \{v_1, v_{mn}\}. \\ \text{Either the jump to } v_{c_2} \text{ or from } v_{c_2} \text{ is not of the best type, unless } v_{c_2} \in \{v_1, v_{mn}\}. \\ \text{Either the jump to } v_{c_3} \text{ or from } v_{c_3} \text{ is not of the best type, unless } v_{c_3} \in \{v_1, v_{mn}\}. \end{cases} \tag{56}$$

Hence, from the first condition we know $\alpha(f) \geq 1$.

Also, unless one of the three non-secluded vertices v_{c_1}, v_{c_2}, and v_{c_3} is in $\{v_1, v_{mn}\}$ (in which case $L(v_1) + L(v_{mn}) + D(v_1) + D(v_{mn}) \geq 4$), it is impossible for v_{c_1}, v_{c_2}, and v_{c_3} to collectively induce only one jump not of the best type, since a single jump can only satisfy at most two of the above conditions. Therefore, we have

$$\sum_{i=1}^{mn-1} d(v_i, v_{i+1}) \leq \left(\frac{1}{2}mn^2 + \frac{1}{2}m^2n - 2\right) - 2 = \frac{1}{2}mn^2 + \frac{1}{2}m^2n - 4.$$

So $span(f) \geq \frac{1}{2}(m^2n + mn^2 - 2mn - 2m - 2n) + 6$.

CASE 5: G has four non-secluded corner vertices v_{c_1}, v_{c_2}, v_{c_3}, and v_{c_4} with $c_1 < c_2 < c_3 < c_4$. Then

$$\begin{cases} \text{Either the jump to } v_{c_1} \text{ or from } v_{c_1} \text{ is not of the best type, unless } v_{c_1} \in \{v_1, v_{mn}\}. \\ \text{Either the jump to } v_{c_2} \text{ or from } v_{c_2} \text{ is not of the best type, unless } v_{c_2} \in \{v_1, v_{mn}\}. \\ \text{Either the jump to } v_{c_3} \text{ or from } v_{c_3} \text{ is not of the best type, unless } v_{c_3} \in \{v_1, v_{mn}\}. \\ \text{Either the jump to } v_{c_4} \text{ or from } v_{c_4} \text{ is not of the best type, unless } v_{c_4} \in \{v_1, v_{mn}\}. \end{cases} \tag{57}$$

We examine the different possibilities for the non-secluded corner vertices v_{c_1}, v_{c_2}, v_{c_3}, and v_{c_4}.

1. If two of the four corner vertices are in $\{v_1, v_{mn}\}$, then $L(v_1)+L(v_{mn})+D(v_1)+D(v_{mn}) \geq 6$ and the remaining two corner vertices would induce a jump not of the best type.
2. If only one of the four corner vertices is in $\{v_1, v_{mn}\}$, then $L(v_1) + L(v_{mn}) + D(v_1) + D(v_{mn}) \geq 4$, and the remaining three corner vertices would induce at least two distinct jumps not of the best type, as was determined in Case 4.
3. If none of the corner vertices are in $\{v_1, v_{mn}\}$, then the four corner vertices would induce at least two distinct jumps not of the best type. However, if $L(v_1) + L(v_{mn}) + D(v_1) + D(v_{mn}) = 2$ and the four corners induce *exactly* 2 jumps not of the best type (in other words, if $c_4 = c_3 + 1$ and $c_2 = c_1 + 1$), then we consider all of the following observations:
   (a) The two jumps not of the best type must be between corner vertices within the same section or region. If the jumps are between corner vertices of the same region, then each jump has a distance $n - 1$. But this is a smaller jump than a jump of length $m-1$ between corner vertices in the same section (since $m \geq n$). Such a jump between corner vertices in the same region therefore further reduces the sum of the distances between consecutive vertices. To avoid this, we set the two jumps that are not of the best type to be between corners in the same section.
   (b) If $c_3 = c_2 + 1$ (so the jump from v_{c_2} to v_{c_3} is of the best type) where v_{c_1} and v_{c_2} are

in one section and v_{c_3} and v_{c_4} are in the other, then v_{c_1} and v_{c_4} will be in opposite regions and sections, which would force an additional jump not of the best type in order to label all vertices. So $c_3 \neq c_2 + 1$, which prevents labeling v_{c_3} immediately after v_{c_2}.

(c) The labeling pattern must begin and end at central vertices. This forces an additional jump not of the best type unless v_1 and v_{mn} are on opposite sections and regions (since no vertices are in more than one section or region). But f already includes two jumps between vertices in the same section, since $c_2 = c_1 + 1$ and $c_4 = c_3 + 1$. So v_1 and v_{mn} must be in opposite regions and sections to avoid another jump not of the best type.

(d) If the two jumps that are not of the best type are between corners in the same section as indicated in (a), then v_{c_4+1} and v_{c_3} are in the same region but opposite sections.

Our observations above indicate that in Case 5, one of the two following statements must hold.

$$\sum_{i=1}^{mn-1} d(v_i, v_{i+1}) \leq \left(\frac{1}{2}mn^2 + \frac{1}{2}m^2n - 2\right) - 3 = \frac{1}{2}mn^2 + \frac{1}{2}m^2n - 5$$

OR

$$\sum_{i=1}^{mn-1} d(v_i, v_{i+1}) \leq \left(\frac{1}{2}mn^2 + \frac{1}{2}m^2n - 2\right) - 2 = \frac{1}{2}mn^2 + \frac{1}{2}m^2n - 4 \quad \text{and} \quad \alpha(f) \geq 1.$$

So $span(f) \geq \frac{1}{2}(m^2n + mn^2 - 2mn - 2m - 2n) + 6$.

Hence, in all 5 cases, $span(f) \geq \frac{1}{2}(m^2n + mn^2 - 2mn - 2m - 2n) + 6$ when m, n are even. □

5 Radio Number of Meshes $M(m,n)$ for $m, n \geq 3$ - Upper Bound

5.1 Upper Bound of $rn(M(m,n))$ for m Even and n Odd

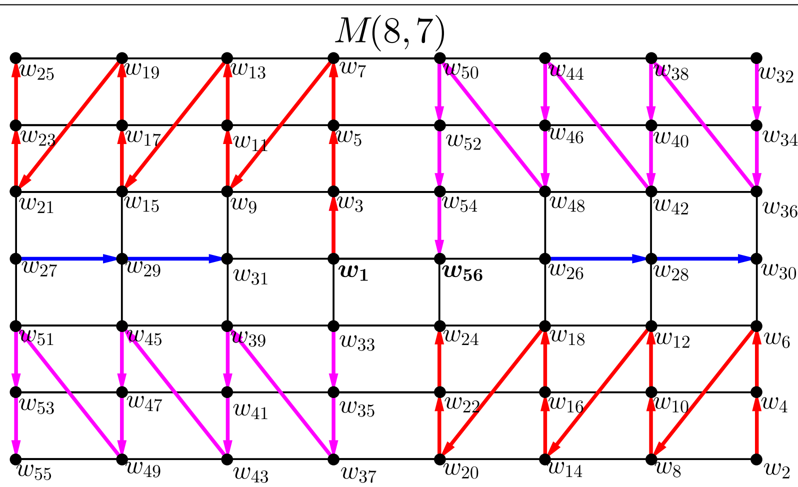
Lemma 5.1. *Let $m = 2l$ and $n = 2k+1$, where $l \geq 2$ and $k \geq 1$. Let $G = M(m,n)$. Then $rn(G) \leq \frac{1}{2}(m^2n + mn^2 - 2mn - m - 2n) + 2$.*

Proof. Let $m = 2l$ and $n = 2k+1$, where $l \geq 2$ and $k \geq 1$. It suffices to find one radio labeling of $G = M(m,n)$ with a span of $\frac{1}{2}(m^2n + mn^2 - 2mn - m - 2n) + 2$. Let $\{w_1, w_2, ..., w_{mn}\}$ be a permutation of $V(G)$ given by the following pattern, which we separate into 3 blocks.

$$w_1 = (l,k+1) \xrightarrow{l+k} (m,1) \xrightarrow{l+k+1} (l,k+2) \xrightarrow{l+k} (m,2) \xrightarrow{l+k+1} (l,k+3) \xrightarrow{l+k} (m,3) \xrightarrow{l+k+1}$$
$$(l,k+4) \xrightarrow{l+k} \dots \xrightarrow{l+k} w_{n-1} = (m,k) \xrightarrow{l+k+1} (l,n) \xrightarrow{2k+l-1} (m-1,1) \xrightarrow{l+k+1} (l-1,k+2) \xrightarrow{l+k}$$
$$(m-1,2) \xrightarrow{l+k+1} (l-1,k+3) \xrightarrow{l+k} (m-1,3) \xrightarrow{l+k+1} (l-1,k+4) \xrightarrow{l+k} \dots \xrightarrow{l+k} w_{4k} =$$
$$(m-1,k) \xrightarrow{l+k+1} (l-1,n) \xrightarrow{2k+l-1} \dots \dots \dots \xrightarrow{2k+l-1} (l+1,1) \xrightarrow{l+k+1} (1,k+2) \xrightarrow{l+k} (l+1,2) \xrightarrow{l+k+1}$$
$$(1,k+3) \xrightarrow{l+k} (l+1,3) \xrightarrow{l+k+1} (1,k+4) \xrightarrow{l+k} \dots \xrightarrow{l+k} (l+1,k) \xrightarrow{l+k+1} (1,n) = w_{mk+1}$$

$$\xrightarrow{l+k+1} w_{mk+2} = (l+2,k+1) \xrightarrow{l+1} (1,k+1) \xrightarrow{l+2} (l+2,k+1) \xrightarrow{l+1} (2,k+1) \xrightarrow{l+2} (l+3,k+1) \xrightarrow{l+1}$$
$$(3,k+1) \xrightarrow{l+2} \dots \xrightarrow{l+2} (2l,k+1) \xrightarrow{l+1} (l-1,k+1) = w_{2l(k+1)-1}$$

$$\xrightarrow{l+k+1} w_{m(k+1)} = (m,n) \xrightarrow{l+k+1} (l,k) \xrightarrow{l+k} (m,n-1) \xrightarrow{l+k+1} (l,k-1) \xrightarrow{l+k} (m,n-2) \xrightarrow{l+k+1}$$
$$(l,k-2) \xrightarrow{l+k} \dots \xrightarrow{l+k} w_{2(l+1)(k+1)-4} = (m,k+2) \xrightarrow{l+k+1} (l,1) \xrightarrow{2k+l-1} (m-1,n) \xrightarrow{l+k+1}$$
$$(l-1,k) \xrightarrow{l+k} (m-1,n-1) \xrightarrow{l+k+1} (l-1,k-1) \xrightarrow{l+k} (m-1,n-2) \xrightarrow{l+k+1} (l-1,k-2) \xrightarrow{l+k} \dots \xrightarrow{l+k}$$
$$w_{2(l+1)(k+1)+2k-4} = (m-1,k+2) \xrightarrow{l+k+1} (l-1,1) \xrightarrow{2k+l-1} \dots \dots \dots \xrightarrow{2k+l-1} (l+1,n) \xrightarrow{l+k+1}$$
$$(1,k) \xrightarrow{l+k} (l+1,n-1) \xrightarrow{l+k+1} (1,k-1) \xrightarrow{l+k} (l+1,n-2) \xrightarrow{l+k+1} (1,k-2) \xrightarrow{l+k} \dots \xrightarrow{l+k}$$
$$(l+1,k+2) \xrightarrow{l+k+1} (1,1) \xrightarrow{l+k} (l+1,k+1) = w_{mn}$$



Above is the labeling pattern described for $M(8,7)$. The red and purple vectors indicate subsequences of vertices whose indices are of equal parity.

Let the quantity above each arrow indicate the distance between the two consecutive vertices. Let $f : V(G) \to \{0,1,2,...\}$ be a function such that $f(w_1) = 0$ and $f(w_{i+1}) - f(w_i) = m + n - 1 - d(w_{i+1}, w_i)$ for all $1 \le i \le mn - 1$.

CLAIM: f is a radio labeling of G.

To prove this, we show that for all $1 \leq i \leq mn - 2$, we have

$$f(w_j) - f(w_i) \geq m + n - 1 - d(w_j, w_i) \text{ for any } j \geq i + 2.$$

By the same justification in the previous section, we have

$$f(w_j) - f(w_i) = f_{j-1} + f_{j-2} + ... + f_{i+1} + f_i.$$

Let $j = i + p$, where $p \geq 2$. We observe 2 cases, using distances indicated in the labeling pattern for f for reference.

CASE 1: $p = 2$.

For convenience we first define the following disjoint subsets of $V(G)$:

- $A_1 = \{(t, k) : l + 2 \leq t \leq 2l\}$
- $A_2 = \{(t, 2k + 1) : 2 \leq t \leq l\}$
- $A_3 = \{(t, k + 2) : l + 2 \leq t \leq 2l\}$
- $A_4 = \{(t, 1) : 2 \leq t \leq l\}$
- $A_5 = \{(l + 1, k), (l - 1, k + 1)\}$
- $A_6 = \{(1, 2k + 1), (l + 1, k + 1)\}$
- $A_7 = \{(t, k + 1) : 1 \leq t \leq l - 2 \text{ or } l + 2 \leq t \leq 2l - 1\}$

Subcase 1a: $w_i \in \bigcup_{t=1}^{4} A_t$. Then according to the labeling pattern, $d(w_j, w_i) = k$.

$$\begin{aligned} \implies f(w_j) - f(w_i) &= 2(m + n - 1) - d(w_i, w_{i+1}) - d(w_{i+1}, w_{i+2}) \\ &= (m + n - 1) + (2l + 2k) - (l + k + 1) - (2k + l - 1) \\ &= (m + n - 1) - k \\ &= (m + n - 1) - d(w_j, w_i). \end{aligned} \tag{58}$$

Subcase 1b: $w_i \in A_5$. Then according to the labeling pattern, $d(w_j, w_i) = 2$.

$$\begin{aligned} \implies f(w_j) - f(w_i) &= 2(m + n - 1) - d(w_i, w_{i+1}) - d(w_{i+1}, w_{i+2}) \\ &= (m + n - 1) + (2l + 2k) - (l + k + 1) - (l + k + 1) \\ &= (m + n - 1) - 2 \\ &= (m + n - 1) - d(w_j, w_i). \end{aligned} \tag{59}$$

Subcase 1c: $w_i \in A_6$. Then according to the labeling pattern, $d(w_j, w_i) = k$.

$$\begin{aligned}\implies f(w_j) - f(w_i) &= 2(m+n-1) - d(w_i, w_{i+1}) - d(w_{i+1}, w_{i+2}) \\ &= (m+n-1) + (2l+2k) - (l+k+1) - (l+1) \\ &= (m+n-1) + k - 2 \\ &\geq (m+n-1) - 1, \text{since } k \geq 1 \\ &\geq (m+n-1) - d(w_j, w_i).\end{aligned} \tag{60}$$

Subcase 1d: $w_i \in A_7$. Then according to the labeling pattern, $d(w_j, w_i) = 1$.

$$\begin{aligned}\implies f(w_j) - f(w_i) &= 2(m+n-1) - d(w_i, w_{i+1}) - d(w_{i+1}, w_{i+2}) \\ &= (m+n-1) + (2l+2k) - (l+1) - (l+2) \\ &= (m+n-1) + 2k - 3 \\ &\geq (m+n-1) - 1, \text{since } k \geq 1 \\ &= (m+n-1) - d(w_j, w_i).\end{aligned} \tag{61}$$

Subcase 1e: $w_i \notin \bigcup_{t=1}^{7} A_t$. Then according to the labeling pattern, $d(w_j, w_i) = 1$.

$$\begin{aligned}\implies f(w_j) - f(w_i) &= 2(m+n-1) - d(w_i, w_{i+1}) - d(w_{i+1}, w_{i+2}) \\ &= (m+n-1) + (2l+2k) - (l+k+1) - (l+k) \\ &= (m+n-1) - 1 \\ &= (m+n-1) - d(w_j, w_i).\end{aligned} \tag{62}$$

Hence, $f(w_j) - f(w_i) \geq m+n-1-d(w_j, w_i)$ whenever $j = i+2$.

CASE 2: $p \geq 3$.

When $p \geq 3$, we can make the following observations about the distances of jumps occurring between w_i and w_j.

1. The jumps alternate between lengths of $l+k+1$ and $l+k$ if the jumps are not to or from vertices on the horizontal axis, the bottom row of Quadrant 3, or the top row of Quadrant 2.
2. The jumps within Block 2 alternate between lengths of $l+1$ and $l+2$. Note that $l+1 \leq l+k$ and $l+2 \leq l+k+1$.

3. The transitioning jumps from Block 1 to Block 2 and from Block 2 to Block 3 are the only two jumps of length $l+k+1$ within the labeling pattern that immediately follow another jump of length $l+k+1$. But since $p \geq 3$ and no three consecutive jumps are of length $l+k+1$, it is true that at most $\lceil \frac{p}{2} \rceil$ jumps are of length $l+k+1$.
4. At most $\lceil \frac{p}{2k} \rceil$ jumps are of length $2k+l-1$. These jumps occur between jumps of length $l+k+1$, so they replace the usual jump of length $l+k$ that is between jumps of length $l+k+1$.

Therefore, we know $\displaystyle\sum_{t=i}^{j-1} d(w_t, w_{t+1}) \leq p(l+k+1) - \left\lceil \frac{p}{2} \right\rceil + \left\lceil \frac{p}{2k} \right\rceil [(2k+l-1)-(l+k)]$. (63)

$$= p(l+k+1) - \left\lceil \frac{p}{2} \right\rceil + \left\lceil \frac{p}{2k} \right\rceil (k-1).$$

$$\begin{aligned}
\implies f(w_j) - f(w_i) &= p(m+n-1) - \sum_{t=i}^{j-1} d(w_t, w_{t+1}) \\
&= m+n-1+(p-1)(2k+2l) - \sum_{t=i}^{j-1} d(w_t, w_{t+1}) \\
&\geq m+n-1+2pl+2pk-2l-2k - \left[p(l+k+1) - \left\lceil \frac{p}{2} \right\rceil + \left\lceil \frac{p}{2k} \right\rceil (k-1) \right] \\
&= m+n-1+pl+pk-2l-2k-p+\left\lceil \frac{p}{2} \right\rceil - \left\lceil \frac{p}{2k} \right\rceil k + \left\lceil \frac{p}{2k} \right\rceil \\
&= m+n-1+(p-2)l-2k+\left(p-\left\lceil \frac{p}{2k} \right\rceil\right)(k-1)+\left\lceil \frac{p}{2} \right\rceil \\
&\geq m+n-1+(p-2)l-2k+\left(p-\left\lceil \frac{p}{2k} \right\rceil\right)(k-1)+2 \quad \text{since } p \geq 3 \\
&= m+n-1+(p-2)l-2(k-1)+\left(p-\left\lceil \frac{p}{2k} \right\rceil\right)(k-1) \\
&> m+n-1+\left(p-\left\lceil \frac{p}{2k} \right\rceil - 2\right)(k-1) \quad \text{since } p \geq 3.
\end{aligned} \tag{64}$$

CLAIM: $\left(p - \lceil \frac{p}{2k} \rceil - 2\right)(k-1) \geq 0$ for all $k \geq 1$.

To prove this claim, we test all integer values of $k \geq 1$.

Case 1: $k=1$. Then $k-1=0$, so $\left(p - \lceil \frac{p}{2k} \rceil - 2\right)(k-1) = 0$.

Case 2: $k \geq 2$.

If $p=3$, then $\left(p - \lceil \frac{p}{2k} \rceil - 2\right) = \left(3 - \lceil \frac{3}{2k} \rceil - 2\right) \geq \left(3 - \lceil \frac{3}{4} \rceil - 2\right) = (3-1-2) = 0$

$$\implies \left(p - \left\lceil \frac{p}{2k} \right\rceil - 2\right)(k-1) \geq \left(p - \left\lceil \frac{p}{2k} \right\rceil - 2\right)(1) = 0, \quad \text{since } k \geq 2. \tag{65}$$

If $p \geq 4$, then $\left(p - \lceil \frac{p}{2k} \rceil - 2\right) \geq \left(p - \lceil \frac{p}{4} \rceil - 2\right) \geq \left(p - \left(\frac{p}{4} + 1\right) - 2\right) = \frac{3p}{4} - 3 \geq \frac{12}{4} - 3 = 0$

$$\implies \left(p - \left\lceil \frac{p}{2k} \right\rceil - 2\right)(k-1) \geq \left(p - \left\lceil \frac{p}{2k} \right\rceil - 2\right)(1) \geq 0, \quad \text{since } k \geq 2. \tag{66}$$

Using this claim, we conclude that

$$\begin{aligned} f(w_j) - f(w_i) &> m + n - 1 + \left(p - \left\lceil \frac{p}{2k} \right\rceil - 2\right)(k-1) \\ &\geq m + n - 1 \\ &> m + n - 1 - d(w_j, w_i). \end{aligned} \tag{67}$$

By Cases 1 and 2, we know that f is a radio labeling of G, proving our claim.

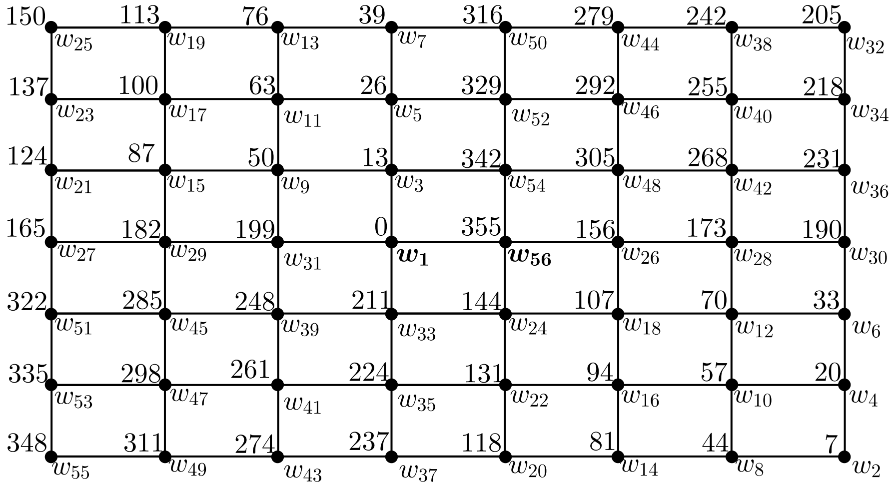
Notice from the labeling pattern that there are five possible distances between consecutively labeled vertices, namely $l+k+1$, $l+k$, $2k+l-1$, $l+1$, and $l+2$, with the number of occurrences $2lk+2$, $2lk-2l+2$, $2l-2$, $l-1$, and $l-2$, respectively.

$$\begin{aligned} \text{Hence, } span(f) &= (mn-1)(m+n-1) - \sum_{i=1}^{mn-1} d(w_i, w_{i+1}) \\ &= (mn-1)(m+n-1) - [(l+k+1)(2lk+2) + (l+k)(2lk-2l+2) + (2k+l-1)(2l-2) \\ &\qquad + (l+1)(l-1) + (l+2)(l-2)] \\ &= (mn-1)(m+n-1) - (4l^2k + 4lk^2 + 4lk + 2l^2 - 1) \\ &= (mn-1)(m+n-1) - \left[m^2\left(\frac{n-1}{2}\right) + 2m\left(\frac{n-1}{2}\right)^2 + 2m\left(\frac{n-1}{2}\right) + \frac{m^2}{2} - 1\right] \\ &= \frac{1}{2}m^2n + \frac{1}{2}mn^2 - \frac{1}{2}m - mn - n + 2 \\ &= \frac{1}{2}(m^2n + mn^2 - 2mn - m - 2n) + 2. \end{aligned} \tag{68}$$

Therefore, $rn(G) \leq \frac{1}{2}(m^2n + mn^2 - 2mn - m - 2n) + 2$, since f is a radio labeling of G with this span.

□

$rn(M(8,7)) = 355$



Above is the optimal radio labeling determined by the labeling pattern described above for $M(8,7)$. The radio number of $M(8,7)$ is 355.

5.2 Upper Bound of $rn(M(m,n))$ for m, n Odd

Lemma 5.2. *Let $m = 2l + 1$ and $n = 2k + 1$. Let $G = M(m, n)$. Then $rn(G) \leq \frac{1}{2}(m^2n + mn^2 - 2mn - m - n) + 2$.*

Proof. Let $m = 2l + 1$ and $n = 2k + 1$. Since $G(m, n)$ is isomorphic to $G(n, m)$ and m, n have the same parity, we may assume without loss of generality that $l \geq k$. Therefore, G has at least as many columns as it does rows.

