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ABSTRACT

Factorizing Bivariate and

Trivariate Polynomials

By

Ronald Chen

This thesis discusses the reducibility of homogeneous trivariate polynomials of degree 2 and 3. In the degree 2 case, we give necessary and sufficient conditions for the reducibility of such a polynomial. When a factorization exists, we show how to find the factors of the polynomial. We also provide necessary and sufficient conditions for the polynomial to be a perfect square.

The question of the reducibility of degree-3 polynomials is more complicated. We don't have a complete answer; we only have partial results. Some information can be obtained from a certain 9 by 3 matrix V whose entries are derived from the coefficients of the polynomial. Specifically,

(1) If the polynomial is reducible, then V has rank 1 or 0.

(2) If V has rank 1, then we have a candidate factor that has to be checked using long division.

(3) If the polynomial factors completely, then V is the zero matrix.

(4) If the polynomial is reducible and V is not the zero matrix, then the polynomial can be factored over the coefficient field.

The converses of these results are not true and we give counterexamples.

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CHAPTER 1

Introduction

This thesis is about the factorizations of homogeneous bivariate and trivariate polynomials having degree 2 or 3. For example, do the following polynomials f and g in the indeterminates x, y, and z factor at all?

$$f = x^2 - 6xy - 2y^2 - 20xz - 6yz + z^2$$

$$g = 2x^3 - 3x^2y + 3xy^2 - y^3 + x^2z - 6xyz + 5y^2z - xz^2 - 7yz^2 + 3z^3$$

For the answers to the reducibility of these two polynomials, see Example 4.57 and Example 5.53. We will find some conditions on the coefficients of these polynomials that determine whether they factor. And if these polynomials factor, do they factor over the coefficient field or over an extension?

Any bivariate homogeneous polynomial will factor completely because of the Fundamental Theorem of Algebra. For example, let F be a field and let

$$f(x, y) = a_3x^3 + a_2x^2y + a_1xy^2 + a_0y^3 \in F[x, y], \quad a_3 \neq 0.$$

Then setting $y = 1$, we get

$$f(x, 1) = a_3x^3 + a_2x^2 + a_1x + a_0.$$

By the Fundamental Theorem of Algebra,

$$f(x, 1) = a_3(x - x_1)(x - x_2)(x - x_3) \tag{1.1}$$

where $x_1, \ldots, x_3 \in \mathbb{C}$ are the roots of $f(x, 1)$. Matching coefficients, we find

$$\begin{aligned} a_2 &= -(x_1 + x_2 + x_3)a_3 \\ a_1 &= (x_1x_2 + x_1x_3 + x_2x_3)a_3 \\ a_0 &= -x_1x_2x_3a_3. \end{aligned} \tag{1.2}$$

Homogenizing each factor in (1.1) by appending a y after each root and using (1.2), we get

$$f(x, y) = a_3(x - x_1y)(x - x_2y)(x - x_3y).$$

Depending on the multiplicities of the roots of $f(x, y)$, $f(x, y)$ can be written in one of the following form:

$$\begin{aligned} f(x, y) &= a_3(x - x_i)^3 \\ f(x, y) &= a_3(x - x_j)^2(x - x_k) \\ f(x, y) &= a_3(x - x_1y)(x - x_2y)(x - x_3y) \end{aligned}$$

for some i and j. The powers of the factors adds to the degree of the polynomial.

In the chapter on bivariate homogeneous polynomials, we have used the known result that a univariate polynomial has a multiple root if and only if its discriminant is zero. This result can be found in, for example, [1, Proposition 34]. And we have also used the known result that a univariate degree 3 polynomial has 3 identical roots if and only if its Hessian is zero. This result can be found in, for example, [9, p. 136].

In the trivariate degree 2 case, the polynomial factors if and only if a certain function R of the coefficients of the polynomial is zero. It will be shown that the polynomial is a square times a constant if and only if R and three functions D_x, D_y, and D_z of the coefficients are zero. The proof is fairly straightforward.

The trivariate degree 3 case is only partially solved. Here are some results that we found.

(1) If the polynomial factors, then a certain 9×3 matrix, which we call V, must have rank 1 or 0.

(2) If V has rank 1, then we have a candidate factor of f that has to be checked using long division.

(3) If the polynomial factors completely, then V is the zero matrix.

(4) If the polynomial factors and $V \neq 0$, then the polynomial can be factored over the coefficient field.

One of the unanswered questions is: When V has rank 1 or 0, does this imply that f is reducible?

CHAPTER 2

Definitions and Preliminary Results

Towards answering those questions that were mentioned earlier, we prove some lemmas and theorems that will be helpful later on. First, we introduce definitions that we will use in the lemmas and theorems. We will use F to denote the coefficient field of the polynomials that appear throughout this thesis.

Definition 2.1. Let F be a field. A *bivariate polynomial* f in x and y is a polynomial in x over the coefficient field $F[y]$. This is written $f \in F[y][x]$ or $f \in F[x, y]$. A *trivariate polynomial* g in x, y, and z is a polynomial in x over the coefficient field $F[z][y]$. This is written $g \in F[z][y][x]$ or $g \in F[x, y, z]$. A *multivariate polynomial* in $x_1, x_2, x_3, \ldots, x_m$ is a polynomial of the ring $F[x_m][x_{m-1}][x_{m-2}] \cdots [x_2][x_1]$.

Every nonconstant multivariate polynomial can be written uniquely as the sum of products of the form $cx_1^{n_1}x_2^{n_2} \cdots x_m^{n_m}$ where $c \in F$ and $x_1, x_2, \ldots, x_m$ are indeterminates. Each $cx_1^{n_1}x_2^{n_2} \cdots x_m^{n_m}$ is called a *monomial*.

Definition 2.2. The *total degree* of a nonzero multivariate monomial $cx_1^{n_1}x_2^{n_2} \cdots x_m^{n_m}$ is the sum $n_1 + n_2 + \cdots + n_m$ of the degrees in the factors. The *total degree* of a nonzero multivariate polynomial f is the highest degree among the monomials of f. A multivariate polynomial is *homogeneous* if every one of its monomials has the same total degree. The *homogeneous degree k component* of a nonzero multivariate polynomial f is the sum of the degree k monomials of f [1, p. 297] [3].

Definition 2.3. Let F be a field. A nonconstant polynomial $f \in F[x]$ is *reducible over F* if $f = gh$ for some nonconstant polynomials g and h in $F[x]$. Otherwise f is *irreducible over* F. Similarly, a nonconstant multivariate polynomial $\bar{f} \in F[x_1, x_2, \ldots, x_m]$ is *reducible over* F if $\bar{f} = \bar{g}\bar{h}$ for some nonconstant polynomials $\bar{g}, \bar{h} \in F[x_1, x_2, \ldots, x_m]$.