When m, n are both odd, a complication with calculations requires that we separate the trivial case of $l = 1$ from the more general case of $l \geq 2$. If $l = 1$, then by assumption $k = 1$, and so $G = M(m, n)$ is a square mesh with 3 rows and 3 columns. We define a function $g : V(G) \to \{0, 1, 2, ...\}$ by the following.

$$\begin{array}{lll} (1,3) \mapsto 15 & (2,3) \mapsto 5 & (3,3) \mapsto 11 \\ (1,2) \mapsto 9 & (2,2) \mapsto 0 & (3,2) \mapsto 17 \\ (1,1) \mapsto 3 & (2,1) \mapsto 13 & (3,1) \mapsto 7 \end{array}$$

From simple calculation it is easy to verify that g as defined above is a radio labeling of $G = M(3, 3)$ with a span of 17. Since $17 = \frac{1}{2}\left[(3)^2(3) + (3)(3)^2 - 2(3)(3) - 3 - 3\right] + 2$, the desired upper bound of $rn(G)$ is achieved when $l = 1$.

To prove the upper bound when $l \geq 2$, it suffices to find one radio labeling of $G = M(m, n)$ with a span of $\frac{1}{2}(m^2n + mn^2 - 2mn - m - n) + 2$. Let $\{w_1, w_2, ..., w_{mn}\}$ be a permutation of $V(G)$ given by the following pattern, which we separate into l blocks. Notice that vertices in this labeling pattern alternate quadrants in diagonal motions, unlike the patterns used in previous cases.

$w_1 = (l + 1, k + 1) \xrightarrow{l+k} (1, 1) \xrightarrow{l+k+1} (l + 1, k + 2) \xrightarrow{l+k+1} (2l + 1, 1) \xrightarrow{l+k+1} (l, k + 1) \xrightarrow{l+k+1} (2l + 1, 2k + 1) \xrightarrow{l+k+1} (l + 1, k) \xrightarrow{l+k+1} w_8 = (1, 2k + 1)$

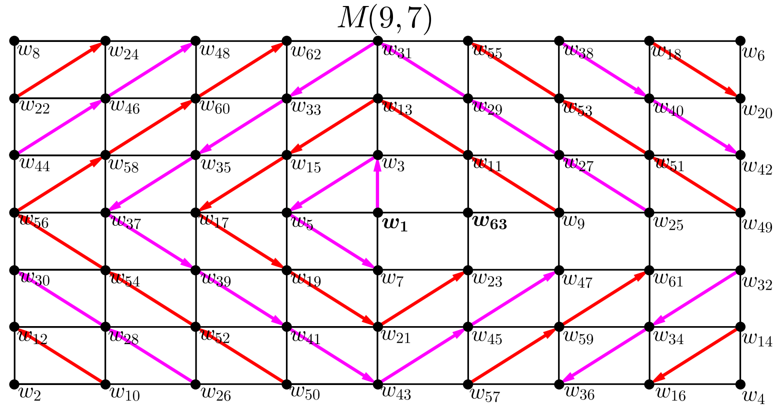
$\xrightarrow{l+k+2} w_9 = (l+3, k+1) \xrightarrow{l+k+1} (2, 1) \xrightarrow{l+k+1} (l+2, k+2) \xrightarrow{l+k+1} (1, 2) \xrightarrow{l+k+1} (l+1, k+3) \xrightarrow{l+k+1} (2l + 1, 2) \xrightarrow{l+k+1} (l, k + 2) \xrightarrow{l+k+1} (2l, 1) \xrightarrow{l+k+1} (l - 1, k + 1) \xrightarrow{l+k+1} (2l, 2k + 1) \xrightarrow{l+k+1} (l, k) \xrightarrow{l+k+1} (2l + 1, 2k) \xrightarrow{l+k+1} (l + 1, k - 1) \xrightarrow{l+k+1} (1, 2k) \xrightarrow{l+k+1} (l + 2, k) \xrightarrow{l+k+1} (2, 2k + 1) = w_{24}$

$\xrightarrow{l+k+2} w_{25} = (l+4, k+1) \xrightarrow{l+k+1} (3,1) \xrightarrow{l+k+1} (l+3, k+2) \xrightarrow{l+k+1} (2,2) \xrightarrow{l+k+1} (l+2, k+3) \xrightarrow{l+k+1} (1,3) \xrightarrow{l+k+1} (l+1, k+4) \xrightarrow{l+k+1} (2l+1, 3) \xrightarrow{l+k+1} (l, k+3) \xrightarrow{l+k+1} (2l, 2) \xrightarrow{l+k+1} (l-1, k+2) \xrightarrow{l+k+1} (2l-1, 1) \xrightarrow{l+k+1} (l-2, k+1) \xrightarrow{l+k+1} (2l-1, 2k+1) \xrightarrow{l+k+1} (l-1, k) \xrightarrow{l+k+1} (2l, 2k) \xrightarrow{l+k+1} (l, k-1) \xrightarrow{l+k+1} (2l+1, 2k-1) \xrightarrow{l+k+1} (l+1, k-2) \xrightarrow{l+k+1} (1, 2k-1) \xrightarrow{l+k+1} (l+2, k-1) \xrightarrow{l+k+1} (2, 2k) \xrightarrow{l+k+1} (l+3, k) \xrightarrow{l+k+1} (3, 2k+1) = w_{48}$

$\vdots$

$\xrightarrow{l+k+2} w_{49} = (2l+1, k+1) \xrightarrow{l+k+1} (l,1) \xrightarrow{l+k+1} (2l, k+2) \xrightarrow{l+k+1} (l-1, 2) \xrightarrow{l+k+1} (2l-1, k+3) \xrightarrow{l+k+1} (l-2, 3) \xrightarrow{l+k+1} \dots \xrightarrow{l+k+1} (2l+1-k, 2k+1) \xrightarrow{l+k+1} (l-k, k+1) \xrightarrow{l+k+1} (2l+1-k, 1) \xrightarrow{l+k+1} (l+1-k, k+2) \xrightarrow{l+k+1} (2l-k+2, 2) \xrightarrow{l+k+1} (l-k+2, k+3) \xrightarrow{l+k+1} \dots \xrightarrow{l+k+1} (2l, k) \xrightarrow{l+k+1} (l, 2k+1) \xrightarrow{k+2} (l+2, k+1) = w_{mn}$

$M(9,7)$



Above is the labeling pattern described for $M(9,7)$. The red and purple vectors indicate subsequences of vertices whose indices are of equal parity.

Let the quantity above each arrow indicate the distance between the two consecutive vertices. Let $f : V(G) \to \{0, 1, 2, ...\}$ be a function such that $f(w_1) = 0$ and $f(w_{i+1}) - f(w_i) = m + n - 1 - d(w_{i+1}, w_i)$ for all $1 \le i \le mn - 1$.

CLAIM: f is a radio labeling of G.

To prove this, we show that for all $1 \le i \le mn-2$, we have

$$f(w_j) - f(w_i) \ge m + n - 1 - d(w_j, w_i) \text{ for any } j \ge i + 2.$$

By the same justification in the previous section, we have

$$f(w_j) - f(w_i) = f_{j-1} + f_{j-2} + ... + f_{i+1} + f_i.$$

Let $j = i + p$, where $p \geq 2$. We observe 2 cases, using distances indicated in the labeling pattern for f for reference.

CASE 1: $p = 2$.

For convenience we first define the following disjoint subsets of $V(G)$:

- $A_1 = \{w_1\} = \{(l+1, k+1)\}$
- $A_2 = \{(1, t) : 1 \leq t \leq k\}$
- $A_3 = \{(2l+1, t) : k+2 \leq t \leq 2k+1\}$
- $A_4 = \{(t, 1) : 2l - k + 2 \leq t \leq 2l + 1\}$
- $A_5 = \{(t, 2k+1) : l + 2 \leq t \leq 2l - k + 1\}$ **Note:** $A_5 = \emptyset$ if $k = l$.
- $A_6 = \{(t, k) : l + 1 \leq t \leq 2l - 1\}$
- $A_7 = \{(t, 2k+1) : 1 \leq t \leq l - 1\}$
- $A_8 = \{w_{mn-2}\} = \{(2l, k)\}$

Subcase 1a: $w_i \in A_1$. Then according to the labeling pattern, $d(w_j, w_i) = 1$.

$$\begin{aligned} \implies f(w_j) - f(w_i) &= 2(m+n-1) - d(w_i, w_{i+1}) - d(w_{i+1}, w_{i+2}) \\ &= (m+n-1) + (2l + 2k + 1) - (l+k) - (l+k+1) \\ &= m + n - 1 \\ &> (m+n-1) - d(w_j, w_i). \end{aligned} \tag{69}$$

Subcase 1b: $w_i \in \bigcup_{t=2}^{3} A_t$. Then according to the labeling pattern, $d(w_j, w_i) = 2l$.

$$\begin{aligned} \implies f(w_j) - f(w_i) &= 2(m+n-1) - d(w_i, w_{i+1}) - d(w_{i+1}, w_{i+2}) \\ &= (m+n-1) + (2l + 2k + 1) - (l+k+1) - (l+k+1) \\ &= (m+n-1) - 1 \\ &> (m+n-1) - 2l \quad \text{since } l \geq 2 \\ &= (m+n-1) - d(w_j, w_i). \end{aligned} \tag{70}$$

Subcase 1c: $w_i \in \bigcup_{t=4}^{5} A_t$.. Then according to the labeling pattern, $d(w_j, w_i) = 2k$.

$$\begin{aligned}
\implies f(w_j) - f(w_i) &= 2(m+n-1) - d(w_i, w_{i+1}) - d(w_{i+1}, w_{i+2}) \\
&= (m+n-1) + (2l+2k+1) - (l+k+1) - (l+k+1) \\
&= (m+n-1) - 1 \\
&> (m+n-1) - 2k, \text{since } k \geq 1 \\
&= (m+n-1) - d(w_j, w_i).
\end{aligned} \tag{71}$$

Subcase 1d: $w_i \in A_6$. Then according to the labeling pattern, $d(w_j, w_i) = 3$.

$$\begin{aligned}
\implies f(w_j) - f(w_i) &= 2(m+n-1) - d(w_i, w_{i+1}) - d(w_{i+1}, w_{i+2}) \\
&= (m+n-1) + (2l+2k+1) - (l+k+1) - (l+k+1) \\
&= (m+n-1) - 1 \\
&> (m+n-1) - 3 \\
&= (m+n-1) - d(w_j, w_i).
\end{aligned} \tag{72}$$

Subcase 1e: $w_i \in A_7$. Then according to the labeling pattern, $d(w_j, w_i) = 2k+1$.

$$\begin{aligned}
\implies f(w_j) - f(w_i) &= 2(m+n-1) - d(w_i, w_{i+1}) - d(w_{i+1}, w_{i+2}) \\
&= (m+n-1) + (2l+2k+1) - (l+k+2) - (l+k+1) \\
&= (m+n-1) - 2 \\
&> (m+n-1) - (2k+1), \text{since } k \geq 1 \\
&= (m+n-1) - d(w_j, w_i).
\end{aligned} \tag{73}$$

Subcase 1f: $w_i \in A_8$. Then according to the labeling pattern, $d(w_j, w_i) = l-1$.

$$\begin{aligned}
\implies f(w_j) - f(w_i) &= 2(m+n-1) - d(w_i, w_{i+1}) - d(w_{i+1}, w_{i+2}) \\
&= (m+n-1) + (2l+2k+1) - (l+k+1) - (k+2) \\
&= (m+n-1) + l - 2 \\
&\geq m+n-1, \text{since } l \geq 2 \\
&> (m+n-1) - d(w_j, w_i).
\end{aligned} \tag{74}$$

Subcase 1g: $w_i \notin \bigcup_{t=1}^{8} A_t$. Then according to the labeling pattern, $d(w_j, w_i) = 2$.

$$\implies f(w_j) - f(w_i) = 2(m+n-1) - d(w_i, w_{i+1}) - d(w_{i+1}, w_{i+2})$$
$$= (m+n-1) + (2l+2k+1) - (l+k+1) - (l+k+1)$$
$$= (m+n-1) - 1 \tag{75}$$
$$> (m+n-1) - 2$$
$$= (m+n-1) - d(w_j, w_i).$$

Hence, $f(w_j) - f(w_i) \geq m+n-1-d(w_j, w_i)$ whenever $j = i+2$.

CASE 2: $p \geq 3$.

Let $S = \{(t, 2k+1) : 1 \leq t \leq l-1\}$. When $p \geq 3$, we can make the following observations about the distances of jumps occurring between w_i and w_j.

1. S is non-empty since $l \geq 2$.
2. The jumps in the labeling pattern are all of length $l+k+1$ except those from a vertex in $\{w_1, w_{mn-1}\} \cup S$.
   (a) Jumps from vertices in S are of length $l+k+2$, since $l \geq 2$. There are $l-1$ jumps of length $l+k+2$ in the labeling pattern.
   (b) The jump from w_1 is of length $l+k$.
   (c) The jump from w_{mn-1} is of length $k+2$. Since $l \geq 2$, we know that $k+2 < l+k+1$.

$$\text{Therefore, we know } \sum_{t=i}^{j-1} d(w_t, w_{t+1}) \leq p(l+k+1) + (l-1). \tag{76}$$

$$
\begin{aligned}
\text{Hence, } f(w_j) - f(w_i) &= p(m+n-1) - \sum_{t=i}^{j-1} d(w_t, w_{t+1}) \\
&= (m+n-1) + (p-1)(2k+2l+1) - \sum_{t=i}^{j-1} d(w_t, w_{t+1}) \\
&\geq (m+n-1) + (p-1)(2k+2l+1) - [p(l+k+1) + (l-1)] \\
&= (m+n-1) + pl + pk - 2l - 2k - l \\
&= (m+n-1) + (p-2)(l+k) - l \\
&> (m+n-1) + (p-2)(l+k) - (l+k) \quad \text{since } k \geq 1 \\
&= (m+n-1) + (p-3)(l+k) \\
&\geq m+n-1 \quad \text{since } p \geq 3 \\
&> (m+n-1) - d(w_i, w_j).
\end{aligned} \tag{77}
$$

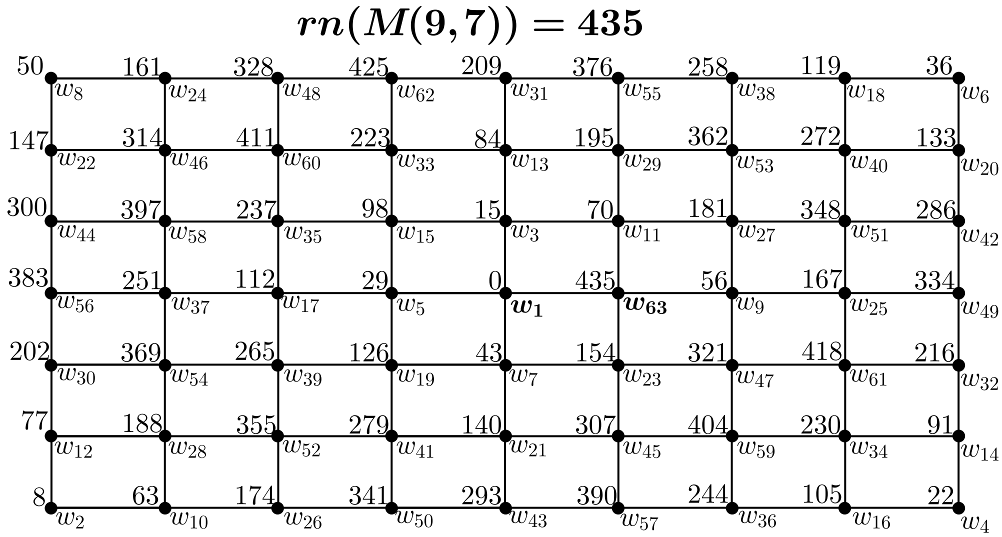
By Cases 1 and 2, we know that f is a radio labeling of G when $l \geq 2$, proving our claim.

Notice from the labeling pattern that there are four possible distances between consecutively labeled vertices, namely $l+k+1$, $l+k+2$, $k+2$, and $l+k$, with the number of occurrences $mn-(l+2)$, $l-1$, 1, 1, respectively.

$$
\begin{aligned}
\text{So } span(f) &= (mn-1)(m+n-1) - \sum_{i=1}^{mn-1} d(w_i, w_{i+1}) \\
&= (mn-1)(m+n-1) - [(l+k+1)(mn-l-2) + (l+k+2)(l-1) + (k+2) + (l+k)] \\
&= (mn-1)(m+n-1) - \left(\frac{m+n}{2}\right)\left(\frac{2mn-m-3}{2}\right) - \left(\frac{m+n+2}{2}\right)\left(\frac{m-3}{2}\right) \\
&\quad - \left(\frac{m+2n+1}{2}\right) \\
&= (mn-1)(m+n-1) - \frac{1}{4}\left[(m+n)(2mn-m-3) + (m+n+2)(m-3) + 2(m+2n+1)\right] \\
&= (mn-1)(m+n-1) - \frac{1}{4}(2m^2n + 2mn^2 - 2m - 2n - 4) \\
&= \frac{1}{2}m^2n + \frac{1}{2}mn^2 - mn - \frac{1}{2}m - \frac{1}{2}n + 2 \\
&= \frac{1}{2}(m^2n + mn^2 - 2mn - m - n) + 2.
\end{aligned} \tag{78}
$$

Therefore, $rn(G) \leq \frac{1}{2}(m^2n + mn^2 - 2mn - m - n) + 2$, since f is a radio labeling of G with this span.

□



Above is the optimal radio labeling determined by the labeling pattern described above for $M(9,7)$. The radio number of $M(9,7)$ is 435.

5.3 Upper Bound of $rn(M(m,n))$ for m,n Even

Lemma 5.3. *Let $m = 2l$ and $n = 2k$. Let $G = M(m,n)$. Then $rn(G) \leq \frac{1}{2}(m^2n + mn^2 - 2mn - 2m - 2n) + 6$.*

Proof. Let $m = 2l$ and $n = 2k$. Since $G(m,n)$ is isomorphic to $G(n,m)$ and m,n have the same parity, we may assume without loss of generality that $l \geq k$. Therefore, G has at least as many columns as it does rows.

When m,n are both even, a complication with calculations requires that we separate the trivial case of $l = 2$ from the more general case of $l \geq 3$. If $l = 2$, then by assumption $k = 2$, and so $G = M(m,n)$ is a square mesh with 4 rows and 4 columns. We define a function $g : V(G) \to \{0, 1, 2, ...\}$ by the following.

$$
\begin{array}{llll}
(1,4) \mapsto 5 & (2,4) \mapsto 18 & (3,4) \mapsto 28 & (4,4) \mapsto 41 \\
(1,3) \mapsto 12 & (2,3) \mapsto 0 & (3,3) \mapsto 46 & (4,3) \mapsto 34 \\
(1,2) \mapsto 25 & (2,2) \mapsto 38 & (3,2) \mapsto 8 & (4,2) \mapsto 21 \\
(1,1) \mapsto 43 & (2,1) \mapsto 31 & (3,1) \mapsto 15 & (4,1) \mapsto 3
\end{array}
$$

From straightforward calculation it is easy to verify that g as defined above is a radio labeling of $G = M(4,4)$ with a span of 46. Since $46 = \frac{1}{2}\left[(4)^2(4) + (4)(4)^2 - 2(4)(4) - 2(4) - 2(4)\right] + 6$, the desired upper bound of $rn(G)$ is achieved when $l = 2$.

To prove the upper bound when m and n are even where $l \geq 3$, it suffices to find one radio labeling of $G = M(m,n)$ with a span of $\frac{1}{2}(m^2n + mn^2 - 2mn - 2m - 2n) + 6$. Let $\{w_1, w_2, ..., w_{mn}\}$ be a permutation of $V(G)$ to define a labeling pattern.

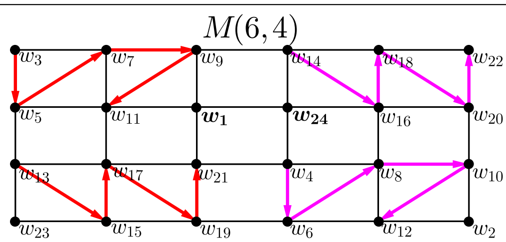
When m and n are both even, there exist slight complications that require a specific labeling for each subcase determined by the parities of l and k. The structures of all these labelings are almost identical, and their commonalities will suffice in proving the desired upper bound. The only slight adjustments involve the corner vertices of each of the four quadrants. Each of the labeling patterns can be systematically separated into two blocks. On each of the sample graphs, the red and purple vectors indicate subsequences of vertices in each quadrant whose indices are of equal parity .

Case 1: l is Odd and $k = 2$.

Then we define f by the following blocks.

$w_1 = (l,3) \xrightarrow{l+2} (m,1) \xrightarrow{m+2} (1,4) \xrightarrow{l+2} (l+1,2) \xrightarrow{l+1} (1,3) \xrightarrow{l+2} (l+1,1) \xrightarrow{l+2} (2,4) \xrightarrow{l+2} (l+2,2) \xrightarrow{l+1} (3,4) \xrightarrow{l+2} (l+3,2) \xrightarrow{l+2} (2,3) \xrightarrow{l+2} (l+2,1) \xrightarrow{l+1} ... \xrightarrow{l+2} (l-1,4) \xrightarrow{l+2} (m-1,2) \xrightarrow{l+1} (l,4) \xrightarrow{l+2} (m,2) \xrightarrow{l+2} (l-1,3) \xrightarrow{l+2} (m-1,1) = w_{2m}$

$\xrightarrow{m-1} w_{2m+1} = (1,2) \xrightarrow{l+2} (l+1,4) \xrightarrow{l+2} (2,1) \xrightarrow{l+2} (l+2,3) \xrightarrow{l+1} (2,2) \xrightarrow{l+2} (l+2,4) \xrightarrow{l+2} (3,1) \xrightarrow{l+2} (l+3,3) \xrightarrow{l+1} ... \xrightarrow{l+2} (l,1) \xrightarrow{l+2} (m,3) \xrightarrow{l+2} (l,2) \xrightarrow{l+2} (m,4) \xrightarrow{m+2} (1,1) \xrightarrow{l+2} (l+1,3) = w_{4m}$



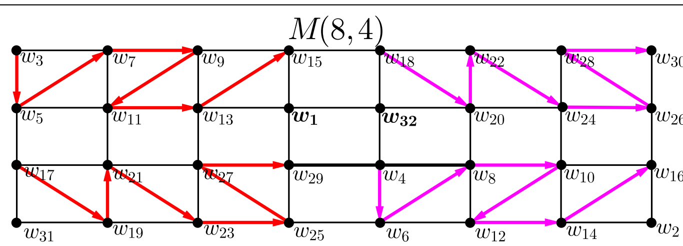
Above is the labeling pattern described for $M(6,4)$.

Case 2: l is Even and $k = 2$.

Then we define f by the following blocks.

$w_1 = (l,3) \xrightarrow{l+2} (m,1) \xrightarrow{m+2} (1,4) \xrightarrow{l+2} (l+1,2) \xrightarrow{l+1} (1,3) \xrightarrow{l+2} (l+1,1) \xrightarrow{l+2} (2,4) \xrightarrow{l+2} (l+2,2) \xrightarrow{l+1} (3,4) \xrightarrow{l+2} (l+3,2) \xrightarrow{l+2} (2,3) \xrightarrow{l+2} (l+2,1) \xrightarrow{l+1} ... \xrightarrow{l+2} (l-2,3) \xrightarrow{l+2} (m-2,1) \xrightarrow{l+1} (l-1,3) \xrightarrow{l+2} (m-1,1) \xrightarrow{l+2} (l,4) \xrightarrow{l+2} (m,2) = w_{2m}$

$\xrightarrow{m-1} w_{2m+1} = (1,2) \xrightarrow{l+2} (l+1,4) \xrightarrow{l+2} (2,1) \xrightarrow{l+2} (l+2,3) \xrightarrow{l+1} (2,2) \xrightarrow{l+2} (l+2,4) \xrightarrow{l+2} (3,1) \xrightarrow{l+2} (l+3,3) \xrightarrow{l+1} ... \xrightarrow{l+1} (l,1) \xrightarrow{l+2} (m,3) \xrightarrow{l+2} (l-1,2) \xrightarrow{l+2} (m-1,4) \xrightarrow{l+1} (l,2) \xrightarrow{l+2} (m,4) \xrightarrow{m+2} (1,1) \xrightarrow{l+2} (l+1,3) = w_{4m}$



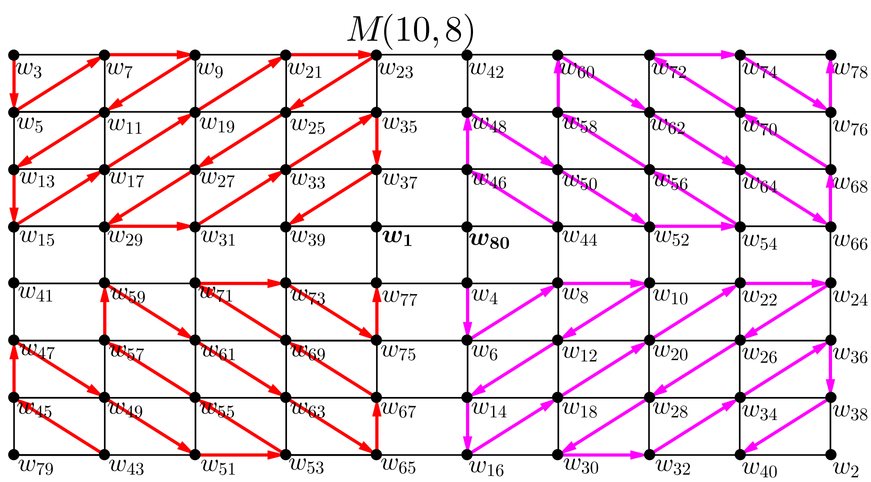
Above is the labeling pattern described for $M(8,4)$.