With the above definitions, we are now ready to prove some theorems that we need in our later discussions. For multivariate polynomials, the (total) degree of a product of two polynomials is the sum of the degrees of the polynomials [7, p. 114]:

Theorem 2.4. *Let F be a field. Let $f, g \in F[x_1, x_2, \ldots, x_n]$ be nonzero. Then* $\deg fg = \deg f + \deg g$.

Proof. We can write f and g using their homogeneous components f_i and g_j as follows

$$f = f_0 + f_1 + \cdots + f_d, \text{ with } \deg f_i = i \text{ for } i = 0, 1, \ldots, d, \text{ and } f_d \neq 0$$

$$g = g_0 + g_1 + \cdots + g_e, \text{ with } \deg g_j = j \text{ for } j = 0, 1, \ldots, e, \text{ and } g_e \neq 0.$$

When we multiply f and g, we get terms of the form $f_i g_j$ having degree $i+j$ and fg can be written using its homogeneous components as follows:

$$fg = f_0g_0 + (f_0g_1 + f_1g_0) + (f_0g_2 + f_1g_1 + f_2g_0) + (f_0g_3 + f_1g_2 + f_2g_1 + f_3g_0) + \cdots + f_dg_e.$$

We know that if D is a domain, then so is $D[x_1, x_2, \ldots, x_n]$ [1, p.235]. So since $F[x_1, x_2, \ldots, x_n]$ is a domain and $f_d \neq 0$ and $g_e \neq 0$, we have $f_dg_e \neq 0$. This is the homogeneous component of fg having the highest degree. Thus, $\deg fg = d + e = \deg f + \deg g$. $\square$

We get another similar theorem when we replace total degree with lower degree.

Definition 2.5. Let the *lower degree*, $\operatorname{lower} f$, of a nonzero polynomial f be the degree of the smallest nonzero homogeneous term of f.

Theorem 2.6. *Let $f, g \in F[x_1, x_2, \ldots, x_n]$. Then* $\operatorname{lower} fg = \operatorname{lower} f + \operatorname{lower} g$.

From the definitions of total degree and lower degree, one sees that:

Fact 2.7.

(1) The lower degree of a polynomial is less than or equal to its total degree, and

(2) A polynomial is homogeneous if and only if its total degree equals its lower degree.

We now prove a commonly accepted fact about inequality, then we will use this to prove that if a homogeneous polynomial factors, then the factors are homogeneous as well.

Lemma 2.8. *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers. If $a_i \leq b_i$ for $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$, then $a_i = b_i$ for $i = 1, 2, \ldots, n$.*

Proof. Suppose, to the contrary, that $a_{i_0} < b_{i_0}$ for some $i_0 \in \{1, 2, \ldots, n\}$. Then because $a_i \leq b_i$ for $i = 1, 2, \ldots, n$, we have $\sum_{i=1}^{n} a_i < \sum_{i=1}^{n} b_i$, which contradicts the assumption that $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$. Thus $a_i = b_i$ for $i = 1, 2, \ldots, n$. $\square$

Theorem 2.9. *Let $f \in F[x_1, x_2, \ldots, x_n]$. If f is homogeneous and $f = gh$ for some $g, h \in F[x_1, x_2, \ldots, x_n]$, then g and h are homogeneous as well.*

Proof. Using Fact 2.7, we know that $\operatorname{lower} g \leq \deg g$ and $\operatorname{lower} h \leq \deg h$. By Theorem 2.4 and Theorem 2.6, we have

$$\begin{aligned} \deg f &= \deg g + \deg h \\ \operatorname{lower} f &= \operatorname{lower} g + \operatorname{lower} h. \end{aligned} \tag{2.10}$$

Since f is homogeneous, we have $\operatorname{lower} f = \deg f$. Thus we have

$$\operatorname{lower} g + \operatorname{lower} h = \deg g + \deg h.$$

By Lemma 2.8, we have $\operatorname{lower} g = \deg g$ and $\operatorname{lower} h = \deg h$. Therefore g and h are homogeneous by Fact 2.7. $\square$

We will need to use the Fundamental Theorem of Algebra whose proof can be found in, for example, [5, p.151]. We want to point out that this is for polynomials of a single variable.

Theorem 2.11 (Fundamental Theorem of Algebra)**.** *Every univariate nonconstant polynomial over $\mathbb{C}$ is completely reducible.*

Since every root corresponds to a linear factor, this is equivalent to the alternative forms of the theorem:

(1) Every univariate nonconstant polynomial over $\mathbb{C}$ has a root in $\mathbb{C}$, or equivalently,

(2) The field $\mathbb{C}$ of complex numbers is an algebraically closed field.

We will also need to use Kronecker's Theorem whose proof can be found in, for example, [2, p. 266].

Theorem 2.12 (Kronecker's Theorem)**.** *Given any nonconstant polynomial, there exist an extension of the base field in which the polynomial factors completely.*

Since every root corresponds to a linear factor, this is equivalent to the alternative form of the theorem:

> Let F be a field and let $f(x)$ be a nonconstant polynomial in $F[x]$. Then there exists an extension field E of F and an $\alpha \in E$ such that $f(\alpha) = 0$.

The following relationship among algebraic structures can be found in, for example, [1, p. 292].

$$\text{fields} \subset \text{Euclidean domains} \subset \text{PIDs} \subset \text{UFDs} \subset \text{integral domains}.$$

Let F be a field. Since F is a field, we have F is a PID. Since F is a PID, we have F is a UFD. We will use the following theorem. The proof can be found in, for example, [2, Theorem 45.29], [3, p. 164], and [4].

Theorem 2.13. *If D is a UFD, then $D[x]$ is a UFD.*

Since F is a UFD, we have $F[x_1]$ is a UFD. Since $F[x_1]$ is a UFD, we have F is a UFD. Continuing this process, we find $F[x_1][x_2][x_3]\cdots[x_{m-1}][x_m]$ is a UFD.

We will use the resultant of two polynomials, so we define it here.