Case 3: l is Odd and $k \geq 4$ is Even.

Then we define f by the following blocks.

$$w_1 = (l,k+1) \xrightarrow{l+k} (m,1) \xrightarrow{m+n-2} (1,n) \xrightarrow{l+k} (l+1,k) \xrightarrow{l+k-1} (1,n-1) \xrightarrow{l+k} (l+1,k-1) \xrightarrow{l+k}$$
$$(2,n) \xrightarrow{l+k} (l+2,k) \xrightarrow{l+k-1} (3,n) \xrightarrow{l+k} (l+3,k) \xrightarrow{l+k} (2,n-1) \xrightarrow{l+k} (l+2,k-1) \xrightarrow{l+k} (1,n-2) \xrightarrow{l+k}$$
$$(l+1,k-2) \xrightarrow{l+k-1} (1,n-3) \xrightarrow{l+k} (l+1,k-3) \xrightarrow{l+k} (2,n-2) \xrightarrow{l+k} (l+2,k-2) \xrightarrow{l+k} (3,n-1) \xrightarrow{l+k}$$
$$(l+3,k-1) \xrightarrow{l+k} (4,n) \xrightarrow{l+k} (l+4,k) \xrightarrow{l+k-1} \ldots \xrightarrow{l+k} (1,k+2) \xrightarrow{l+k} (l+1,2) \xrightarrow{l+k} (1,k+1) \xrightarrow{l+k}$$
$$(l+1,1) \xrightarrow{l+k} (2,k+2) \xrightarrow{l+k} (l+2,2) \xrightarrow{l+k} \ldots\ \ldots\ \ldots \xrightarrow{l+k} (l-k+1,k+1) \xrightarrow{l+k} (m-k+1,1) \xrightarrow{l+k-1}$$
$$(l-k+2,k+1) \xrightarrow{l+k} (m-k+2,1) \xrightarrow{l+k} (l-k+3,k+2) \xrightarrow{l+k} (m-k+3,2) \xrightarrow{l+k} \ldots \xrightarrow{l+k}$$
$$(l,n-1) \xrightarrow{l+k} (m,k-1) \xrightarrow{l+k-1} (l,n-2) \xrightarrow{l+k} (m,k-2) \xrightarrow{l+k} (l-1,n-3) \xrightarrow{l+k} (m-1,k-3) \xrightarrow{l+k}$$
$$\ldots \xrightarrow{l+k} (l,k+3) \xrightarrow{l+k} (m,3) \xrightarrow{l+k-1} (l,k+2) \xrightarrow{l+k} (m,2) \xrightarrow{l+k} (l-1,k+1) \xrightarrow{l+k} (m-1,1) = w_{km}$$

$$\xrightarrow{m+k-3} w_{km+1} = (1,k) \xrightarrow{l+k} (l+1,m) \xrightarrow{n+l-2} (2,1) \xrightarrow{l+k} (l+2,k+1) \xrightarrow{l+k} (1,2) \xrightarrow{l+k}$$
$$(l+1,k+2) \xrightarrow{l+k-1} (1,3) \xrightarrow{l+k} (l+1,k+3) \xrightarrow{l+k} (2,2) \xrightarrow{l+k} (l+2,k+2) \xrightarrow{l+k} (3,1) \xrightarrow{l+k}$$
$$(l+3,k+1) \xrightarrow{l+k} (2,k+2) \xrightarrow{l+k-1} (4,1) \xrightarrow{l+k} (l+4,k+1) \xrightarrow{l+k-1} (3,2) \xrightarrow{l+k} (l+3,k+2) \xrightarrow{l+k}$$
$$\ldots \xrightarrow{l+k} (2,k-1) \xrightarrow{l+k} (l+2,n-1) \xrightarrow{l+k-1} (2,k) \xrightarrow{l+k} (l+2,n) \xrightarrow{l+k} (3,k-1) \xrightarrow{l+k} (l+3,n-1) \xrightarrow{l+k}$$
$$(4,k-2) \xrightarrow{l+k} (l+4,n-2) \xrightarrow{l+k} \ldots\ \ldots\ \ldots \xrightarrow{l+k-1} (l-k+1,k) \xrightarrow{l+k} (m-k+1,n) \xrightarrow{l+k}$$
$$(l-k+2,k-1) \xrightarrow{l+k} (m-k+2,n-1) \xrightarrow{l+k} \ldots \xrightarrow{l+k} (l-1,k) \xrightarrow{l+k} (m-1,n) \xrightarrow{l+k} (l,k-1) \xrightarrow{l+k}$$
$$(m,n-1) \xrightarrow{l+k-1} (k,l) \xrightarrow{l+k} (m,n) \xrightarrow{m+n-2} (1,1) \xrightarrow{l+k} (l+1,k+1) = w_{mn}$$



Above is the labeling pattern described for $M(10,8)$.

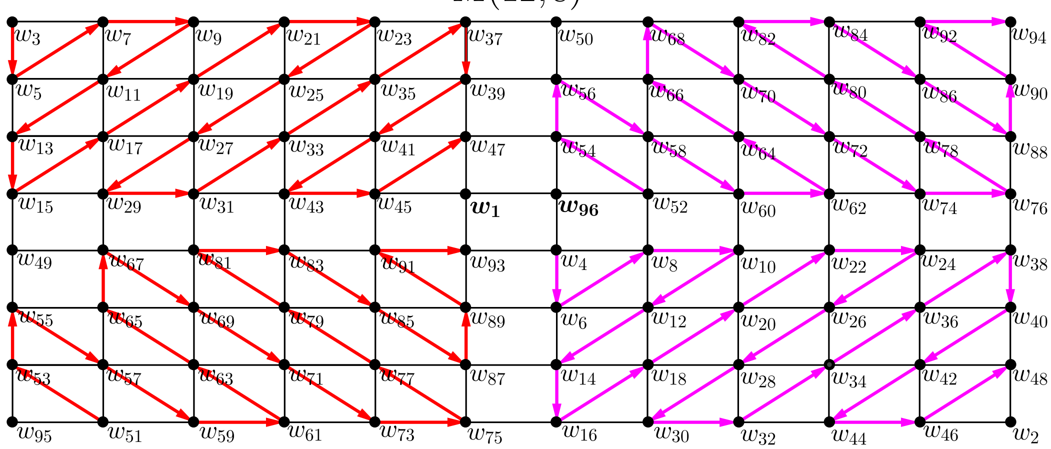
Case 4: l is Even and $k \geq 4$ is Even.

Then we define f by the following blocks.

$$w_1 = (l,k+1) \xrightarrow{l+k} (m,1) \xrightarrow{m+n-2} (1,n) \xrightarrow{l+k} (l+1,k) \xrightarrow{l+k-1} (1,n-1) \xrightarrow{l+k} (l+1,k-1) \xrightarrow{l+k}$$
$$(2,n) \xrightarrow{l+k} (l+2,k) \xrightarrow{l+k-1} (3,n) \xrightarrow{l+k} (l+3,k) \xrightarrow{l+k} (2,n-1) \xrightarrow{l+k} (l+2,k-1) \xrightarrow{l+k} (1,n-2) \xrightarrow{l+k}$$
$$(l+1,k-2) \xrightarrow{l+k-1} (1,n-3) \xrightarrow{l+k} (l+1,k-3) \xrightarrow{l+k} (2,n-2) \xrightarrow{l+k} (l+2,k-2) \xrightarrow{l+k} (3,n-1) \xrightarrow{l+k}$$
$$(l+3,k-1) \xrightarrow{l+k} (4,n) \xrightarrow{l+k} (l+4,k) \xrightarrow{l+k-1} \ldots \xrightarrow{l+k} (1,k+2) \xrightarrow{l+k} (l+1,2) \xrightarrow{l+k} (1,k+1) \xrightarrow{l+k}$$
$$(l+1,1) \xrightarrow{l+k} (2,k+2) \xrightarrow{l+k} (l+2,2) \xrightarrow{l+k} \ldots \ldots \ldots \xrightarrow{l+k-1} (l-k+1,k+1) \xrightarrow{l+k} (m-k+1,1) \xrightarrow{l+k}$$
$$(l-k+2,+2) \xrightarrow{l+k} (m-k+2,2) \xrightarrow{l+k} \ldots \xrightarrow{l+k} (l,n) \xrightarrow{l+k} (m,k) \xrightarrow{l+k} (l,n-1) \xrightarrow{l+k} (m,k-1) \xrightarrow{l+k}$$
$$(l-1,n-2) \xrightarrow{l+k} (m-1,k-2) \xrightarrow{l+k} \ldots \xrightarrow{l+k} (l,n-2) \xrightarrow{l+k} (m,k-2) \xrightarrow{l+k} (l,n-3) \xrightarrow{l+k}$$
$$(m,k-3) \xrightarrow{l+k} \ldots \xrightarrow{l+k} (l-2,k+1) \xrightarrow{l+k} (m-2,1) \xrightarrow{l+k-1} (l-1,k+1) \xrightarrow{l+k} (m-1,1) \xrightarrow{l+k}$$
$$(l,k+2) \xrightarrow{l+k} (m,2) = w_{km}$$

$$\xrightarrow{m+k-3} w_{km+1} = (1,k) \xrightarrow{l+k} (l+1,m) \xrightarrow{n+l-2} (2,1) \xrightarrow{l+k} (l+2,k+1) \xrightarrow{l+k} (1,2) \xrightarrow{l+k}$$
$$(l+1,k+2) \xrightarrow{l+k-1} (1,3) \xrightarrow{l+k} (l+1,k+3) \xrightarrow{l+k} (2,2) \xrightarrow{l+k} (l+2,k+2) \xrightarrow{l+k} (3,1) \xrightarrow{l+k}$$
$$(l+3,k+1) \xrightarrow{l+k} (2,k+2) \xrightarrow{l+k-1} (4,1) \xrightarrow{l+k} (l+4,k+1) \xrightarrow{l+k-1} (3,2) \xrightarrow{l+k} (l+3,k+2) \xrightarrow{l+k}$$
$$\ldots \xrightarrow{l+k} (2,k-1) \xrightarrow{l+k} (l+2,n-1) \xrightarrow{l+k-1} (2,k) \xrightarrow{l+k} (l+2,n) \xrightarrow{l+k} (3,k-1) \xrightarrow{l+k} (l+3,n-1) \xrightarrow{l+k}$$
$$(4,k-2) \xrightarrow{l+k} (l+4,n-2) \xrightarrow{l+k} \ldots \ldots \ldots \xrightarrow{l+k} (l-k+1,k) \xrightarrow{l+k} (m-k+1,n) \xrightarrow{l+k} \ldots \xrightarrow{l+k-1}$$
$$(l,k-1) \xrightarrow{l+k} (m,n-1) \xrightarrow{l+k} (l-1,k) \xrightarrow{l+k} (m-1,n) \xrightarrow{l+k-1} (l,k) \xrightarrow{l+k} (m,n) \xrightarrow{m+n-2}$$
$$(1,1) \xrightarrow{l+k} (l+1,k+1) = w_{mn}$$

$M(12,8)$



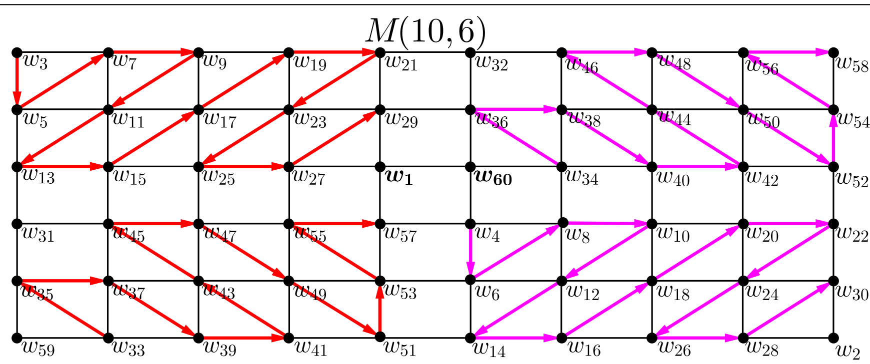
Above is the labeling pattern described for $M(12,8)$.

Case 5: l is Odd and k is Odd.

Then we define f by the following blocks.

$w_1 = (l,k+1) \xrightarrow{l+k} (m,1) \xrightarrow{m+n-2} (1,n) \xrightarrow{l+k} (l+1,k) \xrightarrow{l+k-1} (1,n-1) \xrightarrow{l+k} (l+1,k-1) \xrightarrow{l+k}$
$(2,n) \xrightarrow{l+k} (l+2,k) \xrightarrow{l+k-1} (3,n) \xrightarrow{l+k} (l+3,k) \xrightarrow{l+k} (2,n-1) \xrightarrow{l+k} (l+2,k-1) \xrightarrow{l+k} (1,n-2) \xrightarrow{l+k}$
$(l+1,k-2) \xrightarrow{l+k-1} (1,n-3) \xrightarrow{l+k} (l+1,k-3) \xrightarrow{l+k} (2,n-2) \xrightarrow{l+k} (l+2,k-2) \xrightarrow{l+k} (3,n-1) \xrightarrow{l+k}$
$(l+3,k-1) \xrightarrow{l+k} (4,n) \xrightarrow{l+k} (l+4,k) \xrightarrow{l+k-1} \ldots \xrightarrow{l+k} (1,k+1) \xrightarrow{l+k} (l+1,1) \xrightarrow{l+k-1} (2,k+1) \xrightarrow{l+k}$
$(l+2,1) \xrightarrow{l+k} (3,k+2) \xrightarrow{l+k} (l+3,2) \xrightarrow{l+k} \ldots\ldots\ldots \xrightarrow{l+k} (l-k+1,k+1) \xrightarrow{l+k} (m-k+1,1) \xrightarrow{l+k-1}$
$(l-k+2,k+1) \xrightarrow{l+k} (m-k+2,1) \xrightarrow{l+k} (l-k+3,k+2) \xrightarrow{l+k} (m-k+3,2) \xrightarrow{l+k} \ldots \xrightarrow{l+k} (l,n-1) \xrightarrow{l+k}$
$(m,k-1) \xrightarrow{l+k-1} (l,n-2) \xrightarrow{l+k} (m,k-2) \xrightarrow{l+k} (l-1,n-3) \xrightarrow{l+k} (m-1,k-3) \xrightarrow{l+k} \ldots \xrightarrow{l+k}$
$(l-2,k+1) \xrightarrow{l+k} (m-2,1) \xrightarrow{l+k} (l-1,k+1) \xrightarrow{l+k} (m-1,1) \xrightarrow{l+k} (l,k+2) \xrightarrow{l+k} (m,2) = w_{km}$

$\xrightarrow{m+k-3} w_{2m+1} = (1,k) \xrightarrow{l+k} (l+1,n) \xrightarrow{n+l-2} (2,1) \xrightarrow{l+k} (l+2,k+1) \xrightarrow{l+k} (1,2) \xrightarrow{l+k} (l+$
$1,k+2) \xrightarrow{l+k-1} (1,3) \xrightarrow{l+k} (l+1,k+3) \xrightarrow{l+k} (2,2) \xrightarrow{l+k} (l+2,k+2) \xrightarrow{l+k} (3,1) \xrightarrow{l+k} (l+3,k+$
$1) \xrightarrow{l+k-1} (4,1) \xrightarrow{l+k} (l+4,k+1) \xrightarrow{l+k} (3,2) \xrightarrow{l+k} (l+3,k+2) \xrightarrow{l+k} \ldots \xrightarrow{l+k} (1,k-1) \xrightarrow{l+k}$
$(l+1,n-1) \xrightarrow{l+k} (2,k-1) \xrightarrow{l+k} (l+2,n-1) \xrightarrow{l+k} (3,k-2) \xrightarrow{l+k} (l+3,n-2) \xrightarrow{l+k} (4,k-3) \xrightarrow{l+k}$
$(l+4,n-3) \xrightarrow{l+k} \ldots\ldots\ldots \xrightarrow{l+k-1} (l-k+1,k) \xrightarrow{l+k} (m-k+1,n) \xrightarrow{l+k} (l-k+2,k-1) \xrightarrow{l+k}$
$(m-k+2,n-1) \xrightarrow{l+k} \ldots \xrightarrow{l+k} (l,k-2) \xrightarrow{l+k} (m,n-2) \xrightarrow{l+k-1} (l,k-1) \xrightarrow{l+k} (m,n-1) \xrightarrow{l+k}$
$(l-1,k) \xrightarrow{l+k} (m-1,n) \xrightarrow{l+k-1} (l,k) \xrightarrow{l+k} (m,n) \xrightarrow{m+n-2} (1,1) \xrightarrow{l+k} (l+1,k+1) = w_{mn}$



Above is the labeling pattern described for $M(10,6)$.

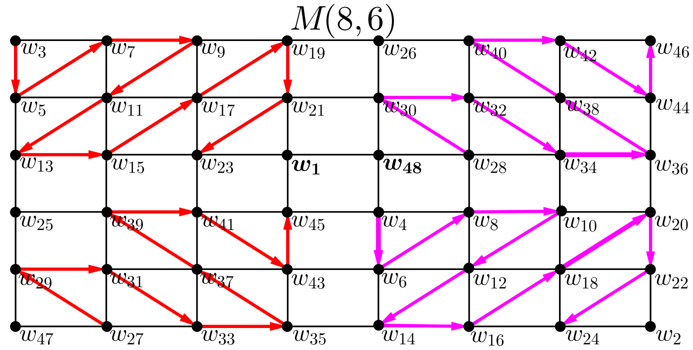
Case 6: l is Even and k is Odd.

Then we define f by the following blocks.

$$w_1 = (l,k+1) \xrightarrow{l+k} (m,1) \xrightarrow{m+n-2} (1,n) \xrightarrow{l+k} (l+1,k) \xrightarrow{l+k-1} (1,n-1) \xrightarrow{l+k} (l+1,k-1) \xrightarrow{l+k}$$
$$(2,n) \xrightarrow{l+k} (l+2,k) \xrightarrow{l+k-1} (3,n) \xrightarrow{l+k} (l+3,k) \xrightarrow{l+k} (2,n-1) \xrightarrow{l+k} (l+2,k-1) \xrightarrow{l+k}$$
$$(1,n-2) \xrightarrow{l+k} (l+1,k-2) \xrightarrow{l+k-1} (1,n-3) \xrightarrow{l+k} (l+1,k-3) \xrightarrow{l+k} (2,n-2) \xrightarrow{l+k} (l+2,k-2) \xrightarrow{l+k}$$
$$(3,n-1) \xrightarrow{l+k} (l+3,k-1) \xrightarrow{l+k} (4,n) \xrightarrow{l+k} (l+4,k) \xrightarrow{l+k-1} \ldots \xrightarrow{l+k} (1,k+1) \xrightarrow{l+k} (l+1,1) \xrightarrow{l+k-1}$$
$$(2,k+1) \xrightarrow{l+k} (l+2,1) \xrightarrow{l+k} (3,k+2) \xrightarrow{l+k} (l+3,2) \xrightarrow{l+k} \ldots\ldots\ldots \xrightarrow{l+k-1} (l-k+1,k+1) \xrightarrow{l+k}$$
$$(m-k+1,1) \xrightarrow{l+k} (l-k+2,k+2) \xrightarrow{l+k} (m-k+2,2) \xrightarrow{l+k} \ldots \xrightarrow{l+k} (l,n) \xrightarrow{l+k} (m,k) \xrightarrow{l+k-1}$$
$$(l,n-1) \xrightarrow{l+k} (m,k-1) \xrightarrow{l+k} (l-1,n-2) \xrightarrow{l+k} (m-1,k-2) \xrightarrow{l+k} \ldots \xrightarrow{l+k} (l,k+3) \xrightarrow{l+k}$$
$$(m,3) \xrightarrow{l+k-1} (l,k+2) \xrightarrow{l+k} (m,2) \xrightarrow{l+k} (l-1,k+1) \xrightarrow{l+k} (m-1,1) = w_{km}$$

$$\xrightarrow{m+k-3} w_{2m+1} = (1,k) \xrightarrow{l+k} (l+1,n) \xrightarrow{n+l-2} (2,1) \xrightarrow{l+k} (l+2,k+1) \xrightarrow{l+k} (1,2) \xrightarrow{l+k} (l+1,k+$$
$$2) \xrightarrow{l+k-1} (1,3) \xrightarrow{l+k} (l+1,k+3) \xrightarrow{l+k} (2,2) \xrightarrow{l+k} (l+2,k+2) \xrightarrow{l+k} (3,1) \xrightarrow{l+k} (l+3,k+1) \xrightarrow{l+k-1}$$
$$(4,1) \xrightarrow{l+k} (l+4,k+1) \xrightarrow{l+k} (3,2) \xrightarrow{l+k} (l+3,k+2) \xrightarrow{l+k} \ldots \xrightarrow{l+k} (1,k-1) \xrightarrow{l+k} (l+1,n-1) \xrightarrow{l+k}$$
$$(2,k-1) \xrightarrow{l+k} (l+2,n-1) \xrightarrow{l+k} (3,k-2) \xrightarrow{l+k} (l+3,n-2) \xrightarrow{l+k} (4,k-3) \xrightarrow{l+k} (l+4,n-3) \xrightarrow{l+k}$$
$$\ldots\ldots\ldots \xrightarrow{l+k} (l-k+1,k) \xrightarrow{l+k} (m-k+1,n) \xrightarrow{l+k-1} (l-k+2,k) \xrightarrow{l+k} (m-k+2,n) \xrightarrow{l+k}$$
$$(l-k+3,k-1) \xrightarrow{l+k} (m-k+3,n-1) \xrightarrow{l+k} \ldots \xrightarrow{l+k-1} (l-1,k) \xrightarrow{l+k} (m-1,n) \xrightarrow{l+k} (l,k-1) \xrightarrow{l+k}$$
$$(m,n-1) \xrightarrow{l+k} (l,k) \xrightarrow{l+k} (m,n) \xrightarrow{m+n-2} (1,1) \xrightarrow{l+k} (l+1,k+1) = w_{mn}$$

$M(8,6)$



Above is the labeling pattern described for $M(8,6)$.

In each block, let the quantity above each arrow indicate the distance between the two consecutive vertices. Let $f : V(G) \to \{0,1,2,...\}$ be a function such that $f(w_1) = 0$ and

$$f(w_{i+1}) - f(w_i) = \begin{cases} m+n-1-d(w_{i+1},w_i) & \text{if } 1 \le i \le mn-1 \text{ unless } i = 2 \text{ or } mn-2; \\ m+n-d(w_{i+1},w_i) & \text{if } i = 2 \text{ or } mn-2. \end{cases}$$

CLAIM: f is a radio labeling of G.

To prove this, we show that for all $1 \leq i \leq mn-2$, we have

$$f(w_j) - f(w_i) \geq m+n-1-d(w_j, w_i) \text{ for any } j \geq i+2.$$

By the same justification in the previous section, we have

$$f(w_j) - f(w_i) = f_{j-1} + f_{j-2} + ... + f_{i+1} + f_i.$$

We make the following observations about the distances of jumps occurring between w_i and w_j.

1. The jumps in the labeling pattern are all of length $l+k$ and $l+k-1$ except those from a vertex in the set $\{w_2, w_{km}, w_{km+2}, w_{mn-2}\}$.
   (a) Jumps from vertices in $\{w_2, w_{mn-2}\}$ are of length $m+n-2$, since these jumps are between antipodal vertices. There are 2 jumps of length $m+n-2$ in the labeling pattern.
   (b) The jump from w_{km} is of length $m+k-3$. Since $m \geq 6$ by assumption, we have $m+k-3 \geq l+k$
   (c) The jump from w_{km+2} is of length $n+l-2$. Since $n \geq 4$ by assumption, we have $n+l-2 \geq l+k$.
2. The jumps of length $l+k-1$ are all from vertices in the right region of G and have a one-to-one correspondence with the vertices w_j in the left region such that w_j and w_{j+2} are adjacent. Quadrant II and Quadrant III each have $(l-1)+(k-1)-1 = l+k-3$ such vertices. Hence, there are $2l+2k-6 = m+n-6$ vertices w_j in $V(G)$ such that $d(w_j, w_{j+1}) = l+k-1$.

Let $j = i+p$, where $p \geq 2$. We observe 3 cases, using distances indicated in the labeling pattern for f for reference.

CASE 1: $p = 2$.

We examine five different cases when $j = i+2$.

Subcase 1a: $i = 1, 2, mn-3$, or $mn-2$. Then according to the labeling pattern, $d(w_j, w_i) = l+k-2$.

$$\begin{aligned}
\implies f(w_j) - f(w_i) &= 2(m+n-1) + 1 - d(w_i, w_{i+1}) - d(w_{i+1}, w_{i+2}) \\
&= (m+n-1) + m + n - (m+n-2) - (l+k) \\
&= (m+n-1) + 2 - (l+k) \\
&= (m+n-1) - (l+k-2) \\
&= (m+n-1) - d(w_j, w_i).
\end{aligned} \tag{79}$$

Subcase 1b: $i = km+1$ or $km+2$. Then according to the labeling pattern, $d(w_j, w_i) = k$.

$$\begin{aligned}
\implies f(w_j) - f(w_i) &= 2(m+n-1) - d(w_i, w_{i+1}) - d(w_{i+1}, w_{i+2}) \\
&= (m+n-1) + m + n - 1 - (l+k) - (n+l-1) \\
&= (m+n-1) + 2 - (l+k) \\
&= (m+n-1) - k \\
&= (m+n-1) - d(w_j, w_i).
\end{aligned} \tag{80}$$

Subcase 1c: $i = km$. Then according to the labeling pattern, $d(w_j, w_i) = n + l - 3$.

$$\begin{aligned}
\implies f(w_j) - f(w_i) &= 2(m+n-1) - d(w_i, w_{i+1}) - d(w_{i+1}, w_{i+2}) \\
&= (m+n-1) + m + n - 1 - (m+k-3) - (l+k) \\
&= (m+n-1) - (l-2) \\
&> (m+n-1) - (n+l-3) \quad \text{since } n > 1 \\
&= (m+n-1) - d(w_j, w_i).
\end{aligned} \tag{81}$$

Subcase 1d: $i = km-1$. Then according to the labeling pattern, $d(w_j, w_i) = l - 1$.

$$\begin{aligned}
\implies f(w_j) - f(w_i) &= 2(m+n-1) - d(w_i, w_{i+1}) - d(w_{i+1}, w_{i+2}) \\
&= (m+n-1) + m + n - 1 - (l+k) - (m+k-3) \\
&= (m+n-1) - (l-2) \\
&> (m+n-1) - (l-1) \\
&= (m+n-1) - d(w_j, w_i).
\end{aligned} \tag{82}$$

Subcase 1e: $i \notin \{1, 2, km-1, km, km+1, km+2, mn-2, mn-2\}$. Then according to the labeling pattern, $d(w_j, w_i) \leq 2$.

$$
\begin{aligned}
\implies f(w_j) - f(w_i) &= 2(m+n-1) - d(w_i, w_{i+1}) - d(w_{i+1}, w_{i+2}) \\
&\geq (m+n-1) + m + n - 1 - (l+k) - (l+k) \\
&= (m+n-1) - 1 \\
&\geq (m+n-1) - d(w_j, w_i) \quad \text{since } d(w_j, w_i) \geq 1.
\end{aligned} \tag{83}
$$

Hence, $f(w_j) - f(w_i) \geq m + n - 1 - d(w_j, w_i)$ whenever $j = i + 2$.

CASE 2: $p = 3$.

We examine five different cases when $j = i + 2$.

Subcase 2a: $i = 1, 2, mn - 3$, or $mn - 4$.

$$
\begin{aligned}
\implies f(w_j) - f(w_i) &= 3(m+n-1) + 1 - \sum_{t=i}^{j-1} d(w_t, w_{t+1}) \\
&= (m+n-1) + 2(m+n-1) + 1 - \sum_{t=i}^{j-1} d(w_t, w_{t+1}) \\
&\geq (m+n-1) + 2(m+n-1) + 1 - [2(l+k) + (m+n-2)] \\
&= (m+n-1) + 2(m+n-1) + 1 - [2m + 2n - 2] \\
&= (m+n-1) + 1 \\
&> (m+n-1) - d(w_j, w_i).
\end{aligned} \tag{84}
$$

Subcase 2b: $i = km - 2$ or $km - 1$.

$$
\begin{aligned}
\implies f(w_j) - f(w_i) &= 3(m+n-1) - \sum_{t=i}^{j-1} d(w_t, w_{t+1}) \\
&= (m+n-1) + 2(m+n-1) - \sum_{t=i}^{j-1} d(w_t, w_{t+1}) \\
&= (m+n-1) + 2(m+n-1) - [2(l+k) + (m+k-3)] \\
&= (m+n-1) + 2(m+n-1) - (2m + 3k - 3) \\
&= (m+n-1) + (k+1) \\
&> (m+n-1) - d(w_j, w_i).
\end{aligned} \tag{85}
$$

Subcase 2c: $i = km$.

$$
\begin{aligned}
\implies f(w_j) - f(w_i) &= 3(m+n-1) - \sum_{t=i}^{j-1} d(w_t, w_{t+1}) \\
&= (m+n-1) + 2(m+n-1) - \sum_{t=i}^{j-1} d(w_t, w_{t+1}) \\
&= (m+n-1) + 2(m+n-1) - [(m+k-3) + (l+k) + (n+l-2)] \\
&= (m+n-1) + 2(m+n-1) - (2m+2n-5) \\
&= (m+n-1) + 3 \\
&> (m+n-1) - d(w_j, w_i).
\end{aligned}
\tag{86}
$$

Subcase 2d: $i = km+1$ or $km+2$.

$$
\begin{aligned}
\implies f(w_j) - f(w_i) &= 3(m+n-1) - \sum_{t=i}^{j-1} d(w_t, w_{t+1}) \\
&= (m+n-1) + 2(m+n-1) - \sum_{t=i}^{j-1} d(w_t, w_{t+1}) \\
&= (m+n-1) + 2(m+n-1) - [2(l+k) + (n+l-2)] \\
&= (m+n-1) + 2(m+n-1) - (3l+2n-2) \\
&= (m+n-1) + l \\
&> (m+n-1) - d(w_j, w_i).
\end{aligned}
\tag{87}
$$

Subcase 2e: $i \notin \{1, 2, km-2, km-1, km, km+1, km+2, mn-4, mn-3\}$.

$$
\begin{aligned}
\implies f(w_j) - f(w_i) &= 3(m+n-1) - \sum_{t=i}^{j-1} d(w_t, w_{t+1}) \\
&= (m+n-1) + 2(m+n-1) - \sum_{t=i}^{j-1} d(w_t, w_{t+1}) \\
&\geq (m+n-1) + 2(m+n-1) - 3(l+k) \\
&= (m+n-1) + (l+k-2) \\
&> (m+n-1) - d(w_j, w_i).
\end{aligned}
\tag{88}
$$

Hence, $f(w_j) - f(w_i) \geq m+n-1 - d(w_j, w_i)$ whenever $j = i+3$.

CASE 3: $p \geq 4$.

Subcase 3a: $i \leq 2$ and $j \geq mn-1$. So $p \geq mn-3$.

$$\text{In this subcase, } \sum_{t=i}^{j-1} d(w_t, w_{t+1}) \leq p(l+k) + [(m+k-3)-(l+k)] + [(n+l-2)-(l+k)]$$
$$+ 2[(m+n-2)-(l+k)]$$
$$= p(l+k) + (l-3) + (k-2) + 2(l+k-2)$$
$$= p(l+k) + 3l + 3k - 9. \tag{89}$$

$$\implies f(w_j) - f(w_i) = p(m+n-1) + 2 - \sum_{t=i}^{j-1} d(w_t, w_{t+1})$$
$$= (m+n-1) + (p-1)(m+n-1) + 2 - \sum_{t=i}^{j-1} d(w_t, w_{t+1})$$
$$\geq (m+n-1) + (p-1)(m+n-1) + 2 - [p(l+k) + 3(l+k-3)]$$
$$= (m+n-1) + pl + pk - p - m - n - 3l - 3k + 12$$
$$= (m+n-1) + (p-3)(l+k) - 2(l+k) - p + 12$$
$$= (m+n-1) + (l+k)(p-5) - p + 12$$
$$\geq (m+n-1) + 5(p-5) - p + 12 \quad \text{since } l \geq 3 \text{ and } p \geq mn-3 \geq 21 > 5$$
$$= (m+n-1) + 4p - 13$$
$$\geq (m+n-1) + 71$$
$$> m+n-1-d(w_j, w_i). \tag{90}$$

Subcase 3b: $i > 2$ or $j < mn-1$.

$$\text{In this subcase, } \sum_{t=i}^{j-1} d(w_t, w_{t+1}) \leq p(l+k) + [(m+k-3)-(l+k)] + [(n+l-2)-(l+k)]$$
$$+ [(m+n-2)-(l+k)]$$
$$= p(l+k) + (l-3) + (k-2) + (l+k-2)$$
$$= p(l+k) + m + n - 7. \tag{91}$$

$$
\begin{aligned}
\implies f(w_j) - f(w_i) &\geq p(m+n-1) - \sum_{t=i}^{j-1} d(w_t, w_{t+1}) \\
&= (m+n-1) + (p-1)(m+n-1) - \sum_{t=i}^{j-1} d(w_t, w_{t+1}) \\
&\geq (m+n-1) + (p-1)(m+n-1) - [p(l+k) + m + n - 7] \\
&= (m+n-1) + pm + pn - p - 2m - 2n - pl - pk + 8 \\
&= (m+n-1) + (p-2)(m+n) - p(l+k) - p + 8 \\
&= (m+n-1) + (2p-4)(l+k) - p(l+k) - p + 8 \\
&= (m+n-1) + (p-4)(l+k) - p + 8.
\end{aligned} \tag{92}
$$

1. If $p = 4$, then $f(w_j) - f(w_i) \geq (m+n-1) + 0 + 4 > m + n - 1 - d(w_j, w_i)$.
2. If $p > 4$,then $f(w_j) - f(w_i) \geq (m+n-1) + (p-4)(5) - p + 8 = (m+n-1) + 4p - 12 > m + n - 1 - d(w_j, w_i)$.

Hence, $f(w_j) - f(w_i) \geq m + n - 1 - d(w_j, w_i)$ whenever $j = i + p$, where $p \geq 4$.

By Cases 1, 2, and 3, we know that f is a radio labeling of G, proving our claim.

From direct calculation, there are five possible distances between consecutively labeled vertices, namely $l + k$, $l + k - 1$, $m + k - 3$, $n + l - 2$, and $m + n - 2$, with the number of occurrences $mn - m - n + 1$, $m + n - 6$, 1, 1, and 2, respectively.

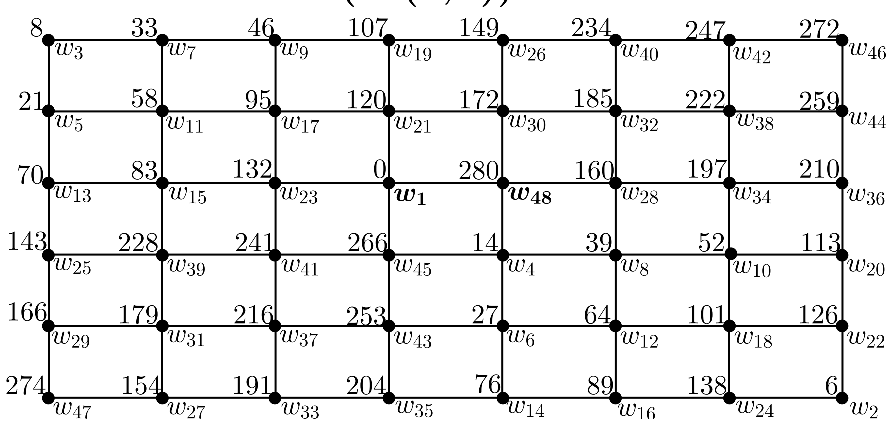
So $span(f) = (mn-1)(m+n-1) - \sum_{i=1}^{mn-1} d(w_i, w_{i+1})$

$$\begin{aligned}
&= (mn-1)(m+n-1) + 2 - [(l+k)(mn-m-n+1) + (l+k-1)(m+n-6) + (m+k-3) \\
&\quad + (n+l-2) + (m+n-2)(2)] \\
&= (mn-1)(m+n-1) + 2 - \left[\left(\frac{m+n}{2}\right)(mn-m-n+1) + \left(\frac{m+n-2}{2}\right)(m+n-6)\right] \\
&\quad - \left[\left(\frac{2m+n-6}{2}\right) + \left(\frac{2n+m-4}{2}\right) + \left(\frac{4m+4n-8}{2}\right)\right] \\
&= (mn-1)(m+n-1) + 2 - \frac{1}{2}[(m+n)(mn-m-n+1) + (m+n-2)(m+n-6)] \\
&\quad - \frac{1}{2}(7n+7m-18) \\
&= (mn-1)(m+n-1) + 2 - \frac{1}{2}[(m^2n+mn^2-7m-7n+12) + (7m+7n-18)] \\
&= (m^2n+mn^2-mn-m-n+1) + 2 - \frac{1}{2}(m^2n+mn^2-6) \\
&= \frac{1}{2}(m^2n+mn^2-2mn-2m-2n) + 6.
\end{aligned} \tag{93}$$

Therefore, $rn(G) \leq \frac{1}{2}(m^2n+mn^2-2mn-2m-2n)+6$, since f is a radio labeling of G with this span.

□

$rn(M(8,6)) = 280$



Above is the optimal radio labeling determined by the labeling pattern described above for $M(8.6)$. The radio number of $M(8,6)$ is 280.

Theorem 5.4. *Let $G = M(m,n)$, where $m,n \geq 3$. Then*

$$rn(G) = \begin{cases} \frac{1}{2}(m^2n + mn^2 - 2mn - m - 2n) + 2 & \text{if } m \text{ is even and } n \text{ is odd;} \\ \frac{1}{2}(m^2n + mn^2 - 2mn - m - n) + 2 & \text{if } m,n \text{ are odd;} \\ \frac{1}{2}(m^2n + mn^2 - 2mn - 2m - 2n) + 6 & \text{if } m,n \text{ are even.} \end{cases}$$

Proof. This result follows immediately from Lemmas 4.2, 4.4, 4.7, 5.1, 5.2, and 5.3. □

6 A Survey of Relevant Results - Trees

Other results by prominent researchers in the calculation of the radio number of various classes of graphs have both motivated and concurred with the work of this thesis. Such results in trees more notably include the radio number of spiders and level-wise regular trees.

6.1 Radio Number of Spiders

Definition 6.1. A *tree* is a connected acyclic graph.

The difficulty of determining the radio number of general trees is that its definition allows too many variations in its structure. As seen in the proof for the lower bound of the radio number for grids, determining a lower bound of the radio number for a graph requires calculating both its diameter and the distances between consecutively labeled vertices of an optimal radio labeling, all of which require more restrictions in the structure of the graph. Thus, we focus on specific types of trees whose structures are far easier to characterize.

Definition 6.2. A *spider* is a tree T that contains at most one vertex of degree greater than or equal to 3, called the *center* of T.

- If $deg(v) \leq 2$ for all $v \in V(T)$, then T is a *path*, and the center of T is the middle vertex of T if the order of T is odd and either of the middle vertices if the order of T is even.
- A *leg* of a spider is a path whose ends are the center and a leaf (a degree-one vertex) of the spider. A spider with m legs (given $m \geq 2$) is denoted by $S_{l_1,l_2,...,l_m}$, where $l_i \in \mathbb{N}$ is the length of the i^{th} leg such that $l_1 \geq l_2 \geq ... \geq l_m$. The j^{th} vertex of the i^{th} leg is denoted $v_{i,j}$, the center is appropriately denoted $v_{0,0}$, and the vertex set of the i^{th} leg is denoted V_i.
- The *level* of a vertex v, denoted $L(v)$, is the distance from the center of T to v. So $L(v_{i,j}) = j$, where $1 \leq i \leq m$ and $0 \leq j \leq l_i$.

Observation 6.1. *From this definition, the following must be true for* $T = S_{l_1,l_2,...,l_m}$.

1. $diam(T) = l_1 + l_2$.
2. $V(T) = V_1 \cup V_2 \cup ... \cup V_m$.
3. $\{v_{0,0}\} = V_1 \cap V_2 \cap ... \cap V_m$. *In fact,* $V_i \cap V_j = \{v_{0,0}\}$ *if* $1 \leq i < j \leq m$.

4. $|V(T)| = l_1 + l_2 + ... + l_m + 1$.

In order to optimize the sum of distances between consecutive vertices in a radio labeling f, we also note that the distance between two vertices $v_{i,j}$ and $v_{i',j'}$ can be characterized as follows.

$$d(v_{i,j}, v_{i',j'}) = \begin{cases} j + j' & if\ i \neq i'; \\ |j - j'| & if\ i = i'. \end{cases}$$

Liu's [4] initial strategy of ordering the vertices by increasing labels and using the summation of $n-1$ inequalities (where $n = |V(G)|$) as the span of a radio labeling f was used in this thesis to prove the general lower bound of the radio number for general grid graphs. This strategy is also used to find a general lower bound of the radio number of various other graphs that are currently investigated, including paths and cycles in [6].

Also, in this thesis the preliminary concepts of *level* and *displacement* of vertices in a grid were both inspired by Liu's concept of the level of vertices in a spider (and a rooted tree in general). Level and displacement were used in this thesis as an integral element of the proof of the lower bound for the radio number of grids. These similar concepts characterize the distance from a vertex to the "middle" of the graph in order to calculate and optimize the sum of the distances between all $|V(G)| - 1$ pairs of consecutively labeled vertices.

To determine a general lower bound of the radio number for spiders and identify spiders that meet this bound, Liu introduced some special notations and a sequence of short lemmas, which we now prove in greater detail. Our arguments differ slightly in some areas, but the structure of the proofs is consistent with the original source.

Definition 6.3. Let f be a radio labeling of G with the vertex ordering given by $0 = f(u_0) < f(u_1) < ... < f(u_{n-1})$. Then for $0 \leq i \leq n-2$,

$$x_i = f(u_{i+1}) - f(u_i) + L(u_{i+1}) + L(u_i) - diam(G) - 1$$

where $L(v_{i,j}) = j$. u_i and u_{i+1} are *consecutive vertices.*

Observation 6.2. *The following are true for* x_i.

1. x_i *must be non-negative, since by definition* $f(u_{i+1}) - f(u_i) \geq diam(G) + 1 - d(u_{i+1}, u_i) \geq diam(G) + 1 - [L(u_{i+1}) + L(u_i)]$.
2. x_i *measures a surplus of distances between consecutive vertices for a radio labeling* f. *If* $x_i = 0$ *and* $d(u_i, u_{i+1}) = L(u_i) + L(u_{i+1})$, *then the equality holds and* u_i *and* u_{i+1} *are on different legs.*
3. *If* u_i *and* u_{i+1} *are on the same leg, then* $x_i \geq 2Min\{L(u_{i+1}), L(u_i)\}$.

Lemma 6.1. *Let* $G = S_{l_1,\ldots l_m}$. *Say* $f : V(G) \to \{0, 1, 2, ...\}$ *satisfies* $f(u_0) < f(u_1) < ... < f(u_{n-1})$. *Then* f *is a radio labeling of* G *if and only if the following two statements hold for every set* $\{u_i, u_{i+1}, ..., u_j\}$ *of consecutive vertices, where* $0 \leq i < j \leq n - 1$.

1. $\sum_{t=i}^{j-1} x_t \geq 2\left(\sum_{t=i+1}^{j-1} L(u_t)\right) - (j - i - 1)(l_1 + l_2 + 1)$.
2. *If* u_i *and* u_j *are on the same leg, then*

$$\sum_{t=i}^{j-1} x_t \geq 2\left(\sum_{t=i+1}^{j-1} L(u_t)\right) - (j - i - 1)(l_1 + l_2 + 1) + 2Min\{L(u_i), L(u_j)\}.$$

Proof. Let $G = S_{l_1,\ldots l_m}$. Say $f : V(G) \to \{0, 1, 2, ...\}$ satisfies $f(u_0) < f(u_1) < ... < f(u_{n-1})$. $(\Longrightarrow)$: Suppose f is a radio labeling, so $f(u_{i+1}) - f(u_i) \geq diam(G) + 1 - d(u_{i+1}, u_i)$ for all $0 \leq i \leq n - 2$. Thus, we have the following $j - i$ equations.

$$\begin{cases} x_i &= f(u_{i+1}) - f(u_i) + L(u_{i+1}) + L(u_i) - diam(G) - 1 \\ x_{i+1} &= f(u_{i+2}) - f(u_{i+1}) + L(u_{i+2}) + L(u_{i+1}) - diam(G) - 1 \\ &\vdots \\ x_{j-2} &= f(u_{j-1}) - f(u_{j-2}) + L(u_{j-1}) + L(u_{j-2}) - diam(G) - 1 \\ x_{j-1} &= f(u_j) - f(u_{j-1}) + L(u_j) + L(u_{j-1}) - diam(G) - 1 \end{cases}$$

$$\text{Hence,} \quad \sum_{t=i}^{j-1} x_t = f(u_j) - f(u_i) + 2\left(\sum_{t=i+1}^{j-1} L(u_t)\right) + L(u_j) + l(u_i) - (j-i)(l_1+l_2+1)$$
$$\geq (l_1+l_2+1) - L(u_i) - L(u_j) + 2\left(\sum_{t=i+1}^{j-1} L(u_t)\right) + L(u_i) + L(u_j) - (j-i)(l_1+l_2+1)$$
$$= 2\left(\sum_{t=i+1}^{j-1} L(u_t)\right) - (j-i-1)(l_1+l_2+1), \text{ which proves (1).} \tag{94}$$

Now if u_i and u_j are on the same leg, then $d(u_i,u_j) = Max\{L(u_i), L(u_j)\} - Min\{L(u_i), L(u_j)\}$.

$$\text{Therefore,} \sum_{t=i}^{j-1} x_t = f(u_j) - f(u_i) + 2\left(\sum_{t=i+1}^{j-1} L(u_t)\right) + L(u_j) + L(u_i) - (j-i)(l_1+l_2+1)$$
$$\geq (l_1+l_2+1) - Max\{L(u_i), L(u_j)\} + Min\{L(u_i), L(u_j)\} + 2\left(\sum_{t=i+1}^{j-1} L(u_t)\right)$$
$$+ L(u_i) + L(u_j) - (j-i)(l_1+l_2+1)$$
$$= 2\left(\sum_{t=i+1}^{j-1} L(u_t)\right) - (j-i-1)(l_1+l_2+1) + 2Min\{L(u_i), L(u_j)\}, \text{ which proves (2).} \tag{95}$$

<u>($\Leftarrow$)</u>: Suppose (1) and (2) hold for every set $\{u_i, u_{i+1}, ..., u_j\}$ of consecutive vertices, where $0 \leq i < j \leq n-1$. We must verify the inequality in the definition of radio labelings by observing two separate cases.

<u>**Case 1:**</u> u_i and u_j are on different legs, so $d(u_i,u_j) = L(u_i) + L(u_j)$. By (1), we have

$$f(u_j) - f(u_i) = \sum_{t=i}^{j-1} x_t - 2\left(\sum_{t=i+1}^{j-1} L(u_t)\right) - L(u_j) - L(u_i) + (j-i)(l_1+l_2+1)$$
$$\geq 2\left(\sum_{t=i+1}^{j-1} L(u_t)\right) - (j-i-1)(l_1+l_2+1) - 2\left(\sum_{t=i+1}^{j-1} L(u_t)\right) - L(u_i) - L(u_j)$$
$$+ (j-i)(l_1+l_2+1)$$
$$= (l_1+l_2+1) - [L(u_i) + L(u_j)] = diam(G) + 1 - d(u_i,u_j). \tag{96}$$

<u>**Case 2:**</u> u_i and u_j are on the same leg, so $d(u_i,u_j) = Max\{L(u_i), L(u_j)\} - Min\{L(u_i), L(u_j)\}$.

Then by (2),

$$\begin{aligned} f(u_j)-f(u_i) &= \sum_{t=i}^{j-1} x_t - 2\left(\sum_{t=i+1}^{j-1} L(u_t)\right) - L(u_j) - L(u_i) + (j-i)(l_1+l_2+1) \\ &\geq 2\left(\sum_{t=i+1}^{j-1} L(u_t)\right) - (j-i-1)(l_1+l_2+1) + 2Min\{L(u_i), L(u_j)\} \\ &\quad + 2\left(\sum_{t=i+1}^{j-1} L(u_t)\right) - L(u_i) - L(u_j) + (j-i)(l_1+l_2+1) \\ &= (l_1+l_2+1) - Max\{L(u_i), L(u_j)\} + Min\{L(u_i), L(u_j)\} \\ &= diam(G) + 1 - d(u_i, u_j). \end{aligned} \tag{97}$$

Thus, f is a radio labeling, which completes our proof of Lemma 6.1. ☐

Notation: Let $G = S_{l_1,l_2,...,l_m}$ such that $l_1 - l_2 \geq 2$.

1. Let $z = \lfloor \frac{l_1-l_2-2}{2} \rfloor$.
2. Let f be a radio labeling of G. For $0 \leq j \leq z$, let t_j be the integer such that $u_{t_j} = v_{1,l_1-j}$, where u_{t_j} denotes the ${t_j}^{th}$ vertex in the labeling sequence for f.