Definition 2.14. Let F be a field. Suppose $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in F[x]$ has roots $x_1, \ldots, x_n$ and $g(x) = b_mx^m + b_{m-1}x^{m-1} + \cdots + b_1x + b_0 \in F[x]$ has roots $y_1, \ldots, y_m$. The *resultant* of $f(x)$ and $g(x)$ [1, p. 621]

$$\operatorname{Res}_x(f(x), g(x)) = a_n^m b_m^n \prod_{\substack{1\le i\le n\\ 1\le j\le m}} (x_i - y_j).$$

This is the same as the determinant of the Sylvester matrix [1, p. 620].

$$\operatorname{Res}_x(f(x), g(x)) = \begin{vmatrix} a_n & a_{n-1} & \cdots & a_0 & & & & \\ & a_n & a_{n-1} & \cdots & a_0 & & & \\ & & a_n & a_{n-1} & \cdots & a_0 & & \\ & & & \ddots & m \text{ rows} & \ddots & & \\ & & & & a_n & a_{n-1} & \cdots & a_0 \\ b_m & b_{m-1} & \cdots & b_0 & & & & \\ & b_m & b_{m-1} & \cdots & b_0 & & & \\ & & b_m & b_{m-1} & \cdots & b_0 & & \\ & & & \ddots & n \text{ rows} & \ddots & & \\ & & & & b_m & b_{m-1} & \cdots & b_0 \end{vmatrix}.$$

CHAPTER 3

Bivariate Homogeneous Polynomials

In this chapter, we give some theorems for the factorization of bivariate homogeneous degree 2 and degree 3 polynomials. These theorems will be used in later chapters to find factorizations of trivariate homogeneous polynomials.

We note that any bivariate homogeneous polynomial is completely reducible.

3.1 Bivariate Homogeneous Degree 2 Polynomials

Any bivariate homogeneous degree 2 polynomial will factor completely because of the Fundamental Theorem of Algebra. The powers of the factors partition the degree; the issue is, what partition is it. Does the polynomial factor as a square, or as two linearly independent factors? The following theorem answers this and it is based on the known result that a univariate polynomial has a multiple root if and only if its discriminant is zero. This result can be found in, for example, [1, Proposition 34].

Theorem 3.1. *Let $f(x, y) = ax^2 + bxy + cy^2 \in F[x, y]$ and let $D = b^2 - 4ac$ be the* ***discriminant*** *of f. Then*

(1) *$D = 0$ if and only if $f(x, y) = A(a_0x + a_1y)^2$ for some A, a_0, a_1. If $\operatorname{char} F \neq 2$, then A, a_0, and a_1 may be chosen in F. Otherwise a_1 may need to be in a quadratic extension of F.*

(2) *$D \neq 0$ if and only if $f = (a_0x + a_1y)(b_0x + b_1y)$ with $a_0b_1 - a_1b_0 \neq 0$ for some a_0, a_1, b_0, and b_1 in a quadratic extension of F.*

Proof. **Case I**: Suppose $a \neq 0$. Let $ax^2 + bx + c$ have roots α_1 and α_2 in some field extension of F. Then

$$\begin{aligned} ax^2 + bx + c &= a(x - \alpha_1)(x - \alpha_2) \\ &= ax^2 - a(\alpha_1 + \alpha_2)x + a\alpha_1\alpha_2 \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} f &= ax^2 + bxy + cy^2 \\ &= a(x - \alpha_1 y)(x - \alpha_2 y) \\ &= ax^2 - a(\alpha_1 + \alpha_2)xy + a\alpha_1\alpha_2 y^2. \end{aligned} \tag{3.3}$$

Matching coefficients, we have $b = -a(\alpha_1 + \alpha_2)$ and $c = a\alpha_1\alpha_2$.

This implies that $\alpha_1 + \alpha_2 = -\frac{b}{a}$, $\alpha_1\alpha_2 = \frac{c}{a}$, and

$$\begin{aligned} D &= a^2(\alpha_1 + \alpha_2)^2 - 4a^2\alpha_1\alpha_2 \\ &= a^2(\alpha_1^2 + 2\alpha_1\alpha_2 + \alpha_2^2) - 4a^2\alpha_1\alpha_2 \\ &= a^2(\alpha_1 - \alpha_2)^2. \end{aligned} \tag{3.4}$$

Then $(\alpha_1 - \alpha_2)^2 = \frac{D}{a^2}$. So $\alpha_1 - \alpha_2 = \frac{\sqrt{D}}{a} \in F\left(\sqrt{D}\right)$. Since $\alpha_1 - \alpha_2 = \frac{\sqrt{D}}{a}$, we have $D = 0$ if and only if $\alpha_1 - \alpha_2 = 0$, if and only if $\alpha_1 = \alpha_2$.

Case I.A: If $D = 0$ and F does not have characteristic 2, then since $\alpha_1 + \alpha_2 = -b/a$ and $\alpha_1 - \alpha_2 = 0$, we have $\alpha_1 = \alpha_2 = -b/2a$. Then (3.3) becomes $f = a\left(x + \frac{b}{2a}y\right)^2 = \frac{1}{4a}(2ax + by)^2$ which has the form $A(a_0x + a_1y)^2$ for some $A, a_0, a_1 \in F$ as claimed.

Case I.B: If $D = 0$ and F has characteristic 2, then $\alpha_1 = \alpha_2$ and (3.3) becomes

$$f = a(x - \alpha_1 y)^2 \tag{3.5}$$

which has the form $A(a_0x + a_1y)^2$ for some $A, a_0 \in F$ and a_1 in a quadratic extension of F.

We come out of these two subcases (case I.A and case I.B) since assertion (1) has been shown. However, we continue with case I where $a \neq 0$ with the goal of showing assertion (2). Towards this goal, we rewrite (3.3) as $f = (ax - a\alpha_1 y)(x - \alpha_2 y)$ which has the form $(a_0x + a_1y)(b_0x + b_1y)$ where $a_0 = a$, $a_1 = -a\alpha_1$, $b_0 = 1$, $b_1 = -\alpha_2$, and $a_0, a_1, b_0, b_1 \in F\left[\sqrt{D}\right]$. Then $a_0b_1 - a_1b_0 = -a\alpha_2 + a\alpha_1 = a(\alpha_1 - \alpha_2)$. By (3.4), we have $D \neq 0$ if and only if $a(\alpha_1 - \alpha_2) \neq 0$. Since $a(\alpha_1 - \alpha_2) = a_0b_1 - a_1b_0$, the condition $a(\alpha_1 - \alpha_2) \neq 0$ is equivalent to $a_0b_1 - a_1b_0 \neq 0$. Thus $D \neq 0$ if and only if $a_0b_1 - a_1b_0 \neq 0$.

Case II: Suppose $a = 0$. Then $f = bxy + cy^2$ and $D = b^2$. Then $D = 0$ if and only if $b = 0$ if and only if $f = cy^2$ which has the form $A(a_0x + a_1y)^2$ where $A, a_0, a_1 \in F$. For the other case, we have $D \neq 0$ and $b \neq 0$ and so $f = bxy + cy^2 = (bx + cy)(0x + y)$ which has the form $(a_0x + a_1y)(b_0x + b_1y)$ where $a_0, a_1, b_0, b_1 \in F \subseteq F\left[\sqrt{D}\right]$ and $a_0b_1 - a_1b_0 = b \neq 0$. □

Case I.B. behaves different because F has characteristic 2. We give an example of this type where $D = 0$, but f does not factor over the coefficient field F. For these conditions to hold, F must have an element that does not have a square root in F. Otherwise, if every element of F has a square root in F, then since $D = 0$ implies that f has a multiple root. By (3.5), we have

$$f = a(x - \alpha_1 y)^2 = a(x^2 - 2\alpha_1 xy + \alpha_1^2 y^2) = ax^2 + a\alpha_1^2 y^2.$$

Since the coefficient field of f is F, we would have $a\alpha_1^2 \in F$ which implies $\alpha_1^2 \in F$ since $a \in F$. Since every element of F has a square root in F, we would have $\alpha_1 \in F$. Since $f = a(x - \alpha_1 y)^2$, f would factor over the coefficient field F which is not what we want. In order to have a coefficient field F that has an element that does have a square root in F, we must choose F to be an infinite field, because in a finite field of characteristic p, every element has a pth root [1, Corollary 36].

Example 3.6. We give an example where $D = 0$, but f does not factor over the coefficient field F. Let $F = \mathbb{Z}_2(s)$, the field of rational functions over $\mathbb{Z}_2$ with the indeterminate s. Note $\operatorname{char} F = 2$. First, we show that $s \in \mathbb{Z}_2(s)$ has no square root in $\mathbb{Z}_2(s)$. If, to the contrary, $\alpha^2 = s$ for some $\alpha \in \mathbb{Z}_2(s)$, then $\deg s = \deg \alpha^2 = 2 \deg \alpha$ is even, but $\deg s = 1$, and we would have a contradiction. So $s \in \mathbb{Z}_2(s)$ has no square root in $\mathbb{Z}_2(s)$. Let $f = x^2 + sy^2$. Then $D = -4s = 0$. Since $\operatorname{char} F = 2$ and $a = 1$, case I.B in the proof of Theorem 3.1 applies. By (3.5), we have

$$f = a(x - \alpha_1 y)^2 = a(x^2 - 2\alpha_1 xy + \alpha_1^2 y^2) = ax^2 + a\alpha_1^2 y^2.$$

Since $a = 1$, we have $f = x^2 + \alpha_1^2 y^2$. Matching coefficients with f, we find $\alpha_1^2 = s$. Since $s \in \mathbb{Z}_2(s)$ has no square root in $\mathbb{Z}_2(s)$, we have $\alpha_1 \notin \mathbb{Z}_2(s) = F$. Since $f = (x - \alpha_1 y)^2$, f does not factor over the coefficient field F.

3.2 Bivariate Homogeneous Degree 3 Polynomials

In this section, we give some theorems for the factorization of bivariate homogeneous degree 3 polynomials. These theorems will be used in later chapters to factor trivariate homogeneous degree 3 polynomials.

Let

$$f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3 \in F[x, y]. \tag{3.7}$$

As discussed before, f will factor completely over an extension of F into 3 linear factors because of the Fundamental Theorem of Algebra. The question is whether f

will factor as three linearly independent factors, or as the product of a constant times a square and a degree 1 factor, or as a constant times a cube. We will show that the discriminant and the Hessian of f determine which of the 3 ways f factors. Define the Hessian H of f by,

$$H = (b^2 - 3ac)x^2 + (bc - 9ad)xy + (c^2 - 3bd)y^2. \tag{3.8}$$

If $\operatorname{char} F \neq 2$, then $H(x,y) = -\frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2\right)$. (In some books, if not most, the quantity inside the outer parentheses is called the Hessian, for example, [8].) The discriminant D of f is

$$D = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd. \tag{3.9}$$

(This agrees with the discriminant for a monic cubic polymial [1, p. 612] when we set $y = 1$ and $a = 1$ and rename the coefficients.) This discriminant D of f is essentially the same as the discriminant D_H of H where H was defined in (3.8). Let's calculate D_H:

$$\begin{aligned} D_H &= (bc - 9ad)^2 - 4(b^2 - 3ac)(c^2 - 3bd) \\ &= -3(b^2c^2 - 4ac^3 - 4b^3d + 18abcd - 27a^2d^2) \\ &= -3D. \end{aligned} \tag{3.10}$$

From this equation, we see that

(1) If $D = 0$, then $D_H = 0$.

(2) If $\operatorname{char} F \neq 3$ and $D_H = 0$, then $D = 0$.

When does f as in (3.7) factor over the base field F as a cube times a constant? The following theorem answers this and it is based on the known result that a univariate degree 3 polynomial has 3 identical roots if and only if its Hessian is zero. This result can be found in, for example, [10, p. 26] and [9, p. 136].

Theorem 3.11. *Let f be a degree 3 bivariate homogeneous polynomial as in* (3.7). *Then $f(x,y) = A(a_0x + a_1y)^3$ for some $A, a_0, a_1 \in F$ if and only if $H = 0$.*

Proof. Suppose $f(x,y) = A(a_0x + a_1y)^3$ for some A, a_0, a_1. Matching coefficients, we find

$$a = a_0^3 A$$

$$b = 3a_0^2 a_1 A$$

$$c = 3a_0 a_1^2 A$$

$$d = a_1^3 A.$$

Plugging these into (3.8), we get $H = 0$.

Conversely, suppose $H = 0$. If $a \neq 0$, then $b^2 - 3ac = 0$ implies $c = \dfrac{b^2}{3a}$, and $bc - 9ad = 0$ implies $d = \dfrac{bc}{9a} = \dfrac{b^3}{27a^2}$. Then

$$\begin{aligned} f &= ax^3 + bx^2y + cxy^2 + dy^3 \\ &= ax^3 + bx^2y + \frac{b^2}{3a}xy^2 + \frac{b^3}{27a^2}y^3 \\ &= a\left(x^3 + 3(x)\left(\frac{b}{3a}y\right)^2 + 3(x)^2\left(\frac{b}{3a}y\right) + \left(\frac{b}{3a}y\right)^3\right) \\ &= a\left(x + \frac{b}{3a}y\right)^3, \end{aligned} \tag{3.12}$$

having the claimed form. If $a = 0$, then $b^2 - 3ac = 0$ implies $b = 0$, and $c^2 - 3bd = 0$ implies $c = 0$. Hence $f = dy^3$, having the claimed form. □

Note that if $a \neq 0$, $D = 0$, and $\operatorname{char} F \neq 3$, then $H = 0$ can be replaced by a simpler one: $p = 0$ where $p = b^2 - 3ac$. We prove this in the following theorem and it comes from known results on the discriminant and the Hessian of a cubic polynomial. These results can be found in, for example, [10, p. 26].