Lemma 6.2. *Let f be a radio labeling for $G = S_{l_1,l_2,...,l_m}$ such that $l_1 - l_2 \geq 2$. Let $n = |V(G)|$.*

1. *If $1 \leq t_j \leq n-2$ for some $j = 0, 1, 2, ..., z$, then*

$$x_{t_j-1} + x_{t_j} \geq l_1 - l_2 - (2j+1) \geq 1.$$

2. *If the first equality holds, then u_{t_j-1} and u_{t_j+1} are on different legs of G, unless one of them is $v_{0,0}$.*

Proof. Suppose $u_{t_j} = v_{1,l_1-j}$ (where $j = 0, 1, 2, ..., z$) for some $1 \leq t_j \leq n-2$. Consider the set $\{u_{t_j-1}, u_{t_j}, u_{t_j+1}\}$ of consecutive vertices. Then $L(u_{t_j}) = l_1 - j$. By Lemma 6.1 and considering

$0 \leq j \leq z = \lfloor \frac{l_1 - l_2 - 2}{2} \rfloor \leq \left(\frac{l_1 - l_2 - 2}{2}\right)$ and $l_1 - l_2 \geq 2$, we obtain the following.

$$\begin{aligned} x_{t_j - 1} + x_{t_j} &\geq 2\left(L(u_{t_j})\right) - [(t_j + 1) - (t_j - 1) - 1](l_1 + l_2 + 1) \\ &= 2(l_1 - j) - (1)(l_1 + l_2 + 1) \\ &= l_1 - l_2 - 2j - 1 \\ &\geq l_1 - l_2 - 2\left(\frac{l_1 - l_2 - 2}{2}\right) - 1 \\ &\geq l_1 - l_2 - (l_1 - l_2 - 2) - 1 = 1. \end{aligned} \tag{98}$$

To prove the second statement, we argue by contraposition. Assume that $u_{t_j - 1}$ and $u_{t_j + 1}$ are on the same leg and that neither of them are the center. Then by Lemma 6.1, we have

$$\begin{aligned} x_{t_j - 1} + x_{t_j} &\geq 2\left(L(u_{t_j})\right) - [(t_j + 1) - (t_j - 1) - 1](l_1 + l_2 + 1) + 2Min\{L(u_{t_j - 1}), L(u_{t_j + 1})\} \\ &> 2(l_1 - j) - (1)(l_1 + l_2 + 1) \\ &= l_1 - l_2 - 2j - 1. \end{aligned} \tag{99}$$

The above inequality is strict because by assumption neither $u_{t_j + 1}$ nor $u_{t_j - 1}$ can be the center, so $Min\{L(u_{t_j - 1}), L(u_{t_j + 1})\}$ must be positive. Thus, the proof for Lemma 6.2 is complete. □

Lemma 6.3. *If there exist j, j' such that $t_{j'} = t_j + 1$ (in other words, $u_{t_{j'}} = u_{t_j + 1}$, so $v_{1, l_1 - j}$ and $v_{1, l_1 - j'}$ are consecutive), then*

$$x_{t_j} > 2(l_1 - l_2 - j' - j - 1).$$

Proof. Assume that j, j' are integers such that $u_{t_{j'}} = u_{t_j + 1}$. Since u_{t_j} and $u_{t_j + 1}$ are both on the longest leg, we have by Lemma 6.1

$$\begin{aligned} x_{t_j} &\geq 2\left(\sum_{t = t_j + 1}^{t_j} L(u_t)\right) - [(t_j + 1) - t_j - 1](l_1 + l_2 + 1) + 2Min\{L(u_{t_j}), L(u_{t_j + 1})\} \\ &= 2Min\{L(u_{t_j}), L(u_{t_j + 1})\} \\ &= 2Min\{l_1 - j, l_1 - j'\} \\ &\geq 2(l_1 - j - j' - l_2) \\ &> 2(l_1 - j - j' - l_2 - 1). \end{aligned} \tag{100}$$

The second inequality holds because $l_1 - j \geq l_1 - j - j'$ and $l_1 - j' \geq l_1 - j - j'$. This completes the proof of Lemma 6.3. $\square$

Lemma 6.4. *Let f be a radio labeling of $G = S_{l_1,l_2,...,l_m}$, where $n = |V(G)|$.*

1. *If one of the following two statements holds*
   (a) *$l_1 - l_2 \leq 1$; or*
   (b) *$l_1 - l_2 \geq 2$ and $1 \leq t_j \leq n-2$ for all $j = 1, 2, ..., z$, then we have*
$$\sum_{i=0}^{n-2} x_i \geq \left\lceil \frac{l_1 - l_2}{2} \right\rceil \left\lfloor \frac{l_1 - l_2}{2} \right\rfloor.$$
2. *If $l_1 - l_2 \geq 2$ and equality holds, then the following three statements must be true:*
   (a) *For all $1 \leq j \leq z$, u_{t_j-1} and u_{t_j+1} are on different legs unless one of them is the center.*
   (b) *For all $0 \leq j < j' \leq z$, v_{1,l_1-j} and $v_{1,l_1-j'}$ are not consecutive vertices.*
   (c) *If $i \notin \{t_j : j = 0, 1, 2, ..., z\} \cup \{t_j - 1 : j = 0, 1, 2, ..., z\}$, then $x_i = 0$.*

Proof. If $l_1 - l_2 \leq 1$, then $\left\lfloor \frac{l_1-l_2}{2} \right\rfloor = 0$, so the result is trivially true (since $x_i \geq 0$ for all i). So we assume that $l_1 - l_2 \geq 2$ and $1 \leq t_j \leq n-2$ for all $j = 1, 2, ..., z$. Let $A = \{t_j : j = 0, 1, 2, ..., z\}$. and $B = \{t_j - 1 : j = 0, 1, 2, ..., z\}$. By Lemma 6.2 and 6.3 (and calculating arithmetic series), we have

$$\begin{aligned}
\sum_{i=0}^{n-2} x_i &= \sum_{t \in A\cup B} x_t + \sum_{t \notin A\cup B} x_t \geq \sum_{t \in A\cup B} x_t \\
&\geq (l_1 - l_2)(z+1) - \sum_{j=0}^{z}(2j+1) \\
&= (l_1 - l_2)(z+1) - [1 + 3 + 5 + ... + (1+2z)] \\
&= (l_1 - l_2)(z+1) - \frac{[1 + (2z+1)](z+1)}{2} \\
&= (l_1 - l_2)(z+1) - (z+1)^2 \\
&= (z+1)(l_1 - l_2 - z - 1) \\
&= \left(\left\lfloor \frac{l_1 - l_2 - 2}{2} \right\rfloor + 1\right)\left(l_1 - l_2 - 1 - \left\lfloor \frac{l_1 - l_2 - 2}{2} \right\rfloor\right) \\
&= \left\lfloor \frac{l_1 - l_2}{2} \right\rfloor \left(\frac{2l_1 - 2l_2 - 2}{2} - \left\lfloor \frac{l_1 - l_2 - 2}{2} \right\rfloor\right) \\
&= \left\lfloor \frac{l_1 - l_2}{2} \right\rfloor \left\lceil \frac{l_1 - l_2}{2} \right\rceil.
\end{aligned} \tag{101}$$

This proves the Part 1 of Lemma 6.4. To prove Part 2, notice that if $l_1 - l_2 \geq 2$ and equality holds, then (a) follows immediately from Lemma 6.2 and (b) follows immediately from the contrapositive of Lemma 6.3. Since x_i is non-negative for all i, if for any $i \notin A \cup B$ we have $x_i > 0$, then $\sum_{t \notin A \cup B} x_t > 0$, which would contradict the assumed equality.

Thus our proof of Lemma 6.4 is complete. □

Lemma 6.5. *Let f be a radio labeling of a spider $G = S_{l_1, l_2, \ldots, l_m}$ with the vertex ordering given by $0 = f(u_0) < f(u_1) < \ldots < f(u_{n-1})$. Then*

$$2\left(\sum_{i=1}^{n-2} L(u_i)\right) + L(u_0) + L(u_{n-1}) \leq \sum_{k=1}^{m} l_k(l_k+1) - 1.$$

Also, the equality holds if and only if $\{u_0, u_{n-1}\} = \{v_{0,0}, v_{t,1}\}$ for some $1 \leq t \leq m$ (in other words, the first and last vertices in the labeling sequence are the center and one of its neighbors).

Proof. Let f be a radio labeling of a spider $G = S_{l_1, l_2, \ldots, l_m}$ with the vertex ordering given by $0 = f(u_0) < f(u_1) < \ldots < f(u_{n-1})$. Then

$$\begin{aligned} 2\left(\sum_{i=1}^{n-2} L(u_i)\right) + L(u_0) + L(u_{n-1}) &= 2\left(\sum_{i=0}^{n-1} L(u_i)\right) - L(u_0) - L(u_{n-1}) \\ &= 2\left(\sum_{k=1}^{m}(1+2+3+\ldots+l_k)\right) - L(u_0) - L(u_{n-1}) \\ &= 2\left(\sum_{k=1}^{m}\left[\frac{l_k(l_k+1)}{2}\right]\right) - L(u_0) - L(u_{n-1}) \qquad (102) \\ &= \sum_{k=1}^{m} l_k(l_k+1) - [L(u_0) + L(u_{n-1})] \\ &\leq \sum_{k=1}^{m} l_k(l_k+1) - 1. \end{aligned}$$

The final inequality holds because $L(u_0) + L(u_{n-1}) \geq 0 + 1 = 1$. Note that the equality holds if and only if $\{L(u_0), L(u_{n-1})\} = \{0, 1\}$ if and only if $\{u_0, u_{n-1}\} = \{v_{0,0}, v_{t,1}\}$ for some $1 \leq t \leq m$ (in other words, the first and last vertices in the labeling sequence of f are the center and one of its neighbors). This completes the proof for Lemma 6.5. □

With Lemmas 6.1, 6.2, 6.3, 6.4, 6.5, we prove a comprehensive theorem for determining a lower bound of the radio numbers of spiders and characterizing radio labelings whose span match this bound.

Recall: Let $G = S_{l_1,l_2,\ldots,l_m}$, where $n = |V(G)|$.

1. Given a radio labeling f of G, we assume that $f(u_0) = 0$ and $span(f) = f(u_{n-1})$.
2. $diam(G) = l_1 + l_2$.
3. $n = 1 + \sum_{k=1}^{m} l_k$.

Theorem 6.6. *Let $G = S_{l_1,l_2,\ldots,l_m}$, where $n = |V(G)|$. Let f be a radio labeling of G. The following statements must hold.*

1. $rn(G) \geq \sum_{k=1}^{m} l_k(l_1 + l_2 - l_k) + \lceil \frac{l_1-l_2}{2} \rceil \lfloor \frac{l_1-l_2}{2} \rfloor + 1$.
2. *$span(f)$ equals this bound if and only if all the following statements are true.*

   (a) $\{u_0, u_{n-1}\} = \{v_{0,0}, v_{s,1}\}$ *for some* $1 \leq s \leq m$ *(in other words, the first and last vertices in the labeling sequence of f are the center of G and one of its neighbors).*

   (b) *If* $l_1 - l_2 \geq 2$, *then* $1 \leq t_j \leq n-2$ *for all* $0 \leq j \leq z$ *(in other words, neither first nor the last vertex in the labeling sequence of f are within the set* $\{v_{1,l_1}, v_{1,l_1-1}, \ldots, v_{1,l_1-z}\}$*).*

   (c) *If* $l_1 - l_2 \geq 2$, *then,* $x_{t_j-1} + x_{t_j} = l_1 - l_2 - (2j+1)$ *for all* $0 \leq j \leq z$.

   (d) *If* $l_1 - l_2 \geq 2$, *then for any* $0 \leq j \leq z$, u_{t_j-1} *and* u_{t_j+1} *belong to different legs, unless one of them is the center.*

   (e) *If* $l_1 - l_2 \leq 1$, *then* $x_i = 0$ *for all* $0 \leq i \leq n-2$. *If* $l_1 - l_2 \geq 2$, *then* $x_i = 0$ *if* $i \notin \{t_j : j = 0, 1, \ldots, z\} \cup \{t_j - 1 : j = 0, 1, \ldots, z\}$.

Proof. Let f be a radio labeling of $G = S_{l_1,l_2,\ldots,l_m}$, where $n = |V(G)|$. To prove the first statement, we write the following $n-1$ equations by the definition of x_i.

$$\begin{cases} f(u_{n-1}) - f(u_{n-2}) &= (l_1 + l_2 + 1) - L(u_{n-1}) - L(u_{n-2}) + x_{n-2} \\ f(u_{n-2}) - f(u_{n-3}) &= (l_1 + l_2 + 1) - L(u_{n-2}) - L(u_{n-3}) + x_{n-3} \\ &\vdots \\ f(u_1) - f(u_0) &= (l_1 + l_2 + 1) - L(u_1) - L(u_0) + x_0 \end{cases}$$

From summing up these $n-1$ equations and applying Lemmas 6.4 and 6.5, we have

$$\begin{aligned} span(f) &= (l_1+l_2+1)(n-1) - 2\left(\sum_{i=1}^{n-2} L(u_i)\right) - L(u_0) - L(u_{n-1}) + \sum_{i=0}^{n-2} x_i \\ &\geq (l_1+l_2+1)\sum_{k=1}^{m} l_k - \sum_{k=1}^{m} l_k(l_k+1) + 1 + \left\lceil \frac{l_1-l_2}{2} \right\rceil \left\lfloor \frac{l_1-l_2}{2} \right\rfloor \\ &= \sum_{k=1}^{m} [l_k(l_1+l_2+1) - l_k(l_k+1)] + 1 + \left\lceil \frac{l_1-l_2}{2} \right\rceil \left\lfloor \frac{l_1-l_2}{2} \right\rfloor \\ &= \sum_{k=1}^{m} l_k(l_1+l_2-l_k) + \left\lceil \frac{l_1-l_2}{2} \right\rceil \left\lfloor \frac{l_1-l_2}{2} \right\rfloor + 1. \end{aligned} \tag{103}$$

Notice that the second equality holds if and only if $(a),(c),(d)$, and (e) all hold by Lemmas 6.2, 6.4, and 6.5. To show that (b) is necessary for equality, we prove the following claim.

CLAIM: If $t_j = 0$ or $t_j = n-1$ for some $j = 0,1,...,z$, then $span(f) > \sum_{k=1}^{m} l_k(l_1+l_2-l_k) + \lceil \frac{l_1-l_2}{2} \rceil \lfloor \frac{l_1-l_2}{2} \rfloor + 1$.

Proof: We observe two cases.

<u>CASE 1</u>: There is exactly one value $0 \leq j \leq z$ such that $u_{t_j} \in \{u_0, u_{n-1}\}$. Then j fails to satisfy the conditions of Lemma 6.2, so from the proof of of Lemma 6.4, we know

$$\sum_{i=0}^{n-2} x_i \geq \left\lceil \frac{l_1-l_2}{2} \right\rceil \left\lfloor \frac{l_1-l_2}{2} \right\rfloor - (l_1-l_2-2j-1).$$

Also, since $L(u_{t_j}) = l_1 - j$, we know that $L(u_0) + L(u_{n-1}) \geq l_1 - j$. Therefore,

$$\begin{aligned} span(f) &= (l_1+l_2+1)(n-1) - 2\left(\sum_{i=1}^{n-2} L(u_i)\right) - L(u_0) - L(u_{n-1}) + \sum_{i=0}^{n-2} x_i \\ &\geq (l_1+l_2+1)\sum_{k=1}^{m} l_k - \sum_{k=1}^{m} l_k(l_k+1) + (l_1-j) + \left\lceil \frac{l_1-l_2}{2} \right\rceil \left\lfloor \frac{l_1-l_2}{2} \right\rfloor - (l_1-l_2-2j-1) \\ &> \sum_{k=1}^{m} l_k(l_1+l_2-l_k) + \left\lceil \frac{l_1-l_2}{2} \right\rceil \left\lfloor \frac{l_1-l_2}{2} \right\rfloor + 1. \end{aligned} \tag{104}$$

<u>CASE 2</u>: There exist $0 \leq j < j' \leq z$ such that $\{u_{t_j}, u_{t_{j'}}\} = \{u_0, u_{n-1}\}$. Then j and j' fail to

satisfy the conditions of Lemma 6.2, so from the proof of of Lemma 6.4, we know

$$\sum_{i=0}^{n-2} x_i \geq \left\lceil \frac{l_1 - l_2}{2} \right\rceil \left\lfloor \frac{l_1 - l_2}{2} \right\rfloor - 2(l_1 - l_2 - j - j' - 1).$$

Also, since $L(u_{t_j}) = l_1 - j$ and $L(u_{t_{j'}}) = l_1 - j'$, we know that $L(u_0) + L(u_{n-1}) \geq (l_1 - j) + (l_1 - j') = 2l_1 - j - j'$. Therefore,

$$\begin{aligned} span(f) &= (l_1 + l_2 + 1)(n-1) - 2\left(\sum_{i=1}^{n-2} L(u_i)\right) - L(u_0) - L(u_{n-1}) + \sum_{i=0}^{n-2} x_i \\ &\geq (l_1 + l_2 + 1)\sum_{k=1}^{m} l_k - \sum_{k=1}^{m} l_k(l_k + 1) + (2l_1 - j - j') + \left\lceil \frac{l_1 - l_2}{2} \right\rceil \left\lfloor \frac{l_1 - l_2}{2} \right\rfloor \\ &\quad - 2(l_1 - l_2 - j - j' - 1) \\ &> \sum_{k=1}^{m} l_k(l_1 + l_2 - l_k) + \left\lceil \frac{l_1 - l_2}{2} \right\rceil \left\lfloor \frac{l_1 - l_2}{2} \right\rfloor + 1, \text{ proving the claim.} \end{aligned} \tag{105}$$

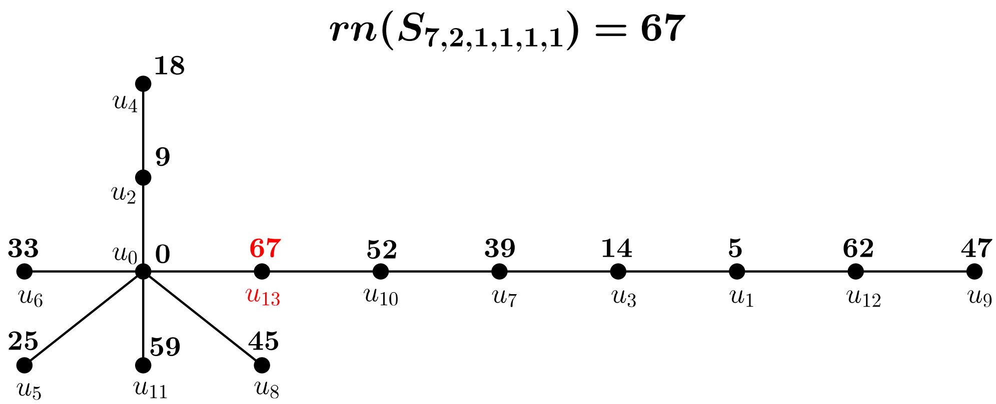
Thus, the proof for Theorem 6.6 is complete, and we have established a general lower bound for the radio number of spiders and characterized radio labelings that achieve this bound. □

Liu's result [4] for the radio number of spiders is as follows.

Theorem 6.7. *Let $G = S_{l_1, l_2, \ldots, l_m}$ be a spider. If $l_1 = l_2$, then*

$$rn(G) = \begin{cases} \sum_{i=1}^{m} l_i(2l_1 - l_i) + 1 & if\ m \geq 3\ or\ l_1 = 1; \\ \sum_{i=1}^{m} l_i(2l_1 - l_i) + 2 & otherwise. \end{cases}$$

If $l_1 > l_2$, then $rn(G) = \sum_{i=1}^{m} l_i(l_1 + l_2 - l_i) + \lceil \frac{l_1 - l_2}{2} \rceil \lfloor \frac{l_1 - l_2}{2} \rfloor + 1$ if and only if $\sum_{i=2}^{m} l_i \geq \frac{l_1 + l_2 - 1}{2}$.



Above is an optimal radio labeling for the spider $S_{7,2,1,1,1,1}$. This spider achieves the lower bound established in Theorem 6.6.

6.2 Radio Number of Level-Wise Regular Trees

A fundamental property of trees is that every tree T has a central element, depending on the parity of $diam(T)$, the length of a longest path in T.

1. If $diam(T)$ is even, then T has a *central vertex* $r \in V(T)$, and we say $L_0 = \{r\}$.
2. If $diam(T)$ is odd, then T has a *central edge* $r'r'' \in E(T)$, and we say $L_0 = \{r', r''\}$.

It is important to note that in the context of general trees T, the *center* of T is a vertex with minimum eccentricity. That is not necessarily the case for the center $v_{0,0}$ of a spider, which is defined as the unique vertex with degree at least 3 (if such a vertex exists).

Halasz and Tuza provide the following definitions [1] of several special types of trees.

Definition 6.4. Let T be a tree with order n.

1. The *level sets* of T are the sets $L_i = \{v \in V \mid d(u, L_0) = i\}$ for $1 \leq i \leq h = \lfloor \frac{1}{2} diam(T) \rfloor$, where $d(v, L_0)$ denotes the minimum distance from v to an element of L_0.

2. An *i-vertex* is a vertex $v \in L_i$, where $0 \leq i \leq h$.
3. T is *level-wise regular* if all $i-$vertices have equal degree for each $i = 0, 1, 2, ..., h$, and this common degree of i-vertices is denoted m_i.
   (a) If $diam(T) = 2h$, then $|L_0| = 1$, and we denote the tree uniquely determined by $(m_0, m_1, ..., m_{h-1})$ as $T^1_{m_0,m_1,...,m_{h-1}}$.
   (b) If $diam(T) = 2h + 1$, then $|L_0| = 2$, and we denote the tree uniquely determined by $(m_0, m_1, ..., m_{h-1})$ as $T^2_{m_0,m_1,...,m_{h-1}}$.
4. A *complete m-ary tree* is a level-wise regular tree represented by $m_0 = m$, $m_i = m + 1$ if $1 \leq i \leq h - 1$, and $m_h = 1$.
5. An *internally m-regular complete tree* is a level-wise regular tree represented by $m_i = m$ if $0 \leq i \leq h - 1$ and $m_h = 1$. So all vertices except leaves have degree m.

The radio number of complete m-ary trees was previously completely determined by Li, Mak, and Zhou [3]. The radio number of $(m + 1)$-regular complete trees, as defined above, follows from Halasz and Tuza's formula [1] for the radio number of level-wise regular trees.

To prove this formula, we introduce the following definitions.

Definition 6.5. Let G be a graph. Let $\vec{j} = (j_1, j_2, ..., j_d)$ be a d-tuple of integers such that $1 \leq d \leq diam(G)$.

1. The d^{th} *power of* G, denoted G^d, is the graph such that

$$V(G) = V(G^d) \text{ and } uv \in E(G^d) \text{ if and only if } d_G(u, v) \leq d.$$

   We denote the edge set of G^d by E^d.
2. The *weight function* $w_j : E^d \to \{j_1, j_2, ..., j_d\}$ is defined by

$$w_{\vec{j}}(u, v) = j_i \text{ if and only if } d_G(u, v) = i \quad \text{for all } uv \in E^d.$$

Powers of graphs should not be confused with the *Cartesian product* of graphs. Although the notation representations for both are identical, it is important to recognize the context under which the graph is discussed to avoid confusion or ambiguity. In this subsection, the exponential notation represents powers of graphs as defined above.

Observation 6.3. *Given a graph G with diameter d, if we set $j_i = d+1-i$ for $i = 1, 2, ..., d$ and define a weight function $w_j : E^d \to \{j_1, j_2, ..., j_d\}$ by $w_{\vec{j}}(u, v) = j_i$ if and only if $d_G(u, v) = i$ for all $uv \in E^d$, then a function $f : V(G) \to \{0, 1, 2, ...\}$ is a radio labeling if and only if it satisfies*

$$|f(u) - f(v)| \geq w_{\vec{j}}(u, v) \text{ for all } uv \in E^d.$$

.

Definition 6.6. Given an understood $\vec{j} = (j_1, j_2, ..., j_d)$ for a graph G, the *weighted d^{th} power graph* for G, denoted $G(V, E, w)$ or G^d_w, is the graph satisfying

$$V(G^d_w) = V(G)\ ,\ E(G^d_w) = E(G^d), \text{ and } w(u, v) = j_i \text{ if and only if } d_G(u, v) = i.$$

For the graph G with diameter d, we henceforth take $\vec{j} = (j_1, j_2, ..., j_d)$, where $d_G(u, v) = i$ and $j_i = d + 1 - i$ for each $1 \leq i \leq d$ and $u, v \in V(G)$.

Observation 6.4. *By taking the above definition of $\vec{j}$, G^d_w is a complete graph with edge weights $w(u, v) = d + 1 - i$, where $i = d_G(u, v)$ for all distinct $u, v \in V(G)$.*

Definition 6.7. Let G be a graph.

1. A *Hamiltonian Path* P of G is a path that spans $V(G)$, i.e. P contains all vertices of G.
2. Suppose G has diameter d, and let w be a weight function on E^d. Then the *weighted length of a path* P in G^d_w is the sum of the weights of the edges in P.