Theorem 3.13. *Let $p = b^2 - 3ac$. If $a \neq 0$, then*

(1) *$H = 0$ implies $p = 0$.*

(2) *If $D = 0$ and F does not have characteristic 3, then $p = 0$ implies $H = 0$.*

Proof. H can be written using p as follows

$$H = px^2 + (bc - 9ad)xy + (c^2 - 3bd)y^2. \tag{3.14}$$

To see (1) indirectly, we note that if $p \neq 0$, then $H \neq 0$.

To show (2), suppose $p = 0$. Then $b^2 - 3ac = 0$ (by definition of p) and $D = (bc - 9ad)^2$. If $D = 0$, then $bc - 9ad = 0$. Multiplying $bc - 9ad = 0$ by b, we have $b^2c - 9abd = 0$. Multiplying $b^2 - 3ac = 0$ by c, we have $b^2c - 3ac^2 = 0$. Then $(b^2c - 9abd) - (b^2c - 3ac^2) = -9abd + 3ac^2 = -3a(3bd - c^2) = 0$. Since $a \neq 0$ and F does not have characteristic 3, we have $3bd - c^2 = 0$. Thus $H = 0$. □

We give an example of Theorem 3.11.

Example 3.15. Let $f = x^3 + 3x^2y + 3xy^2 + y^3$. Matching coefficients with (3.7), we find $a = 1$, $b = 3$, $c = 3$, and $d = 1$. Plugging these into (3.8), we find $H = 0$. By Theorem 3.11, we have $f = (x + y)^3$. For this polynomial, $p = b^2 - 3ac = 0$ which is a simpler calculation than using (3.8) to calculate H.

When does f as in (3.7) factor into three independent linear factors? The following theorem answers this and it is based on the known result that a univariate polynomial has a multiple root if and only if its discriminant is zero. This result can be found in, for example, [1, Proposition 34] and [10, p. 26].

Theorem 3.16. *Let f be a degree 3 bivariate homogeneous polynomial as in (3.7). Then*

$$f(x,y) = (a_0x + a_1y)(b_0x + b_1y)(c_0x + c_1y) \tag{3.17}$$

for some a_0, a_1, b_0, b_1, c_0, c_1 in a degree 6, or less, extension of F with $a_0b_1 - a_1b_0 \neq 0$, $a_0c_1 - a_1c_0 \neq 0$, and $b_0c_1 - b_1c_0 \neq 0$, if and only if $D \neq 0$.

Proof. First we prove that f is a product of three linear factors—independent of D and a—because of the Fundamental Theorem of Algebra. Suppose $a \neq 0$. Let $\alpha_1, \alpha_2, \alpha_3$ be the roots of $ax^3 + bx^2 + cx + d$ in an extension of F of degree at most 6. Then

$$ax^3 + bx^2 + cx + d = a(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$$

and

$$\begin{aligned} f &= a(x - \alpha_1 y)(x - \alpha_2 y)(x - \alpha_3 y) \\ &= (ax - a\alpha_1 y)(x - \alpha_2 y)(x - \alpha_3 y) \end{aligned} \tag{3.18}$$

which is a product of three linear factors. If $a = 0$, then by Theorem 3.1

$$\begin{aligned} f &= y(bx^2 + cxy + dy^2) \\ &= y(a_0x + a_1y)(b_0x + b_1y) \end{aligned} \tag{3.19}$$

for some a_0, a_1, b_0, and b_1 in a quadratic extension of F. Thus f is a product of three linear factors.

So independent of D and a, f has the form

$$f(x,y) = (a_0x + a_1y)(b_0x + b_1y)(c_0x + c_1y). \tag{3.20}$$

Matching coefficients, we find

$$\begin{aligned} a &= a_0 b_0 c_0 \\ b &= a_1 b_0 c_0 + a_0 b_1 c_0 + a_0 b_0 c_1 \\ c &= a_1 b_1 c_0 + a_1 b_0 c_1 + a_0 b_1 c_1 \\ d &= a_1 b_1 c_1. \end{aligned} \tag{3.21}$$

Plugging these into (3.9), we get

$$D = (a_0 b_1 - a_1 b_0)^2 (a_0 c_1 - a_1 c_0)^2 (b_0 c_1 - b_1 c_0)^2. \tag{3.22}$$

From this equation, we have $D \neq 0$ if and only if $a_0 b_1 - a_1 b_0 \neq 0$, $a_0 c_1 - a_1 c_0 \neq 0$, and $b_0 c_1 - b_1 c_0 \neq 0$. □

We give an example of degree 3 bivariate homogeneous polynomial f that factors as in (3.17).

Example 3.23. Let $f = x^3 - xy^2 + y^3 \in F[x, y]$. Then matching coefficients, we find $a = 1$, $b = 0$, $c = -1$, and $d = 1$. Plugging these into (3.9), we find $D = -23$. By Theorem 3.16,

$$f = (x - \alpha y)(x - \beta y)(x - \gamma y)$$

for some α, β, and γ in a degree 6, or less, extension of F with $-\beta + \alpha \neq 0$, $-\gamma + \alpha \neq 0$, and $-\gamma + \beta \neq 0$.

When does f, as in (3.7), factor as a square times a linear factor? The following theorem answers this and it is based on the known result that a univariate polynomial has a multiple root if and only if its discriminant is zero. This result can be found in, for example, [1, Proposition 34]. And we have also used the known result that a univariate degree 3 polynomial has 3 identical roots if and only if its Hessian is zero. This result can be found in, for example, [10, p. 26] and [9, p. 136]. First, we rewrite f as a multiple of H plus a remainder in two ways:

$$(b^2-3ac)^2 f = \left(\underbrace{a(b^2-3ac)}_{b_0 \in F} x + \underbrace{\left(b^3 - 4abc + 9a^2 d\right)}_{b_1 \in F} y \right) H - D\left(3axy^2 + by^3\right). \quad (3.24)$$

$$(c^2-3bd)^2 f = \left(\underbrace{\left(c^3 - 4bcd + 9ad^2\right)}_{b_0 \in F} x + \underbrace{d(c^2-3bd)}_{b_1 \in F} y \right) H - D\left(3dx^2 y + cx^3\right). \quad (3.25)$$

These two equations will be used to prove the following theorem.

Theorem 3.26. *Let f be a degree 3 bivariate homogeneous polynomial as in (3.7). Suppose the coefficient field F has characteristic other than 2 and 3. Then the following are equivalent.*

(1) $f = (a_0 x + a_1 y)^2 (b_0 x + b_1 y)$ *for some* $a_0, a_1, b_0, b_1 \in F$ *such that* $a_0 b_1 - a_1 b_0 \neq 0$.

(2) $H \neq 0$ *and* $D = 0$.

(3) $H \neq 0$ *and* $H = A(a_0 x + a_1 y)^2$ *for some* $A, a_0, a_1 \in F$.

Proof. First we show (2) $\Leftrightarrow$ (3). By (3.10), we have that $D = 0$ is equivalent to $D_H = 0$ since $\operatorname{char} F \neq 3$. By Theorem 3.1, we have that $D_H = 0$ is equivalent to $H = A(a_0 x + a_1 y)^2$ for some $A, a_0, a_1 \in F$ since $\operatorname{char} F \neq 2$.

To show (1) $\Rightarrow$ (2), suppose $f = (a_0 x + a_1 y)^2 (b_0 x + b_1 y)$ for some $a_0, a_1, b_0, b_1 \in$

F such that $a_0b_1 - a_1b_0 \neq 0$. Matching coefficients, we get

$$\begin{aligned} a &= a_0^2 b_0 \\ b &= a_0(2a_1b_0 + a_0b_1) \\ c &= a_1(a_1b_0 + 2a_0b_1) \\ d &= a_1^2 b_1. \end{aligned} \tag{3.27}$$

Plugging these into (3.9) and (3.8), we find $D = 0$ and $H = (a_0b_1 - a_1b_0)^2(a_0x + a_1y)^2$. Since $a_0b_1 - a_1b_0 \neq 0$, we have $a_0 \neq 0$ or $a_1 \neq 0$. So $H \neq 0$.

We now show that (3) $\Rightarrow$ (1). Suppose $H \neq 0$ and $H = A(a_0x + a_1y)^2$ for some $A, a_0, a_1 \in F$. As we have shown at the beginning of the proof, this is equivalent to $D_H = 0$ and $D = 0$. We note that either $b^2 - 3ac \neq 0$ or $c^2 - 3bd \neq 0$, since otherwise, (3.10) together with $D_H = 0$ would imply that $bc - 9ad = 0$, and by (3.8), H would be zero contrary to our assumption. Since $b^2 - 3ac \neq 0$ or $c^2 - 3bd \neq 0$, and $D = 0$ and $H \neq 0$, we have that (3.24) and (3.25) imply that f is the product of H (which equals $A(a_0x + a_1y)^2$) and a linear factor $b_0x + b_1y$ where b_0 and b_1 are the underbraced coefficients in (3.24) and (3.25). That is,

$$f(x, y) = A(a_0x + a_1y)^2(b_0x + b_1y)$$

where $b_0, b_1 \in F$. The constant A can be absorbed in the coefficients b_0 and b_1 so that f has the claimed form $f(x, y) = (a_0x + a_1y)^2(b_0x + b_1y)$ where $a_0, a_1, b_0, b_1 \in F$.

It remains to prove that $a_0b_1 - a_1b_0 \neq 0$. Suppose, to the contrary, $a_0b_1 - a_1b_0 = 0$. Then $f(x, y) = B(a_0x + a_1y)^3$ for some $B \in F$, and by Theorem 3.11, we would have $H = 0$, which would contradict our assumption. $\square$

We give an example of a degree 3 bivariate homogeneous polynomial having the three equivalent conditions in Theorem 3.26.

Example 3.28. Let $f = x^3 - x^2y - xy^2 - y^3 \in Q[x,y]$. Then matching coefficients with (3.7), we find $a = 1$, $b = -1$, $c = -1$, and $d = 1$. Plugging these into (3.8) and (3.9), we find $H = 4(x-y)^2$ and $D = 0$. By Theorem 3.26, $f = (a_0x+a_1y)^2(b_0x+b_1y)$ for some $a_0, a_1, b_0, b_1 \in F$ such that $a_0b_1 - a_1b_0 \neq 0$. This agrees with the factorization $f = (x-y)^2(x+y)$.

Here is an example that shows that without $\operatorname{char} F \neq 3$, condition (3) of Theorem 3.26 might not imply conditions (1) or (2).

Example 3.29. Let $F = \mathbb{Z}_3$ and let $f = x^2y + xy^2 = xy(x+y) \in F[x,y]$. Matching coefficients with (3.7), we find $a = d = 0$ and $b = c = 1$. Then

$$\begin{aligned} H &= (b^2 - 3ac)x^2 + (bc - 9ad)xy + (c^2 - 3bd)y^2 \\ &= x^2 + xy + y^2 \\ &= (x-y)^2. \end{aligned}$$

An easy calculation shows that $D = 1$. In this example, condition (1) of Theorem 3.26 is false, condition (2) is false, and condition (3) is true.

CHAPTER 4

Trivariate Homogeneous Degree 2 Polynomials

In this chapter, we consider the reducibility of polynomials of the form

$$f = C_0 z^2 + C_1 xz + C_2 yz + C_4 x^2 + C_3 xy + C_5 y^2 \tag{4.1}$$

over some base field F containing the constants C_i in the indeterminates x, y and z. Throughtout this chapter, we will assume that $\operatorname{char} F \neq 2$. To see if the polynomial f reduces over F or over some extension of F, the following quantity is important. Let

$$R = -C_1C_2C_3 + C_0C_3^2 + C_2^2C_4 + C_1^2C_5 - 4C_0C_4C_5. \tag{4.2}$$

The goal is to show that

- If f is reducible over F or over some extension of F, then $R = 0$, and
- If $R = 0$, then f is reducible over a quadratic extension of F.

We have looked through a number of literatures that are related to this subject, in particular, literatures on ternery quadratic forms, for example, [11], [12], and [13]. It is well known that f can be written in the form

$$f = \begin{bmatrix} x & y & z \end{bmatrix} M \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

for some symmetric 3×3 matrix M. But we have not yet seen the quantity R being given in the literatures that we have looked at.

Where is R coming from? What's the intuition? There are two possible explanations. One of them is that R comes from the discriminant of the symmetric matrix of the quadratic form f in the basis (x, y, z), see equation (4.18). The second explanation comes from the remainder of f divided by a general linear polynomial. We first assume that f factors, and then derive consequences. So suppose

$$f = (a_0 z + a_1 x + a_2 y)(b_0 z + b_1 x + b_2 y) \tag{4.3}$$

for some a_0, a_1, a_2, b_0, b_1, and b_2 in an extension of F. Suppose $a_0 \neq 0$. Then using polynomial long division to divide f by $a_0 z + a_1 x + a_2 y$, treating both as polynomial in z, we find

$$a_0^2 f = (a_0 z + a_1 x + a_2 y)Q + \widehat{R} \tag{4.4}$$

where

$$\begin{aligned} Q &= (-a_1 C_0 + a_0 C_1)x + (-a_2 C_0 + a_0 C_2)y + a_0 C_0 z \\ \widehat{R} &= K_1 x^2 + K_2 xy + K_3 y^2 \\ K_1 &= a_1^2 C_0 - a_0 a_1 C_1 + a_0^2 C_4 \\ K_2 &= 2a_1 a_2 C_0 - a_0 a_2 C_1 - a_0 a_1 C_2 + a_0^2 C_3 \\ K_3 &= a_2^2 C_0 - a_0 a_2 C_2 + a_0^2 C_5. \end{aligned} \tag{4.5}$$