Lemma 6.8. *Let G be a graph. Then $rn(G)$ is at least as large as the minimum weighted length of a Hamiltonian Path in G^d_w, where $d = diam(G)$.*

Proof. Let G be a graph with order $n = |V(G)|$ and diameter d. Let f be a radio labeling of G, so f is one-to-one. Let G^d_w be the weighted power graph of G, so $w(u, v) = d + 1 - i$, where $i = d_G(u, v)$, for all distinct $u, v \in V(G)$. Since f is one-to-one, f induces a unique linear ordering of $V(G)$ given by

$$0 = f(u_1) < f(u_2) < ... < f(u_{n-1}) < f(u_n) = span(f).$$

From Observation 6.3, we know that $f(u_{k+1}) - f(u_k) \geq w(u_{k+1}, u_k)$ for all $1 \leq k \leq n-1$. This produces the following $n-1$ inequalities.

$$\begin{cases} f(u_n) - f(u_{n-1}) & \geq w(u_n, u_{n-1}) \\ f(u_{n-1}) - f(u_{n-2}) & \geq w(u_{n-1}, u_{n-2}) \\ & \vdots \\ f(u_3) - f(u_2) & \geq w(u_3, u_2) \\ f(u_2) - f(u_1) & \geq w(u_2, u_1) \end{cases}$$

Thus, $span(f) = f(u_n) \geq \sum_{k=1}^{n-1} w(u_{k+1}, u_k)$, where $u_1, u_2, ..., u_{n-1}, u_n$ is the Hamiltonian Path in G_w^d corresponding to the f-ordering of $V(G) = V(G_w^d)$. Therefore, by definition of radio number, $rn(G) \geq \text{Min}\left\{\sum_{t=1}^{n-1} w(v_{t+1}, v_t) : v_1, v_2, ..., v_n \text{ is a Hamiltonian Path in } G_w^d\right\}$, which completes the proof of Lemma 6.8. $\square$

It is important to remember that although a radio labeling f of G induces a Hamiltonian Path $v_1, v_2, ..., v_n$ in G_w^d that satisfies $|f(v_{k+1}) - f(v_k)| \geq w(v_{k+1}, v_k)$ for all $1 \leq k \leq n-1$, the converse is not true. In other words, given a Hamiltonian Path $v_1, v_2, ..., v_n$ in G_x^d, a function $f : V(G) \to \{0, 1, ...\}$ that satisfies $|f(v_{k+1}) - f(v_k)| \geq w_{\bar{j}}(v_{k+1}, v_k)$ for all $1 \leq k \leq n-1$ is not necessarily a radio labeling of G. This is because the Hamiltonian Path criterion only satisfies the necessary inequality for every pair of consecutive vertices and not necessarily globally for all distinct $u, v \in V(G)$. Therefore, the above equality does not always hold.

Definition 6.8. Consider a directed graph T.

1. T is a *tournament* if there is exactly one directed edge connecting any two vertices.
2. A tournament T is *transitive* if adjacency is transitive in T; in other words, if $u \backsim v$ and $v \backsim w$ in T, then $u \backsim w$ in T.

Proposition 6.1. *Let T be a tournament.*

1. *T has a Hamiltonian path.*
2. *T is transitive if and only if the Hamiltonian path in T is unique.*

Though there are several details and cases to consider in the proof of (2), a general sketch of the argument is as follows:

1. <u>(⇒)</u>: If T is transitive and yet contains two distinct Hamiltonian paths P_1 and P_2, then by transitivity the first two vertices u and v that differ in the orderings of P_1 and P_2 will necessarily be mutually adjacent, since both paths span T by definition. This violates the requirements of a tournament.
2. <u>(⇐)</u>: If T contains a unique Hamiltonian path $v_1, v_2, ..., v_n$, then if we assume there is a vertex v_m that is incident to a vertex v_k of smaller index (thus violating transitivity), then the existence of this edge creates a second Hamiltonian path in T, a contradiction.

The importance of this proposition is that there must exist a bijection between the directed Hamiltonian paths of G_w^d and the transitive orientations of G_w^d. Any transitive orientation of G_w^d therefore induces a unique Hamiltonian path.

Lemma 6.9. *Let G be a graph with diameter d. Then $rn(G)$ is equal to the smallest possible weighted length of a longest directed path taken over all transitive orientations of G_w^d.*

Proof. Let G be a graph with order n and diameter d.

1. Let $l_T := \max\left\{\sum_{i=1}^{m-1} w(u_{i+1}, u_i) : u_1, u_2, ..., u_m \text{ is a path in } T\right\}$ for each transitive orientation of T of G_w^d.
2. Let $M = \min\left\{l_T : T \text{ is a transitive orientation of } G_w^d\right\}$.

We wish to show that $rn(G) = M$.

<u>$rn(G) \leq M$</u>: Let T be a transitive orientation of G_w^d, and let $v_1, v_2, ..., v_n$ be the unique Hamiltonian path in T. We show that $rn(G) \leq l_T$. Define a map $f : V(G) \to \{0, 1, 2, ...\}$ recursively by

$$f(v_i) = \begin{cases} 0 & \text{if } i = 1; \\ d + 1 + \max\{f(v_t) - d(v_i, v_t) : 1 \leq t \leq i - 1\} & \text{if } 2 \leq i \leq n. \end{cases}$$

Notice for all $1 \leq j < i \leq n$, we have

$$d + 1 + \max\{f(v_t) - d(v_i, v_t) : 1 \leq t \leq i - 1\} \geq d + 1 + f(v_j) - d(v_i, v_j).$$

Thus, for each $1 \le j < i \le n$, by definition f satisfies $f(v_i) - f(v_j) \ge d + 1 - d(v_i, v_j)$, and so f is a radio labeling of G. In fact, for each $2 \le i \le n$ equality is achieved for some $j_i < i$, so by backtracking we obtain a monotone decreasing sequence of labels of vertices that define a directed path $v_1 = u_1, u_2, ..., u_m = v_n$ (not necessarily Hamiltonian) from v_1 to v_n. By definition of the weight function w on E^d, $w(u_{i+1}, u_i) = d + 1 - d(u_{i+1}, u_i)$ for each $1 \le i \le m - 1$. We therefore have the following equations by the definition of f.

$$\begin{cases} f(u_m) - f(u_{m-1}) & = w(u_m, u_{m-1}) \\ f(u_{m-1}) - f(u_{m-2}) & = w(u_{m-1}, u_{m-2}) \\ & \vdots \\ f(u_3) - f(u_2) & = w(u_3, u_2) \\ f(u_2) - f(u_1) & = w(u_2, u_1) \end{cases}$$

The weighted length of $u_1, u_2, ..., u_m$ is $\sum_{i=1}^{m-1} w(u_{i+1}, u_i) = f(u_m) = f(v_n)$, so we have $rn(G) \le f(v_n) \le l_T$. Since T is an arbitrary transitive orientation of G_w^d, we have $rn(G) \le M$.

$\underline{rn(G) \ge M}$: Let $f : V(G) \to \{0, 1, 2, ..., rn(G)\}$ be an optimal radio labeling of G, so $span(f) = rn(G)$. Then f is necessarily one-to-one, and we can uniquely order the vertices $v_1, v_2, ..., v_n$ such that

$$0 = f(v_1) < f(v_2) < ... < f(v_n) = rn(G).$$

We construct a tournament by orienting the edges of G_w^d from smaller index to larger index, so $v_j \backsim v_i$ if and only if $1 \le j < i \le n$. Since inequality is a transitive relation on natural numbers, we know that adjacency so defined is a transitive relation on the vertices of G_w^d. Thus, this orientation T of G_w^d is transitive and by definition induces the directed Hamiltonian path $v_1, v_2, ..., v_n$.

Now let P be a heaviest weighted directed path $u_1, u_2, ..., u_m$ in T (not necessarily Hamiltonian), so $l_T = \sum_{i=1}^{m-1} w(u_{i+1}, u_i)$. Since f is a radio labeling of G by assumption, we have $f(u_{i+1}) - f(u_i) \ge d + 1 - d(u_{i+1}, u_i) = w(u_{i+1}, u_i)$ for all $1 \le i \le m - 1$. We therefore have the following system of inequalities.

$$\begin{cases} f(u_m) - f(u_{m-1}) & \geq w(u_m, u_{m-1}) \\ f(u_{m-1}) - f(u_{m-2}) & \geq w(u_{m-1}, u_{m-2}) \\ & \vdots \\ f(u_3) - f(u_2) & \geq w(u_3, u_2) \\ f(u_2) - f(u_1) & \geq w(u_2, u_1) \end{cases}$$

By summing up these inequalities, we have $f(u_m) \geq \sum_{i=1}^{m-1} w(u_{i+1}, u_i) = l_T$. Since the largest element of the codomain of f is $rn(G)$, we know

$$rn(G) \geq f(u_m) \geq \sum_{i=1}^{m-1} w(u_{i+1}, u_i) = l_T \geq M.$$

Thus, $rn(G) = M$, which proves Lemma 6.9. $\square$

Notation: Given an h-tuple $(m_0, m_1, ..., m_{h-1})$, we use the following notation for level-wise regular trees.

$$T^1 = T^1_{m_0, m_1, \ldots, m_{h-1}} \text{ and } T^2 = T^2_{m_0, m_1, \ldots, m_{h-1}}$$

Theorem 6.10. *Suppose $h \geq 1$ and $m_i \geq 2$ for all $0 \leq i \leq h-1$.*

1. *If $d = 2h$, then $rn(T^1) \geq (d+1)(n-1) + 1 - 2 \sum_{i=1}^{h} \left[i \cdot m_0 \prod_{j=1}^{i-1} (m_j - 1) \right]$.*
2. *If $d = 2h+1$, then $rn(T^2) \geq d(n-1) - 4 \sum_{i=1}^{h} \left[i \cdot \prod_{j=0}^{i-1} (m_j - 1) \right]$.*

Proof. Suppose $h \geq 1$ and $m_i \geq 2$ for all $0 \leq i \leq h-1$. Let d denote the diameter of the graph. Consider two cases.

Case 1: $d = 2h$, so the center L_0 of $T^1 = T^1_{m_0, m_1, \ldots, m_{h-1}}$ is a singleton. For each vertex v of T^1, let $\alpha(v) := d(v, L_0)$. Because T^1 is level-wise regular, vertices with equal level have equal degree. Also, the center has exactly m_0 children and each i-vertex (where $1 \leq i \leq h-1$) has $m_i - 1$ children. Thus, for each $1 \leq i \leq h$, we have

$$|L_i| = m_0 \prod_{j=1}^{i-1} (m_j - 1).$$

CLAIM: Let $0 \le i' \le i'' \le h$. If $v' \in L_{i'}$ and $v'' \in L_{v''}$, then $d(v', v'') \le i' + i''$.

To prove this claim, we consider two possibilities.

1. If v' is on the path from v'' to L_0, then $d(v', v'') = i'' - i' \le i'' + i'$, with equality holding if and only if $v' \in L_0$.
2. If v' is not on the path from v'' to L_0, then if w is the closest common ancestor of v' and v'', then $d(v', v'') = d(v', w) + d(w, v'') \le i' + i''$, with equality holding if and only if $w \in L_0$.

This proves the claim. By the claim, $w(v', v'') \ge d + 1 - d(v', v'') \ge d + 1 - (i' + i'')$ in $(T^1)^d_w$.

We define a function l on $V(T^1)$ by $l(v) := h + \frac{1}{2} - \alpha(v) = l_i$, where $v \in L_i$. Notice that $l(v) = \frac{d+1}{2} - \alpha(v)$, since d is even. Observe now that if $P = v_1, v_2, ..., v_n$ is a Hamiltonian path in $(T^1)^d_w$, then for $1 \le j \le n-1$, we have

$$\begin{aligned} w(v_j, v_{j+1}) &\ge d + 1 - (\alpha(v_j) + \alpha(v_{j+1})) \\ &= \left[h + \frac{1}{2} - \alpha(v_j)\right] + \left[h + \frac{1}{2} - \alpha(v_{j+1})\right] \\ &= l(v_j) + l(v_{j+1}). \end{aligned} \tag{106}$$

Thus, we have the following system of inequalities.

$$\begin{cases} w(v_1, v_2) & \ge l(v_1) + l(v_2) \\ w(v_2, v_3) & \ge l(v_2) + l(v_3) \\ & \vdots \\ w(v_{n-2}, v_{n-1}) & \ge l(v_{n-2}) + l(v_{n-1}) \\ w(v_{n-1}, v_n) & \ge l(v_{n-1}) + l(v_n). \end{cases}$$

By summing up and noting that each $l(v_i)$ appears twice except for $l(v_1)$ and $l(v_n)$, we have

$$\sum_{j=1}^{n-1} w(v_j, v_{j+1}) \ge l(v_1) + \sum_{j-2}^{n-1} l(v_j) + l(v_n) \ge 2\sum_{j=1}^{n} l(v_j) - l(v_1) - l(v_n). \tag{107}$$

We can now apply Lemma 6.8, from which we have

$$
\begin{aligned}
rn(T^1) &\geq \text{Min}\left\{\sum_{j=1}^{n-1} w(v_j, v_{j+1}) : v_1, v_2, ..., v_n \text{ is a Hamiltonian path of } (T^1)_w^d\right\} \\
&\geq \text{Min}\left\{\sum_{j=1}^{n} 2l(v_j) - l(v_1) - l(v_n) : v_1, v_2, ..., v_n \text{ is a Hamiltonian path of } (T^1)_w^d\right\} \\
&= \text{Min}\left\{\sum_{j=1}^{n} [d+1-2\alpha(v_j)] - d - 1 + \alpha(v_1) + \alpha(v_n) : v_1, ..., v_n \text{ is Hamiltonian in } (T^1)_w^d\right\} \\
&= \sum_{j=1}^{n} [d+1-2\alpha(v_j)] - d - 1 + 0 + 1 \\
&= n(d+1) - d - 2\sum_{j=1}^{n} \alpha(v_j) \\
&= n(d+1) - d - 2\sum_{i=1}^{h} i|L_i| \\
&= (d+1)(n-1) + 1 - 2\sum_{i=1}^{h}\left[i \cdot m_0 \prod_{j=1}^{i-1}(m_j - 1)\right].
\end{aligned} \tag{108}
$$

This completes the proof for Case 1.

Case 2: $d = 2h$, so the center L_0 of $T^2 = T^1_{m_0, m_1, ..., m_{h-1}}$ contains exactly two adjacent vertices. For each vertex v of T^2, let $\alpha(v) := d(v, L_0)$. Because T^2 is also level-wise regular, vertices with equal level have equal degree. Also, each of the two central vertices has exactly $m_0 - 1$ children and each i-vertex (where $1 \leq i \leq h-1$) also has $m_i - 1$ children. Thus, for each $1 \leq i \leq h$, we have

$$|L_i| = 2\prod_{j=0}^{i-1}(m_j - 1).$$

By the same argument as in Case 1 (and taking account the additional edge at level 0 in T^2, we know if $v' \in L_{i'}$ and $v'' \in L_{i''}$, then $d(v', v'') \leq i' + i'' + 1$. Thus in $(T^2)_w^d$, we know $w(v', v'') \geq d + 1 - d(v', v'') \geq d + 1 - (i' + i' + 1') = d - (i' + i'')$.

As in Case 1, we define a function l on $V(T^2)$ by $l(v) := h + \frac{1}{2} - \alpha(v) = l_i$, where $v \in L_i$. Notice that $l(v) = \frac{d}{2} - \alpha(v)$, since d is odd. Observe now that if $P = v_1, v_2, ..., v_n$ is a Hamiltonian path in $(T^2)^d_w$, then for $1 \le j \le n-1$, we have

$$\begin{aligned} w(v_j, v_{j+1}) &\ge d - (\alpha(v_j) + \alpha(v_{j+1})) \\ &= 2h + 1 - \alpha(v_j) - \alpha(v_{j+1}) \\ &= \left[h + \frac{1}{2} - \alpha(v_j)\right] + \left[h + \frac{1}{2} - \alpha(v_{j+1})\right] \\ &= l(v_j) + l(v_{j+1}). \end{aligned} \tag{109}$$

Notice that although the parity of d alters the computations, the result is identical to our result in Case 1. Thus, by summing up edge weights as in Case 1, we still obtain

$$\sum_{j=1}^{n-1} w(v_j, v_{j+1}) \ge 2\sum_{j=1}^{n} l(v_j) - l(v_1) - l(v_n). \tag{110}$$

By applying Lemma 6.8 again, we obtain

$$\begin{aligned} rn(T^2) &\ge \text{Min}\left\{\sum_{j=1}^{n-1} w(v_j, v_{j+1}) : v_1, v_2, ..., v_n \text{ is a Hamiltonian path of } (T^2)^d_w\right\} \\ &\ge \text{Min}\left\{\sum_{j=1}^{n} 2l(v_j) - l(v_1) - l(v_n) : v_1, v_2, ..., v_n \text{ is a Hamiltonian path of } (T^2)^d_w\right\} \\ &= \text{Min}\left\{\sum_{j=1}^{n} [d - 2\alpha(v_j)] - d + \alpha(v_1) + \alpha(v_n) : v_1, v_2, ..., v_n \text{ is Hamiltonian in } (T^2)^d_w\right\} \\ &= \sum_{j=1}^{n} [d - 2\alpha(v_j)] - d + 0 + 0 \\ &= nd - 2\sum_{j=1}^{n} \alpha(v_j) - d \\ &= d(n-1) - 2\sum_{i=1}^{h} i|L_i| \\ &= d(n-1) - 2\sum_{i=1}^{h}\left[i \cdot 2\prod_{j=0}^{i-1}(m_j - 1)\right] \\ &= d(n-1) - 4\sum_{i=1}^{h}\left[i \cdot \prod_{j=0}^{i-1}(m_j - 1)\right]. \end{aligned} \tag{111}$$

This completes the proof for Case 2. □

If we make a stronger assumption that $m_i \geq 3$ for all $0 \leq i \leq h-1$, then it is possible to construct a radio labeling whose span meets the lower bound established in Theorem 6.10. This will establish the radio number for all level-wise regular trees that meet this requirement.

Theorem 6.11. *Suppose $h \geq 1$ and $m_i \geq 3$ for all $0 \leq i \leq h-1$.*

1. *If $d = 2h$, then* $rn(T^1) = (d+1)(n-1) + 1 - 2\sum_{i=1}^{h}\left[i \cdot m_0 \cdot \prod_{j=1}^{i-1}(m_j - 1)\right]$.
2. *If $d = 2h+1$, then* $rn(T^2) = d(n-1) - 4\sum_{i=1}^{h}\left[i \cdot \prod_{j=0}^{i-1}(m_j - 1)\right]$.

In other words, the equalities of Theorem 6.10 hold if $m_i \geq 3$ for all $0 \leq i \leq h-1$.

Proof. Suppose $h \geq 1$ and $m_i \geq 3$ for all $1 \leq i \leq h-1$. Let d denote the diameter of the graph. Consider two cases.

Case 1: Suppose $d = 2h$, so the center of T^1 is a single vertex. To prove equality for the radio number of T^1, we must construct a radio labeling f of T^1 whose span meets the lower bound established in Theorem 6.10. For each $1 \leq i \leq h-1$ and each vertex v in L_i, we enumerate the edges between v and its children from 0 to $m_i - 2$. Mark the edges between L_0 and its children with integers from 0 to $m_0 - 1$.

Now let $v \in L_i$. There is a unique path from L_0 to v since T^1 is a tree, so v can be uniquely represented by the following i-tuple:

$$a(v) := (a_0(v), a_1(v), ..., a_{i-1}(v))$$

where $a_j(v)$ denotes the number assigned to the edge between the j^{th} and $(j+1)^{th}$ levels on the (v, L_0)-path. Now for each $1 \leq i \leq h$, define a map $s_i : L_i \to \mathbb{Z}$ by

$$s_i(v) := a_0 + a_1 m_0 + a_2 m_0(m_1 - 1) + ... + a_{i-1} m_0(m_1 - 1)...(m_{i-2} - 1).$$

If we let $m'_j = \begin{cases} m_0 & if\ j = 0; \\ m_j - 1 & if\ 1 \leq j \leq h-1 \end{cases}$ then we have $s_i(v) = \sum_{k=0}^{i-1}\left[a_k(v) \cdot \prod_{j=0}^{k-1} m'_j\right]$.

Notice that for each i, s_i is a bijection between L_i and the set $\{0, 1, 2, ..., |L_i| - 1\}$. So each s_i induces a unique ordering of the vertices in L_i from smallest to largest s_i-value. Using the ordering $L_0 \to L_h \to L_{h-1} \to ... \to L_1$ of the levels of T^1 and listing the vertices within each level according the function s_i, we uniquely determine an ordering $v_1, v_2, ..., v_{n-1}, v_n$ of $V(T^1)$. Now define a recursive map $f : V(T^1) \to \{0, 1, 2, ...\}$ by

$$f(v_i) := \begin{cases} 0 & if\ i = 0; \\ f(v_{i-1}) + l(v_{i-1}) + l(v_i) & if\ 1 \le i \le n \end{cases}$$

where $l(v) := h + \frac{1}{2} - d(v, L_0)$. For convenience, denote $d(v, L_0)$ by $\alpha(v)$ for each $v \in V(T^1)$. By definition, $l(v) \ge \frac{1}{2}$ for all $v \in V(T^1)$ and $l(v_{i-1}) + l(v_i) = 2h + 1 - \alpha(v_{i-1}) - \alpha(v_i) = d + 1 - \alpha(v_{i-1}) - \alpha(v_i)$ for all $1 \le i \le n$. A critical observation now is that for any level i, any $|L_i|$ consecutive vertices in $\bigcup_{k=i}^{h} L_k$ have mutually distinct ancestors in L_i. Thus, if two vertices $v_p, v_q \in \bigcup_{k=i}^{h} L_k$ have a common ancestor $z \in L_i$, then $|p - q| \ge |L_i| = \prod_{j=0}^{i-1} m'_j$.

To verify that f satisfies the radio labeling inequality, let $v_p \in L_{i'}$ and $v_{p+k} \in L_{i''}$, where $k \ge 1$. Let $z \in L_i$ be the lowest common ancestor of v_p and v_{p+k}, so $d(v_p, v_{p+k}) = i' + i'' - 2i$ and $\{v_p, v_{p+1}, ..., v_{p+k}\} \subseteq \bigcup_{t=i}^{h} L_t$ based on the ordering defined on $V(T^1)$. Then we have the following system of k inequalities.

$$\begin{cases} f(v_{p+k}) - f(v_{p+k-1}) & = l(v_{p+k}) + l(v_{p+k-1}) & \ge l(v_{p+k}) + \frac{1}{2} \\ f(v_{p+k-1}) - f(v_{p+k-2}) & = l(v_{p+k-1}) + l(v_{p+k-2}) & \ge \frac{1}{2} + \frac{1}{2} = 1 \\ & \vdots & \\ f(v_{p+2}) - f(v_{p+1}) & = l(v_{p+2}) + l(v_{p+1}) & \ge \frac{1}{2} + \frac{1}{2} = 1 \\ f(v_{p+1}) - f(v_p) & = l(v_{p+1}) + l(v_p) & \ge \frac{1}{2} + l(v_p). \end{cases}$$

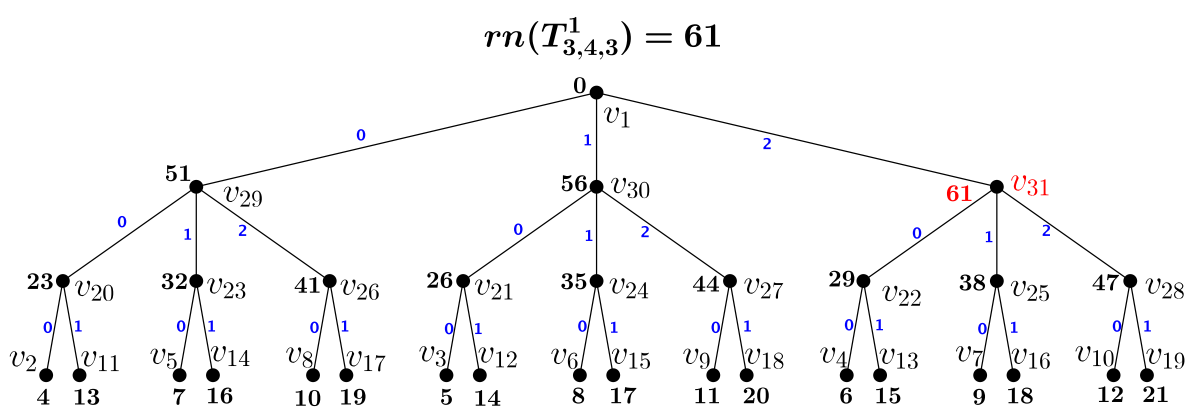
By summing up these inequalities and taking into account the minimum values of k and m_i for each $0 \le i \le h-1$, we have

$$
\begin{aligned}
f(v_{p+k}) - f(v_p) &\geq l(v_{p+k}) + l(v_p) + k - 1 \\
&= \left(h + \frac{1}{2} - i'\right) + \left(h + \frac{1}{2} - i''\right) + k - 1 \\
&\geq d + 1 - i' - i'' + |L_i| - 1 \\
&= d + 1 - i' - i'' + m_0 \prod_{j=1}^{i-1}(m_j - 1) - 1 \\
&\geq d + 1 - i' - i'' + 3 \cdot 2^{i-1} - 1 \\
&\geq d + 1 - i' - i'' + 2i \\
&= d + 1 - (i' + i'' - 2i) \\
&= d + 1 - d(v_p, v_{p+k}).
\end{aligned} \tag{112}
$$

Thus, f is a radio labeling of T^1 as claimed. Furthermore, using the standard telescoping argument for calculating the span of radio labelings, we have by definition of f that

$$
\begin{aligned}
span(f) = f(v_n) &= (d+1)(n-1) - 2\sum_{i=1}^{n} \alpha(v_i) + \alpha(v_1) + \alpha(v_n) \\
&= (d+1)(n-1) - 2\sum_{i=1}^{h} \left[i \cdot m_0 \cdot \prod_{j=1}^{i-1}(m_j - 1) \right] + 0 + 1 \\
&= (d+1)(n-1) + 1 - 2\sum_{i=1}^{h} \left[i \cdot m_0 \cdot \prod_{j=1}^{i-1}(m_j - 1) \right].
\end{aligned} \tag{113}
$$

This matches the lower bound for $rn(T^1)$ found in Theorem 6.10, so Case 1 is proved.



Above is an optimal radio labeling for the level-wise regular tree $T^1_{3,4,3}$. Edges are marked and vertices are ordered as in the proof of Case 1 of Theorem 6.11. $T^1_{3,4,3}$ has a radio number 61.

Case 2: Suppose $d = 2h+1$, so the center of T^2 is contains two vertices. To prove equality for the radio number of T^2, we must construct a radio labeling f of T^2 whose span meets the lower bound established in Theorem 6.10.

Let e denote the central edge of T^2. Then $T^2 \setminus e$ has two isomorphic level-wise regular components, say T' and T'' with diameter $2h$. The difference between the trees discussed in Case 1 and T' or T'' is that the central vertices of T' and T'' have degree $m_0 - 1$, not m_0.

To prove that $rn(T^2)$ is bounded above by the lower bound established in Theorem 6.10, we will establish an ordering $v_1, v_2, ..., v_n$ of $V(T^2)$ such that the same recursive map f from Case 1 satisfies the required inequality for radio labelings between any two distinct vertices in $V(T^2)$.

Our ordering will meet the following initial requirements.

1. $\{v_1, v_n\} = L_0$, so $\alpha(v_1) = \alpha(v_n) = 0$.
2. The vertices in the labeling will alternate between T' and T'', beginning with T'. Thus, T' will contain all odd-indexed vertices, and T'' will contain all even-indexed vertices.
3. The levels of T' are ordered by $L_0 \to L_h \to L_{h-1} \to ... \to L_2 \to L_1$.
4. The levels of T'' are ordered by $L_1 \to L_2 \to L_3 \to ... \to L_h \to L_0$, the reverse order of levels in T'.

For each $1 \leq i \leq h$, let L'_i denote $V(T') \cap L_i$, and similarly let L''_i denote $V(T'') \cap L_i$, so $|L'_i| = |L''_i| = \frac{1}{2}|L_i|$. Analogous to Case 1, for each level i there are bijections $s'_i : L'_i \to \{0, 1, ..., |L'_i| - 1\}$ and $s''_i : L'_i \to \{0, 1, ..., |L''_i| - 1\}$ that when applied to each level of $V(T^2)$ yield a cumulative ordering $v_1, v_3, v_5, ...v_{n-1}$ of $V(T')$ and a cumulative ordering $v_2, v_4, ..., v_n$ of $V(T'')$ such that the following hold.

1. Any $|L'_i|$ odd consecutively labeled vertices in $\displaystyle\bigcup_{k=i}^{h} L'_k$ have mutually distinct ancestors in L'_i.
2. Any $|L''_i|$ even consecutively labeled vertices in $\displaystyle\bigcup_{k=i}^{h} L''_k$ have mutally distinct ancestors in L''_i.

Thus, since $L_i' \cap L_i'' = \varnothing$, we have that any $|L_i|$ consecutively labeled vertices in $\bigcup_{k=i}^{h} L_k$ have mutually distinct ancestors in L_i. Consequently, if two vertices $v_p, v_q \in \bigcup_{k=i}^{h} L_k$ have a common ancestor $z \in L_i$, then $|p - q| \geq |L_i|$.

With this ordering $v_1, v_2, ..., v_n$ of $V(T^2)$ established we verify that the recursive map from Case 1 is a radio labeling for T^2. Let $v_p \in L_{i'}$ and $v_{p+k} \in L_{i''}$, where $k \geq 1$. Observe two cases.

Case 2a: Suppose v_p and v_{p+k} are on different branches of T^2, so $d(v_p, v_{p+k}) = i' + i'' + 1$. Then by the telescoping argument from Case 1 and the fact that $d = 2h + 1$, we have

$$\begin{aligned} f(v_{p+k}) - f(v_p) &\geq l(v_{p+k}) + l(v_p) + k - 1 \\ &= \left(h + \frac{1}{2} - i'\right) + \left(h + \frac{1}{2} - i''\right) + k - 1 \\ &= d - i' - i'' + k - 1 \\ &= d + k - (i' + i'' + 1) \\ &\geq d + 1 - d(v_p, v_{p+k}). \end{aligned} \tag{114}$$

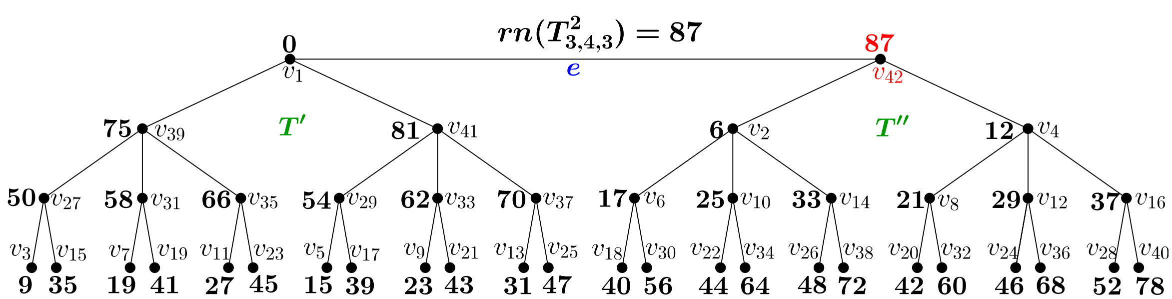
Case 2b: Suppose v_p and v_{p+k} are on the same branch of T^2. Let $z \in L_i$ be the lowest common ancestor of v_p and v_{p+k}, so $d(v_p, v_{p+k}) = i' + i'' - 2i$. Then again by telescoping, we have

$$\begin{aligned}
f(v_{p+k}) - f(v_p) &\geq l(v_{p+k}) + l(v_p) + k - 1 \\
&= \left(h + \frac{1}{2} - i'\right) + \left(h + \frac{1}{2} - i''\right) + k - 1 \\
&\geq d - i' - i'' + |L_i| - 1 \\
&= d - i' - i'' + 2\prod_{j=0}^{i-1}(m_j - 1) - 1 \\
&\geq d - i' - i'' + 2 \cdot 2^i - 1 \\
&= d + 1 - i' - i'' + 2(2^i - 1) \\
&\geq d + 1 - i' - i'' + 2i \\
&= d + 1 - (i' + i'' - 2i) \\
&= d + 1 - d(v_p, v_{p+k}).
\end{aligned} \tag{115}$$

Thus, since both cases satisfy the radio labeling inequality, we have that f is a radio labeling of T^2. Furthermore, using the standard telescoping argument for calculating the span of radio labelings, we have by definition of f that

$$\begin{aligned}
span(f) = f(v_n) &= d(n-1) - 2\sum_{i=1}^{n} \alpha(v_i) + \alpha(v_1) + \alpha(v_n) \\
&= d(n-1) - 2\sum_{i=1}^{h} \left[i \cdot 2 \cdot \prod_{j=0}^{i-1}(m_j - 1) \right] + 0 + 0 \\
&= d(n-1) - 4\sum_{i=1}^{h} \left[i \cdot \prod_{j=0}^{i-1}(m_j - 1) \right].
\end{aligned} \tag{116}$$

This matches the lower bound for $rn(T^2)$ established in Theorem 6.10, so the proof for Case 2 is complete. This completes the proof for Theorem 6.11. □



Above is an optimal radio labeling for the level-wise regular tree $T^2_{3,4,3}$. Vertices are ordered as in the proof of Case 2 of Theorem 6.11. $T^2_{3,4,3}$ has a radio number 87.

Corollary 6.11.1. *Let T be an internally $(m+1)$-regular complete tree (where $m \geq 3$) with diameter $d \geq 3$, so the height $h = \left\lfloor \frac{d}{2} \right\rfloor$ and the degree parameters are given by $m_0 = m_1 = ... = m_{h-1} = m+1$. Then*

1. *If $d = 2h$, then*

$$\begin{aligned} rn(T) &= 1 + \sum_{i=0}^{h-1} \left[m^i (m+1)(d-1-2i) \right] \\ &= m^h + \frac{4m^{h+1} - 2hm^2 - 4m + 2h}{(m-1)^2}. \end{aligned} \tag{117}$$

2. *If $d = 2h+1$, then*

$$\begin{aligned} rn(T) &= \sum_{i=0}^{h} \left[2m^i (d-2i) \right] - d \\ &= 2m^h + \frac{6m^{h+1} - 2m^h - (2h-1)m^2 - 4m + 2h + 1}{(m-1)^2}. \end{aligned} \tag{118}$$

Proof. Let T be an internally $(m+1)$-regular complete tree (where $m \geq 3$) with diameter $d \geq 3$. By setting $m_i = m+1$ for all $0 \leq i \leq h-1$ in the formulas established in Theorem 6.11, the result follows in each case with lengthy but elementary algebraic simplification. □

7 A Survey of Relevant Results - Graphs with Cycles

Our final survey involves other graphs that do not contain unique paths between all pairs of vertices, particularly r^{th} power paths and Hamming graphs. It is here that we investigate radio labelings that are bijections from a vertex of size n onto a set of n consecutive natural numbers in greater detail.

7.1 Radio Number of r^{th}-Power Paths

Recall: The r^{th} *power of a graph* G , denoted G^r, is the graph constructed from G by inserting an edge between any pair of vertices whose distance is at most r in G..

Thus, a *fourth-power path* P_n^4 is the fourth power of a path P_n of n vertices. The radio number of fourth-power paths is almost completed determined by Lo and Alegria in [7], and bounds are established for the single case in which the exact radio number of P_n^4 is not yet fully determined. Preceding results for r^{th} power paths include Liu and Xie's proof of the radio number of all square paths in [5] and Sooryanarayana's proof of the radio number of third-power paths in [9].

Liu and Xie [5] have the following result on the radio number of square paths P_n^2.

Theorem 7.1. *Let P_n^2 be a square path on n vertices, and let $k = \lfloor \frac{n}{2} \rfloor$. Then*

$$rn(P_n^2) = \begin{cases} k^2 + 2 & if\ n \equiv 1\ (mod\ 4)\ and\ n \geq 9; \\ k^2 + 1 & otherwise. \end{cases}$$

Following Liu and Xie's result is Sooryanarayana's [9] solution for the radio number of third-power paths P_n^3, also known as the *cube* of P_n.

Theorem 7.2. *Let P_n^3 be the cube of a path on n vertices, where $n \geq 6$ and $n \neq 7$. Then*

$$rn(P_n^3) = \begin{cases} \frac{n^2+12}{6} & if\ n \equiv 0\ (mod\ 6); \\ \frac{n^2-2n+19}{6} & if\ n \equiv 1\ (mod\ 6); \\ \frac{n^2+2n+10}{6} & if\ n \equiv 2\ (mod\ 6); \\ \frac{n^2+15}{6} & if\ n \equiv 3\ (mod\ 6); \\ \frac{n^2-2n+16}{6} & if\ n \equiv 4\ (mod\ 6); \\ \frac{n^2+2n+13}{6} & if\ n \equiv 5\ (mod\ 6). \end{cases}$$

An immediate commonality between grids, spiders, level-wise regular trees, and r^{th} power paths is the concept of *center* and *level* in relation to distances between vertices. However, it is important to note that the level function on the vertex set of P_n^r defines the level of a vertex v to be the P_n-distance from v to a center of P_n, not its distance in P_n^r.

The same system used by Lo and Alegria of representing a radio labeling f by blocks that indicate both the ordering of vertices *and* the distances between consecutive vertices is adopted several times in this thesis as a means to prove the upper bound of the radio number of a grid graph. The advantage of adopting this systematic notation is that the pattern of labeling becomes apparent in the sequencing of the vertices in each block and that the distances between consecutive vertices can more easily be summed by calculating the number of occurrences of each distance in each block.

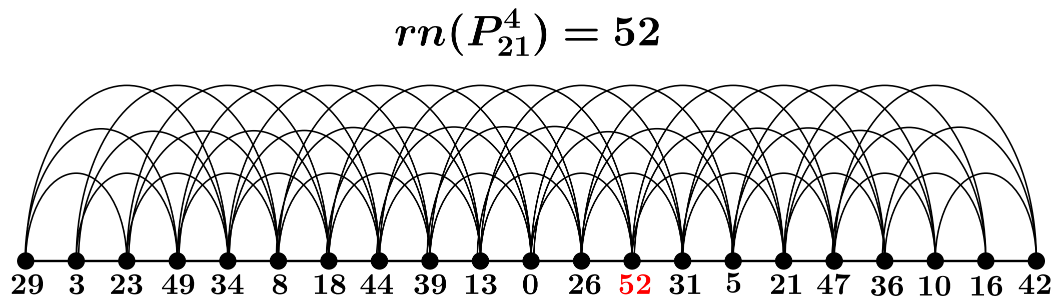
Lo and Alegria's result [7] on the radio number of the fourth-power path P_n^4 is as follows.

Theorem 7.3. *Let $G = P_n^4$ be a fourth-power path on n vertices (where $n \geq 6$), and let $k = \lceil \frac{n-1}{4} \rceil$, so $k = diam(G)$. Then*

$$rn(G) = \begin{cases} 2k^2+1 & if\ n \equiv 0, 3, 6, or\ 7\ (mod\ 8)\ or\ n = 9; \\ 2k^2+2 & if\ n \equiv 4\ or\ 5\ (mod\ 8); \\ 2k^2 & if\ n \equiv 2\ (mod\ 8). \end{cases}$$

If $n \equiv 1 (mod\ 8)$ and $n \geq 17$ (where n is of the form $8q+1$), then

$$2k^2 + 2 \leq rn(P_{8q+1}^4) \leq 2k^2 + q.$$



Above is an optimal radio labeling for the fourth-power path P_{21}^4. Following Case 2 of Theorem 7.3, P_{21}^4 has a radio number 52.

When calculating the radio number of fourth-power paths of order n with Theorem 7.3, it is acceptable to consider only the cases with $n \geq 6$, since P_1^4, P_2^4, P_3^4, P_4^4, and P_5^4 are all complete graphs and therefore trivially have a radio number of $n-1$. Any proper coloring suffices as a radio labeling of complete graphs. The following subsection generalizes the types of graphs that satisfy this property.

7.2 Radio Number of Hamming Graphs and Graceful Radio Labelings

Definition 7.1. Let $n_1, n_2, ..., n_d \in \mathbb{N}$, where $d \geq 2$ and $n_i \geq 2$ for every $1 \leq i \leq d$. A *Hamming Graph* is a graph H of the form $K_{n_1} \square K_{n_2} \square ... \square K_{n_d}$, the Cartesian Product of d complete graphs of orders $n_1, n_2, ..., n_d$ respectively. If $n_1 = n_2 = ... n_d = n$, then we denote H by ${K_n}^d$.

Observation 7.1. *Let G_1, G_2 be graphs.*

1. $d_{G_1 \square G_2}((u, v), (u', v')) = d_{G_1}(u, u') + d_{G_2}(v, v')$.
2. $diam(G_1 \square G_2) = diam(G_1) + diam(G_2)$.

In finding optimal radio labelings for Hamming Graphs, Niedzialomski uses a slightly altered definition [8] of radio labelings and span than the definitions used throughout this thesis.

Definition 7.2. Let G be a graph.

1. A *radio labeling* of G is a function $f : V(G) \to \mathbb{N}$ such that $|f(u) - f(v)| \geq diam(G) + 1 - d(u, v)$ for all distinct $u, v \in V(G)$.
2. The *span* of a radio labeling f, denoted *span*(f), is $\text{Max}\{f(u) : u \in V(G)\}$.
3. The *radio number* of G is the minimum span of any radio labeling of G.

The subtle difference between Niedzialomski's definition of radio labeling and the definition otherwise used throughout this thesis is the codomain of f; in this new definition, 0 is not a possible label. Also, the span of f is now the largest *label* and not the largest *separation* between labels. However, the inequality requirement is identical in both definitions, so this slight modification of the definition does not change the process of determining the radio number of Hamming Graphs. In this section we are to determine the span of optimal radio labelings of Hamming Graphs H. Notice by this new definition that for a graph G of order n, $rn(G) \geq n$, since radio labelings are injective maps.

Definition 7.3. Let G be a graph with order n, and let f be a radio labeling of G.

1. f is a *consecutive radio labeling* of G if $f(V(G)) = \{1, 2, ..., n\}$.
2. If a consecutive radio labeling of G exists, then G is a *radio graceful graph*.

Observation 7.2. *Let G be a graph of order n.*

1. *G is radio graceful if $rn(G) = n$.*

2. *Since any radio labeling f is one-to-one, if f is indeed a consecutive radio labeling of G, then f is onto $\{1, 2, ..., n\}$.*

3. *If $V(G) = \{v_1, v_2, ..., v_n\}$, then G is radio graceful if and only if there exists an ordering $x_1, x_2, ..., x_n$ of elements of $V(G)$ such that for all $\Delta \in \{1, 2, ..., diam(G)\}$ and $i \in \{1, 2, ..., n - \Delta\}$,*

$$d(x_i, x_{i+\Delta}) \geq diam(G) - \Delta + 1.$$

4. *If the diameter of G is 2, then any labeling f on G such that consecutive vertices in the labeling sequence of f are not adjacent (in other words, $d(x_i, x_{i+1}) \geq 2$ for $1 \leq i \leq |V(G)| - 1$) will satisfy the requirement in (3) of a graceful labeling of G. Since nonadjacent vertices in G are adjacent in G^c (the* complement *of G), this implies G is radio graceful if and only if G^c contains a Hamiltonian Path, which is used as the ordering of the vertices of a graceful labeling of G.*

Examples:

1. Complete graphs K_n are radio graceful. Since $diam(K_n) = d_{K_n}(u, v) = 1$ for all distinct $u, v \in V(K_n)$, we know that if $u, v \in V(K_n)$ are distinct, then $|f(u) - f(v)| \geq diam(K_n) + 1 - d(u, v) = 1 + 1 - 1 = 1$. So any proper vertex coloring suffices as a radio labeling of K_n, provided the colors are in $\{1, 2, ..., n\}$.

2. The Petersen graph is radio graceful, since the diameter of the Petersen graph is 2, and its complement contains a Hamiltonian Path.

General Strategy for Proving Radio Gracefulness: Let G be a graph with order n.

1. Define a list of elements $\{x_1, x_2, ...x_n\}$ of $V(G)$.

2. Prove that the list is an *ordering* of $V(G)$, that is, there is neither any repetition nor exclusion.

3. Prove that the order induces a consecutive radio labeling.

We use this strategy in a moderately revised proof of the radio number of Hamming Graphs of the form ${K_n}^t$, where $n \geq 3$.

Definition 7.4. Let $C = G_1 \square G_1 \square ... \square G_t$. For $x_i \in V(C)$, let $x_i = (x_{i_1}, x_{i_2}, ..., x_{i_t})$. Then we define a function $\pi : V(C) \times V(C) \to \{0, 1, 2, ..., t\}$ by

$$\pi(x_i, x_j) = \sum_{k=1}^{t} \pi_k(x_i, x_j), \text{ where } \pi_k(x_i, x_j) = \begin{cases} 1 & if\ x_{i_k} = x_{j_k}; \\ 0 & otherwise. \end{cases}$$

Observation 7.3. *If $C = G^t$, then*

$$t - \pi(x_i, x_j) \le d_{G^t}(x_i, x_j) = \sum_{k=1}^{t} d_G(x_{i_k}, x_{j_k}) \le diam(G)\left(t - \pi(x_i, x_j)\right).$$

From this inequality, if $H = K_{n_1} \square K_{n_2} \square ... \square K_{n_t}$, then

$$t - \pi(x_i, x_j) \le d_H(x_i, x_j) \le diam(K_{n_\alpha})\left(t - \pi(x_i, x_j)\right) = (1)\left(t - \pi(x_i, x_j)\right)$$

which implies

$$d_H(x_i, x_j) = t - \pi(x_i, x_j).$$

Lemma 7.4. *Let G be a graph with $n = |V(G)|$. For $t \in \mathbb{N}$, let $\{x_1, x_2, ..., x_{n^t}\}$ be an ordering of $V(G^t)$ that induces a consecutive radio labeling of G^t (so G^t is radio graceful). Then*

$$\pi(x_i, x_j) \le \frac{|i-j| - 1}{diam(G)} \text{ for all } i, j \in \{1, 2, ..., n^t\}.$$

Proof. Suppose the above conditions hold. Then there exists a map $f : V(G^t) \to \{1, 2, ..., n^t\}$ defined by $f(x_i) = i$ for every $1 \le i \le n^t$ that satisfies $|f(x_i) - f(x_j)| \ge diam(G^t) + 1 - d_{G^t}(x_i, x_j)$ for all i, j.

$\implies |i - j| \ge t\,(diam(G)) + 1 - d_{G^t}(x_i, x_j)$ for all i, j.

$\implies t\,(diam(G)) + 1 - |i - j| \le d_{G^t}(x_i, x_j)$ for all i, j.

From Observation 7.3, $d_{G^t}(x_i, x_j) \le diam(G)\left(t - \pi(x_i, x_j)\right)$, so $t\ diam(G) + 1 - |i - j| \le t\ diam(G) - \pi(x_i, x_j)(diam(G))$. Therefore,

$$\pi(x_i, x_j) \le \frac{|i-j| - 1}{diam(G)}, \text{ since } diam(G) > 0.$$

□

Lemma 7.5. *Let $H = K_{n_1} \square K_{n_2} \square ... \square K_{n_t}$ be of order N. Then an ordering $x_1, x_2, ...x_N$ of $V(H)$ induces a consecutive radio labeling of H if and only if $\pi(x_i, x_j) \leq |i - j| - 1$ for all $i, j \in [N]$.*

Proof. Let $H = K_{n_1} \square K_{n_2} \square ... \square K_{n_t}$ be of order N.

<u>$(\implies)$</u> : Suppose the ordering $x_1, x_2, ..., x_N$ induces a consecutive radio labeling of H. Then there exists a function $f : V(H) \to [N]$ defined by $f(x_i) = i$ for all i satisfying $|f(x_i) - f(x_j)| \geq diam(H) + 1 - d_H(x_i, x_j)$ for all $i, j \in [N]$.

$\implies |i - j| \geq t + 1 - d_H(x_i, x_j)$ for all $i, j \in [N]$.

$\implies t + 1 - |i - j| \leq d_H(x_i, x_j)$ for all $i, j \in [N]$.

Since $d_H(x_i, x_j) \leq t - \pi(x_i, x_j)$, we know that $t + 1 - |i - j| \leq t - \pi(x_i, x_j)$ for all $i, j \in [N]$. So $\pi(x_i, x_j) \leq |i - j| - 1$ for all $i, j \in [N]$.

<u>$(\impliedby)$</u>: Suppose $\pi(x_i, x_j) \leq |i - j| - 1$ for all $i, j \in [N]$. Then $t + \pi(x_i, x_j) \leq t + |i - j| - 1$ for all $i, j \in [N]$. Note that $diam(H) = t$ and $t - \pi(x_i, x_j) = d_H(x_i, x_j)$.

$\implies diam(H) - |i - j| + 1 = t - |i - j| + 1 \leq t - \pi(x_i, x_j) = d_H(xi_i, x_j)$ for all $i, j \in [N]$.

$\implies |i - j| \geq diam(H) + 1 - d_H(x_i, x_j)$ for all $i, j \in [N]$.

Hence, if we define a function $f : V(H) \to [N]$ by $f(x_i) = i$ for all $i \in [N]$, then f satisfies the inequality for a radio labeling, which completes the proof. $\square$

<u>**Step 1**</u>: Define the vertices $x_1, x_2, ...x_{n^t}$ of $H = {K_n}^t$.

Let $n \geq 3$ and $V(K_n) = \{v_1, v_2, ...v_n\}$. Let $t \in [n]$. We recursively describe vertices as the rows of n^{t-1} matrices of order n x t.

Definition 7.5. Assume the following notations.