Since f factors as in (4.3), we have $\widehat{R} = 0$ and so $K_1 = K_2 = K_3 = 0$. Let S be the resultant of K_1 and K_2, treating both as polynomial in a_1 and assuming their leading coefficients are nonzero. Then

$$\begin{aligned} S = a_0^2 \big(&-a_2^2 C_0 C_1^2 + a_0 a_2 C_1^2 C_2 - a_0^2 C_1 C_2 C_3 + a_0^2 C_0 C_3^2 + 4a_2^2 C_0^2 C_4 \\ &-4a_0 a_2 C_0 C_2 C_4 + a_0^2 C_2^2 C_4\big). \end{aligned}$$

So assuming nonzero leading coefficients, $K_1 = 0$ and $K_2 = 0$ have a common solution for a_1 if and only if $S = 0$. Since $a_0 \neq 0$, the other part of S without the factor of a_0^2

equals zero. Call this other part T, i.e.,

$$T = -a_2^2C_0C_1^2 + a_0a_2C_1^2C_2 - a_0^2C_1C_2C_3 + a_0^2C_0C_3^2 + 4a_2^2C_0^2C_4 - 4a_0a_2C_0C_2C_4 + a_0^2C_2^2C_4$$

and $T = 0$. Similarly, using resultant and assuming nonzero leading coefficients, we find that $T = 0$ and $K_3 = 0$ have a common solution for a_2 if and only if

$$a_0^4C_0^2(\underbrace{-C_1C_2C_3 + C_0C_3^2 + C_2^2C_4 + C_1^2C_5 - 4C_0C_4C_5}_{R})^2 = 0.$$

Suppose $C_0 \neq 0$, then since $a_0 \neq 0$, we have $R = 0$.

Towards proving that $R = 0$ is a necessary and sufficient condition for f to factor, we first introduce a symmetry diagram that will come in handy when we make a symmetry argument. If we permute the indeterminates x, y, and z of f, then the new polynomial is reducible if and only if the original one is. To see how the coefficients C_i permutes when the indeterminates are permuted, we give a symmetry diagram in Figure 4.1. For example, interchanging x and y in f means that we interchange C_4

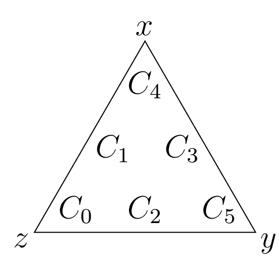

Figure 4.1: Symmetry diagram for the coefficients of degree 2 polynomial.

and C_5, C_1 and C_2, but C_0 and C_3 are unchanged. Note that R is unchanged by these changes or any other symmetry.

Lemma 4.6. *If f is reducible over F or over some extension of F, then $R = 0$.*

Proof. If f is reducible, then f is a product of two homogeneous degree 1 polynomials. Hence f has the form $f = (a_0z + a_1x + a_2y)(b_0z + b_1x + b_2y)$ where $a_0, a_1, a_2, b_0, b_1, b_2$

are in F or in some extension of F. Expanding this expression gives

$$
\begin{aligned}
f &= (a_0 z + a_1 x + a_2 y)(b_0 z + b_1 x + b_2 y) \\
&= \underbrace{a_0 b_0}_{C_0} z^2 + (\underbrace{a_0 b_1 + a_1 b_0}_{C_1}) xz + (\underbrace{a_0 b_2 + a_2 b_0}_{C_2}) yz + \underbrace{a_1 b_1}_{C_4} x^2 \\
&\quad + (\underbrace{a_1 b_2 + a_2 b_1}_{C_3}) xy + \underbrace{a_2 b_2}_{C_5} y^2.
\end{aligned}
$$

Matching coefficients, we find

$$
\begin{aligned}
C_0 &= a_0 b_0 \\
C_1 &= a_0 b_1 + a_1 b_0 \\
C_2 &= a_0 b_2 + a_2 b_0 \\
C_4 &= a_1 b_1 \\
C_3 &= a_1 b_2 + a_2 b_1 \\
C_5 &= a_2 b_2.
\end{aligned} \tag{4.7}
$$

Plugging the expressions for the C_i from above into the expression for R, we get

$$\begin{aligned}
R &= -C_1C_2C_3 + C_0C_3^2 + C_2^2C_4 + C_1^2C_5 - 4C_0C_4C_5 \\
&= -(a_0b_1 + a_1b_0)(a_0b_2 + a_2b_0)(a_1b_2 + a_2b_1) + a_0b_0(a_1b_2 + a_2b_1)^2 \\
&\quad + (a_0b_2 + a_2b_0)^2a_1b_1 + (a_0b_1 + a_1b_0)^2a_2b_2 - 4a_0b_0a_1b_1a_2b_2 \\
&= \big(- a_1a_2^2b_0^2b_1 - a_0a_2^2b_0b_1^2 - a_1^2a_2b_0^2b_2 - 2a_0a_1a_2b_0b_1b_2 \\
&\quad - a_0^2a_2b_1^2b_2 - a_0a_1^2b_0b_2^2 - a_0^2a_1b_1b_2^2\big) \\
&\quad + \big(a_0a_2^2b_0b_1^2 + 2a_0a_1a_2b_0b_1b_2 + a_0a_1^2b_0b_2^2\big) \\
&\quad + \big(a_1a_2^2b_0^2b_1 + 2a_0a_1a_2b_0b_1b_2 + a_0^2a_1b_1b_2^2\big) \\
&\quad + \big(a_1^2a_2b_0^2b_2 + 2a_0a_1a_2b_0b_1b_2 + a_0^2a_2b_1^2b_2\big) \\
&\quad - 4a_0a_1a_2b_0b_1b_2 \\
&= 0. \qquad \square
\end{aligned}$$

It takes more work and more steps to show that if $R = 0$, then f is reducible over a quadratic extension of F. Toward this goal, some important quantities are defined as follows. If we set $x = 0$, then $f = C_0z^2 + C_2yz + C_5y^2$. We define D_x as the discriminant of this polynomial, i.e.,

$$D_x = C_2^2 - 4C_0C_5. \tag{4.8}$$

Similarly, if we set $y = 0$, then $f = C_0z^2 + C_1xz + C_4x^2$ and we define D_y as

$$D_y = C_1^2 - 4C_0C_4. \tag{4.9}$$

Similarly, if we set $z = 0$, then $f = C_4x^2 + C_3xy + C_5y^2$ and we define D_z as

$$D_z = C_3^2 - 4C_4C_5. \tag{4.10}$$

Some more important quantities are defined below. They occur in parts of equations that show the relationship between f and the discriminants defined earlier.