1. $V(K_n) = \{v_1, v_2, ..., v_n\}$.
2. $\sigma = (v_1, v_2, ..., v_n)$, a permutation in the symmetric group $S_{V(K_n)}$.
3. For $2 \leq k \leq n^{t-1}$, $p_k = \text{Max}\{\alpha \in \mathbb{Z} : n^\alpha | k-1\}$.

Then the matrices $A^{(1)}, A^{(2)}, ..., A^{(n^{t-1})}$ are defined recursively as follows.

$$A^{(1)} = \begin{bmatrix} v_1 & v_1 & v_1 & \dots & v_1 \\ v_2 & x_2 & v_2 & \dots & v_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_n & v_n & v_n & \dots & v_n \end{bmatrix} \text{ and } {a_{i,j}}^{(k)} = \begin{cases} \sigma\left({a_{i,j}}^{(k-1)}\right) & if\ j = t - p_k; \\ {a_{i,j}}^{(k-1)} & otherwise. \end{cases}$$

<u>Example</u>: Consider $A = K_4 = \{v_1, v_2, v_3, v_4\}$.

$$A^{(1)} = \begin{bmatrix} v_1 & v_1 & v_1 \\ v_2 & v_2 & v_2 \\ v_3 & v_3 & v_3 \\ v_4 & v_4 & v_4 \end{bmatrix}, \ A^{(2)} = \begin{bmatrix} v_1 & v_1 & \sigma(v_1) \\ v_2 & v_2 & \sigma(v_2) \\ v_3 & v_3 & \sigma(v_3) \\ v_4 & v_4 & \sigma(v_4) \end{bmatrix} = \begin{bmatrix} v_1 & v_1 & v_2 \\ v_2 & v_2 & v_3 \\ v_3 & v_3 & v_4 \\ v_4 & v_4 & v_1 \end{bmatrix} \text{ since } t - p_2 = 3 - 0 = 3$$

$$A^{(3)} = \begin{bmatrix} v_1 & v_1 & v_3 \\ v_2 & v_2 & v_4 \\ v_3 & v_3 & v_1 \\ v_4 & v_4 & v_2 \end{bmatrix}, \ A^{(4)} = \begin{bmatrix} v_1 & v_1 & v_4 \\ v_2 & v_2 & v_1 \\ v_3 & v_3 & v_2 \\ v_4 & v_4 & v_3 \end{bmatrix}, \ A^{(5)} = \begin{bmatrix} v_1 & v_2 & v_4 \\ v_2 & v_3 & v_1 \\ v_3 & v_4 & v_2 \\ v_4 & v_1 & v_3 \end{bmatrix} \text{ since } t - p_5 = 3 - 1 = 2$$

Observation 7.4. *Let* $2 \leq k \leq n^{t-1}$.

1. *All columns of* $A^{(k)}$ *are the same as the columns of* $A^{(k-1)}$ *except the column at which* $j = t - p_k$.
2. *Since* $k - 1 < k \leq n^{t-1}$, n^{t-1} *does not divide* $k - 1$, *and so* $p_k \leq t - 2$. *Thus,* $j \geq 2$, *and so* σ *is never applied to the first column of any* $A^{(k)}$. *In other words,* $A^{(1)}, A^{(2)}, ..., A^{(n^{t-1})}$ *all have the same first column.*

Definition 7.6. Let $V({K_n}^t) = \{x_1, x_2, ..., x_{n^t}\}$. If $i = bn + c$ (where $c \in [n]$), then

$$x_i = x_{bn+c} = \left({a_{c,1}}^{(b+1)}, {a_{c,2}}^{(b+1)}, ..., {a_{c,t}}^{(b+1)} \right).$$

In other words, x_{bn+c} is the c^{th} row of $A^{(b+1)}$.

Step 2: Verify $x_1, x_2, ..., x_{n^t}$ is an ordering of $V({K_n}^t)$.

We show that $x_1, x_2, ...x_{n^t}$ has no repeated vertices (so $x_i = x_j$ if and only if $i = j$) and does not omit any elements of $V({K_n}^t)$. To simplify this task, notice first that Column 1 is fixed in each $A^{(k)}$ and also that for $2 \leq i \leq n$, $1 \leq j \leq t$, and $1 \leq k \leq n^{t-1}$, we have ${a_{i,j}}^{(k)} = \sigma\left({a_{i-1,j}}^{(k)}\right)$. Thus, the first row of each matrix uniquely defines all other rows in the matrix.

Observation 7.5. *Since the first row of each matrix uniquely defines all other rows of the same matrix, it suffices to show that no two matrices have the same first row.*

Definition 7.7. For ${K_n}^t$, take $A^{(1)}, ..., A^{(n^{t-1})}$ as defined above.

1. Let $A = [a_{i,j}]$ be the $n^{t-1} \times t$ matrix defined by $a_{i,j} = {a_{1,j}}^{(i)}$, so the i^{th} row of A is the first row of $A^{(i)}$. Explicitly, the i, j^{th} entry of A is defined recursively by

$$a_{i,j} = \begin{cases} v_1 & \textit{if } j = 1; \\ \sigma\left(a_{i-1,\, j}\right) & \textit{if } j = t - p_i\,; \\ a_{i-1,j} & \textit{otherwise.} \end{cases}$$

2. A *j-block* is any of the vectors

$$\begin{bmatrix} a_{1,\, j} \\ a_{2,\, j} \\ \vdots \\ a_{n^{t-1},\, j} \end{bmatrix}, \begin{bmatrix} a_{n^{t-j}+1,\, j} \\ a_{n^{t-j}+2,\, j} \\ \vdots \\ a_{2n^{t-j},\, j} \end{bmatrix}, \ldots\ldots, \begin{bmatrix} a_{(n^{j-1}-1)n^{t-j}+1,\, j} \\ a_{(n^{j-1}-1)n^{t-j}+2,\, j} \\ \vdots \\ a_{n^{t-1},\, j} \end{bmatrix}$$

3. The c^{th} *j-block* , denoted $\beta(c, j)$, is the vector

$$\begin{bmatrix} a_{(c-1)n^{t-j}+1,\, j} \\ a_{(c-1)n^{t-j}+2,\, j} \\ \vdots \\ a_{cn^{t-j},\, j} \end{bmatrix}$$

4. The *scope* of $\beta(c, j)$ is the set of consecutive integers $\{(c-1)n^{t-j}+1, (c-1)n^{t-j}+2, ..., cn^{t-j}\}$. In essence, the scope of $\beta(c, j)$ is the set $\{i : a_{i,j}$ is an entry of $\beta(c, j)\}$.

Observation 7.6. *For each j, there are n^{j-1} j-blocks, each of dimension n^{t-j}.*

Lemma 7.6. *Let $1 \leq j \leq t$ and $1 \leq c \leq n^{j-1}$. Then all entries of $\beta(c, j)$ are identical.*

Proof. Let $1 \leq j \leq t$ and $1 \leq c \leq n^{j-1}$. If $j = t$, then $\beta(c, j)$ has $n^{t-t} = n^0 = 1$ entry, which is a trivial case. So assume that $1 \leq j \leq t-1$. Note that the only $i \in S(\beta(c, j))$ that satisfies $n^{t-j}|i-1$ is $i = (c-1)n^{t-j}+1$, the top row of $\beta(c, j)$. So if $i \neq (c-1)n^{t-j}+1$ (in other words, if $a_{i,j}$ is not the top of $\beta(c, j)$), then $p_i \neq t-j$ (so $j \neq t-p_i$).

$\implies a_{i,j} = a_{i-1,j}$ by the definition of the matrix A.

Therefore, all entries of $\beta(c, j)$ are identical to its top entry, which completes the proof of Lemma 7.6. □

Observation 7.7. *Since all entries of $\beta(c, j)$ are identical, any entry of $\beta(c, j)$ can represent $\beta(c, j)$. So $\sigma(\beta(c, j))$ makes sense, since it is uniquely defined by any entry in $\beta(c, j)$.*

Lemma 7.7. *Let $2 \leq j \leq t$ and $1 \leq c \leq n^{j-1}-1$. Then $\beta(c, j) = \beta(c+1, j)$ if and only if $n|c$.*

Proof. Let $2 \leq j \leq t$ and $1 \leq c \leq n^{j-1}-1$. Let $a_{i,j}$ be the top entry of $\beta(c+1, j)$, so $i = cn^{t-j}+1$. Then $n^{t-j}|i-1$, so $p_i \geq t-j$. From Lemma 7.6, we have

$$\begin{aligned}\beta(c+1,j)=\beta(c,j) &\iff \text{The bottom entry of } \beta(c,j) \text{ equals the top entry of } \beta(c+1,j)\\ &\iff a_{i-1,j}=a_{i,j}\\ &\iff j\neq t-p_i\\ &\iff p_i>t-j\\ &\iff n^{1+t-j}|i-1\\ &\iff n|c,\ \text{since } i-1=cn^{t-j}.\end{aligned} \tag{119}$$

This completes the proof of Lemma 7.7. □

Lemma 7.8. *Let $2 \le j \le t$ and $1 \le m \le n^{j-2}$. Then $\beta(x,j) \neq \beta(y,j)$ if $(m-1)n+1 \le x < y \le mn$.*

Proof. Let $2 \le j \le t$ and $1 \le m \le n^{j-2}$. Suppose $(m-1)n+1 \le x < y \le mn$, so $y = x+\Delta$ for some $0 < \Delta < n$. By Lemma 7.7, since n does not divide z if $(m-1)n+1 \le z \le mn-1$, we have

$$\beta(mn,j)=\sigma\left(\beta(mn-1,j)\right)=\sigma^2\left(\beta(mn-2,j)\right)=...=\sigma^{n-1}\left(\beta((m-1)n+1,j)\right).$$

Thus, $\beta(y,j)=\beta(x+\Delta,j)=\sigma^{\Delta}\left(\beta(x,j)\right)\neq\beta(x,j)$, since $0<\Delta<n=|\sigma|$. This completes the proof of Lemma 7.8. □

Lemma 7.9. *If two rows of A share their first k entries, then they are in the scope of the same k-block.*

Proof. We proceed with induction on k. The base case ($k=1$) is trivial, since there exists a unique 1-block, so every row is in the scope of this 1-block.

For the inductive step, assume that the result holds for some $k \in \mathbb{N}$. Suppose the x^{th} and y^{th} rows share the first $k+1$ entries (and therefore the same first k entries). Then the x^{th} and

y^{th} rows are in the scope of the same k-block, say the m^{th} k-block, by the inductive hypothesis.

By Lemma 7.8, $\{\beta((m-1)n+1, k+1), ..., \beta(mn, k+1)\}$ are n distinct $(k+1)$-blocks with scope $\beta(m,k)$. So the x^{th}, and y^{th} rows can be in the scope of exactly one of these $(k+1)$-blocks, since they share the same $(k+1)^{th}$ entry. Thus, they are in the scope of the same $(k+1)$-block, which completes the induction and proves Lemma 7.9. $\square$

Lemma 7.10. *The list $x_1, x_2, ..., x_{n^t}$ is an ordering of $V({K_n}^t)$.*

Proof. Suppose the x^{th} and y^{th} rows of A have equal entries. Then by Lemma 7.9, they are in the scope of the same t-block. This has only $n^{t-t} = n^0 = 1$ entry, so $x = y$; in other words, A has no identical rows. By definition, the rows of A are the top rows of $A^{(1)}, A^{(2)}, ..., A^{(n^{t-1})}$, so none of the top rows of $A^{(1)}, A^{(2)}, ..., A^{(n^{t-1})}$ are identical.

$\Longrightarrow$ None of the rows x_i of $A^{(1)}, A^{(2)}, ..., A^{(n^{t-1})}$ are identical by by Observation 6.7.

Hence, the sequence $x_1, x_2, ..., x_{n^t}$ is an ordering of $V({K_n}^t)$, which proves Lemma 7.10. $\square$

Step 3: Prove that the ordering $x_1, x_2, ..., x_{n^t}$ induces a consecutive radio labeling of ${K_n}^t$. This gives the first main result for Hamming Graphs.

Theorem 7.11. *Let $n \geq 3$ and $t \leq n$. Then ${K_n}^t$ is radio graceful.*

Proof. Let $H = {K_n}^t$, where $n \geq 3$ and $t \leq n$. Let $x_1, x_2, ..., x_{n^t}$ be the previously defined ordering of $V(H)$. Define $f : V(H) \to \mathbb{N}$ by $f(x_i) = i$ for all $x_i \in V(H)$. By Lemma 7.5, using $j = i + \Delta$ (where $\Delta > 0$), it suffices to prove

$$\pi(x_i, x_{i+\Delta}) \leq \Delta - 1 \text{ for all } \Delta \in [n^t - 1] \text{ and } i \in [n^t - \Delta]$$

since $\Delta - 1 = |i - j| - 1$.

By Lemma 7.10, x_i and $x_{i+\Delta}$ are distinct (since $\Delta > 0$), so not all coordinates of x_i and $x_{i+\Delta}$ are equal.

$$\implies \pi(x_i, x_{i+\Delta}) \leq t-1.$$

Hence, we only need to prove $\pi(x_i, x_{i+\Delta}) \leq \Delta - 1$ for all $\Delta \in [t-1]$, not $[n^t - 1]$.

Let x_i be some row in $A^{(k)}$, and let $\Delta \leq t-1$. Since $A^{(k)}$ has n rows and $\Delta \leq t-1 < n$, we know $x_{i+\Delta}$ is either in $A^{(k)}$ or $A^{(k+1)}$. We examine both cases.

CASE 1: $x_{i+\Delta}$ is in $A^{(k)}$.

Note that for each $1 \leq j \leq t$, we know $\sigma^\Delta \left({a_{i,j}}^{(k)}\right) \neq {a_{i,j}}^{(k)}$, since $0 < \Delta < n = |\sigma|$.

Since ${a_{i+\Delta,j}}^{(k)} = \sigma^\Delta \left({a_{i,j}}^{(k)}\right)$ for each $1 \leq j \leq t$, we have ${a_{i+\Delta,j}}^{(k)} \neq {a_{i,j}}^{(k)}$ for each $1 \leq j \leq t$. Thus, $\pi(x_i, x_{i+\Delta}) = 0 \leq \Delta - 1$.

CASE 2: $x_{i+\Delta}$ is in $A^{(k+1)}$ We consider two subcases based on the value of Δ.

Subcase 2a: $\Delta > 1$.

Recall that all but exactly one of the columns of $A^{(k)}$ and $A^{(k+1)}$ are identical. Since $\Delta < n$ and n is the number of rows of $A^{(k)}$, we know x_i and $x_{i+\Delta}$ are in different rows of their respective matrices. Thus, at least $t-1$ of the entries of x_i and $x_{i+\Delta}$ differ. Hence, $\pi(x_i, x_{i+\Delta}) \leq 1 \leq \Delta - 1$, since $\Delta \geq 2$.

Subcase 2b: $\Delta = 1$.

In this sub case, $x_{i+\Delta} = x_{i+1}$, so we know the following:

1. x_i is the last row of $A^{(k)}$.
2. $x_{i+\Delta}$ is the first row of $A^{(k+1)}$.

Let $j' = t - p_{k+1}$. Recall that ${a_{i,j}}^{(k+1)} = \sigma\left({a_{i,j}}^{(k)}\right)$ if and only if $j = j'$. Thus, we have

$$x_{i+\Delta} = x_{i+1} = \left({a_{1,1}}^{(k+1)}, ..., {a_{1,j'}}^{(k+1)}, ..., {a_{1,t}}^{(k+1)}\right) = \left({a_{1,1}}^{(k)}, ..., \sigma\left({a_{1,j'}}^{(k)}\right), ..., {a_{1,t}}^{(k)}\right)$$

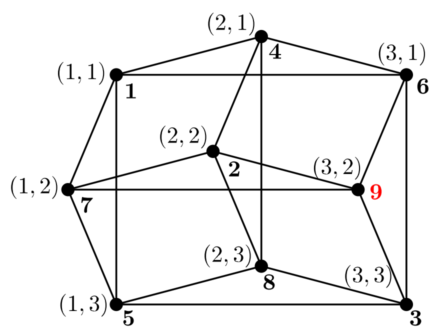
and

$$x_i = \left({a_{n,1}}^{(k)}, ..., {a_{n,j'}}^{(k)}, ..., {a_{n,t}}^{(k)}\right) = \left(\sigma^{-1}\left({a_{1,1}}^{(k)}\right), ..., \sigma^{-1}\left({a_{1,j'}}^{(k)}\right), ..., \sigma^{-1}\left({a_{1,t}}^{(k)}\right)\right).$$

But $|\sigma| = n \geq 3$, so for each j, ${a_{1,j}}^{(k)} \neq \sigma^{-1}\left({a_{1,j}}^{(k)}\right)$ and $\sigma\left({a_{1,j'}}^{(k)}\right) \neq \sigma^{-1}\left({a_{1,j'}}^{(k)}\right)$.

Thus, $\pi(x_i, x_{i+\Delta}) = 0 \leq \Delta - 1$, since $\Delta = 1$. This exhausts all all cases, and so our proof of Theorem 7.11 is complete. □

$$\boldsymbol{rn(K_3^2) = 9}$$



Above is a consecutive (hence optimal) radio labeling for K_3^2, a radio graceful Hamming graph. The ordering of $V(K_3^2)$ satisfies the characterization of consecutive radio labelings established in Lemma 7.5.

Lemma 7.12. *Given a graph G, let $d = diam(G)$ and $n = |V(G)|$. Let*

$$s = 1 + \sum_{k=d}^{n-1} (n-k) \left\lfloor \frac{k}{d} \right\rfloor. \tag{120}$$

If $t \geq s$, then G^t is not radio graceful.

Proof. Let G be a graph with diameter d and a vertex set $V(G) = \{v_1, v_2, ..., v_n\}$, so $n = |V(G)|$. Let $s = 1 + \sum_{k=d}^{n-1} (n-k) \left\lfloor \frac{k}{d} \right\rfloor$. Let $t \geq s$. Suppose (to the contrary) that $x_1, x_2, ...x_{n^t}$ is an ordering of $V(G^t)$ that induces a graceful labeling $f : V(G^t) \to [n^t]$, where $f(x_i) = i$ for each $x_i \in V(G^t)$. From Lemma 7.4, we know

$$\pi(x_i, x_j) \leq \frac{|i-j|-1}{d} \text{ for all } i, j \in [n^t].$$

Define a function $\Pi : V(G^t) \times V(G^t) \to \mathbb{Z}$ by

$$\Pi(x_i, x_j) = Min \left\{ t, \frac{|i-j|-1}{d} \right\} \text{ for all } (x_i, x_j) \in V(G^t) \times V(G^t).$$

Note from this definition that the maximum number of coordinates x_i can have in common with any prior vertex in the ordering of f is $\sum_{j=1}^{i-1} \Pi(x_i, x_j)$, since $\pi(x_i, x_j) \leq \Pi(x_i, x_j)$ for all j. Thus, the first $n+1$ vertices $x_1, x_2, ...x_{n+1}$ can agree on at most $\sum_{i=2}^{n+1} \left(\sum_{j=1}^{i-1} \Pi(x_i, x_j) \right)$ coordinates. So if M is the number of coordinate agreements within $x_1, x_2, ..., x_{n+1}$, then

$$\begin{aligned}
M &\leq \sum_{i=2}^{n+1} \left(\sum_{j=1}^{i-1} \Pi(x_i, x_j) \right) \\
&\leq \sum_{i=2}^{n+1} \left(\sum_{j=1}^{i-1} \left\lfloor \frac{|i-j|-1}{d} \right\rfloor \right) \\
&= \sum_{i=2}^{2} \left(\sum_{j=1}^{i-1} \left\lfloor \frac{|i-j|-1}{d} \right\rfloor \right) + \sum_{i=3}^{n+1} \left(\sum_{j=1}^{i-1} \left\lfloor \frac{|i-j|-1}{d} \right\rfloor \right) \\
&= \sum_{j=1}^{2} \left\lfloor \frac{|3-j|-1}{d} \right\rfloor + \sum_{j=1}^{3} \left\lfloor \frac{|4-j|-1}{d} \right\rfloor + ... + \sum_{j=1}^{n} \left\lfloor \frac{|(n+1)-j|-1}{d} \right\rfloor \\
&= \sum_{j=1}^{2} \left\lfloor \frac{2-j}{d} \right\rfloor + \sum_{j=1}^{3} \left\lfloor \frac{3-j}{d} \right\rfloor + ... + \sum_{j=1}^{n} \left\lfloor \frac{n-j}{d} \right\rfloor \\
&= \sum_{k=1}^{1} \left\lfloor \frac{k}{d} \right\rfloor + \sum_{k=1}^{2} \left\lfloor \frac{k}{d} \right\rfloor + ... + \sum_{k=1}^{n-1} \left\lfloor \frac{k}{d} \right\rfloor \\
&= (n-1) \left\lfloor \frac{1}{d} \right\rfloor + (n-2) \left\lfloor \frac{2}{d} \right\rfloor + ... + \left\lfloor \frac{n-1}{d} \right\rfloor \\
&= \sum_{k=1}^{n-1} (n-k) \left\lfloor \frac{k}{d} \right\rfloor \\
&= \sum_{k=d}^{n-1} (n-k) \left\lfloor \frac{k}{d} \right\rfloor = s - 1 = t.
\end{aligned} \tag{121}$$

Thus, the first $n+1$ vertices contain fewer than t coordinate agreements. This indicates that there must exist at least one coordinate in which none of the first $n+1$ coordinates agree, say the p^{th} coordinate. So the set P of p^{th} coordinates of $x_1, x_2, ...x_{n+1}$ has $n+1$ distinct elements, which is impossible because $P \subseteq \{v_1, v_2, ..., v_n\}$. Therefore, G^t has no graceful labeling, which completes the proof of Lemma 7.12. ☐

Lemma 7.12 leads us to our second main result for Hamming Graphs.

Theorem 7.13. *Let $n \geq 3$ and $t \geq 1 + \frac{n(n^2-1)}{6}$. Then ${K_n}^t$ is not radio graceful.*

Proof. From Lemma 7.12, we know that ${K_n}^t$ is not radio graceful for any $t \geq s$, where

$$\begin{aligned} s &= 1 + \sum_{k=diam(K_n)}^{n-1} (n-k) \left\lfloor \frac{k}{diam(K_n)} \right\rfloor \\ &= 1 + \sum_{k=1}^{n-1} (n-k)(k) \\ &= 1 + n \left(\sum_{k=1}^{n-1} k \right) - \sum_{k=1}^{n-1} k^2 \\ &= 1 + n \cdot \frac{(n+1)(n)}{2} + \frac{(n-1)(n)\left[2(n-1)+1\right]}{6} \\ &= 1 + \frac{3n^2(n-1) - n(n-1)(2n-1)}{6} \\ &= 1 + \frac{n(n+1)(3n-2n+1)}{6} = 1 + \frac{n(n^2-1)}{6}. \end{aligned} \tag{122}$$

Thus, ${K_n}^t$ is not radio graceful if $t \geq 1 + \frac{n(n^2-1)}{6}$. ☐

Whether ${K_n}^t$ is radio graceful for $n+1 \leq t \leq \frac{n(n^2-1)}{6}$ remains open. Recent attempts were made to find a consecutive radio labeling for ${K_3}^4$, the smallest example of a Hamming Graph ${K_n}^t$ whose radio gracefulness (or lack thereof) remains unproven. We have thus far determined using programming techniques a sequence of 79 vertices of ${K_3}^4$ (i.e. 79 4-tuples of $\{1, 2, 3\}$) that satisfy the requirements for consecutive vertices of a graceful labeling. However, whether it is possible to find such a sequence of all 81 vertices of ${K_3}^4$ as needed remains unknown.

References

[1] V. Halasz and Z. Tuza, "Distance-Constrained Labeling of Complete Trees," *Discrete Math*, 338 (2015), no. 8, 1398-1406.

[2] W. K. Hale, "Frequency Assignment: Theory and Applications," *Proc. IEEE*, 68 (1980), 1497–1514.

[3] Li, Xiangwen; Mak, Vicky; Zhou, "Sanming Optimal Radio Labellings of Complete m-ary Trees," *Discrete Appl. Math*, 158 (2010), no. 5, 507?515.

[4] D. Liu, "Radio Numbers for Trees," *Discrete Mathematics*, 308 (2008), 1153-1164.

[5] D. Liu and M. Xie, "Radio Number for Square Paths," *Ars Combin.* 90 (2009), 307-319.

[6] D. Liu and X. Zhu, "Multi-level Distance Labelings and Radio Number for Paths and Cycles," *SIAM Journal on Discrete Mathematics*, 19 (2005), 610-611.

[7] M. Lo and L. Alegria, "Radio Number for Fourth-Power Paths," *Involve 9* (2016), no. 2, 317-332.

[8] A. Niedzialomski, "Radio Graceful Hamming Graphs." *Discuss. Math. Graph Theory* 36 (2016), no. 4, 1007-1020.

[9] B. Sooryanarayana, M. Vishu Kumar, K. Manjula, "Radio Number of Cube of a Path," *Int. J. Math. Comb.* 1 (2010), 5-29.