$$E_x = C_1C_3 - 2C_2C_4 \tag{4.11}$$

$$E_y = C_2C_3 - 2C_1C_5 \tag{4.12}$$

$$E_z = C_1C_2 - 2C_0C_3 \tag{4.13}$$

$$L_x = C_1z + 2C_4x + C_3y \tag{4.14}$$

$$L_y = C_2z + C_3x + 2C_5y \tag{4.15}$$

$$L_z = 2C_0z + C_1x + C_2y. \tag{4.16}$$

We will prove some preliminary lemmas about the relationship between f, R, and the above quantities and use these to prove that if $R = 0$, then f is reducible over a quadratic extension of F. Towards showing these relationships, we start by re-writing f using matrix equations. $2f$ can be re-written as $2f = \begin{bmatrix} x & y & z \end{bmatrix} M \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where

$$M = \begin{bmatrix} 2C_4 & C_3 & C_1 \\ C_3 & 2C_5 & C_2 \\ C_1 & C_2 & 2C_0 \end{bmatrix}. \tag{4.17}$$

Calculating the determinant of M is straightforward.

$$\begin{aligned} \det M &= 8C_0C_4C_5 + C_1C_2C_3 + C_1C_2C_3 - 2C_2^2C_4 - 2C_0C_3^2 - 2C_1^2C_5 \\ &= -2R. \end{aligned} \tag{4.18}$$

Using row one of M to calculate $\det M$, we get

$$\begin{aligned} \det M &= 2C_4 \begin{vmatrix} 2C_5 & C_2 \\ C_2 & 2C_0 \end{vmatrix} - C_3 \begin{vmatrix} C_3 & C_2 \\ C_1 & 2C_0 \end{vmatrix} + C_1 \begin{vmatrix} C_3 & 2C_5 \\ C_1 & C_2 \end{vmatrix} \\ &= -2C_4D_x + C_3E_z + C_2E_y. \end{aligned} \tag{4.19}$$

Since $\det M = -2R$, we have $2R = 2C_4D_x - C_3E_z - C_2E_y$.

Similarly, using row two of M to calculate $\det M$, we get

$$\begin{aligned}\det M &= -C_3\begin{vmatrix} C_3 & C_1 \\ C_2 & 2C_0 \end{vmatrix} + 2C_5\begin{vmatrix} 2C_4 & C_1 \\ C_1 & 2C_0 \end{vmatrix} - C_2\begin{vmatrix} 2C_4 & C_3 \\ C_1 & C_2 \end{vmatrix} \\ &= C_3E_z - 2C_5D_y + C_2E_x.\end{aligned} \tag{4.20}$$

Since $\det M = -2R$, we have $2R = -C_3E_z + 2C_5D_y - C_2E_x$.

Similarly, using row three of M to calculate $\det M$, we get

$$\begin{aligned}\det M &= C_1\begin{vmatrix} C_3 & C_1 \\ 2C_5 & C_2 \end{vmatrix} - C_2\begin{vmatrix} 2C_4 & C_1 \\ C_3 & C_2 \end{vmatrix} + 2C_0\begin{vmatrix} 2C_4 & C_3 \\ C_3 & 2C_5 \end{vmatrix} \\ &= C_1E_y + C_2E_x - 2C_0D_z.\end{aligned} \tag{4.21}$$

Since $\det M = -2R$, we have $2R = 2C_0D_z - C_1E_y - C_2E_x$.

Calculating the adjoint of M is straightforward.

$$\begin{aligned}\operatorname{adj} M &= \begin{bmatrix} 4C_0C_5 - C_2^2 & C_1C_2 - 2C_0C_3 & C_2C_3 - 2C_1C_5 \\ C_1C_2 - 2C_0C_3 & 4C_0C_4 - C_1^2 & C_1C_3 - 2C_2C_4 \\ C_2C_3 - 2C_1C_5 & C_1C_3 - 2C_2C_4 & 4C_4C_5 - C_3^2 \end{bmatrix} \\ &= \begin{bmatrix} -D_x & E_z & E_y \\ E_z & -D_y & E_x \\ E_y & E_x & -D_z \end{bmatrix}\end{aligned} \tag{4.22}$$

$$\operatorname{adj}\operatorname{adj} M = \begin{bmatrix} D_yD_z - E_x^2 & E_xE_y + D_zE_z & E_xE_z + D_yE_y \\ E_xE_y + D_zE_z & D_xD_z - E_y^2 & E_yE_z + D_xE_x \\ E_xE_z + D_yE_y & E_yE_z + D_xE_x & D_xD_y - E_z^2 \end{bmatrix}. \tag{4.23}$$

We prove a lemma that gives an alternate way of calculating $\operatorname{adj}\operatorname{adj} M$.

Lemma 4.24. *Let F be a field, and let n be a natural number greater than or equal to 2, and let A be an $n \times n$ matrix whose entries are elements of F. Then* $\operatorname{adj}\operatorname{adj} A = (\det A)^{n-2}A$.

Proof. Let

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ \multicolumn{5}{c}{\dotfill} \\ x_{n1} & x_{n2} & x_{n3} & \dots & x_{nn} \end{bmatrix}$$

where each x_{ij} is an indeterminate. Then $\det X \in F[x_{11}, x_{12}, x_{13}, \dots, x_{nn}]$. We start from a known result that can be found in, for example [6, p.137]:

$$(\operatorname{adj} X)X = X(\operatorname{adj} X) = (\det X)I. \tag{4.25}$$

Taking the determinant on both sides of the equation, we have

$$(\det X)(\det \operatorname{adj} X) = (\det X)^n. \tag{4.26}$$

Since $F[x_{11}, x_{12}, x_{13}, \dots, x_{nn}]$ is a domain and $\det X$ is not the zero polynomial, we have

$$(\det \operatorname{adj} X) = (\det X)^{n-1}. \tag{4.27}$$

Replacing X by $\operatorname{adj} X$ in (4.25), we have

$$\begin{aligned} (\operatorname{adj}(\operatorname{adj} X)) \operatorname{adj} X &= (\det \operatorname{adj} X)I \\ &= (\det X)^{n-1} I \qquad \text{by (4.27)}. \end{aligned} \tag{4.28}$$

Multiplying both sides by X on the right, we have

$$(\operatorname{adj}(\operatorname{adj} X))(\operatorname{adj} X)X = (\det X)^{n-1} X. \tag{4.29}$$

Since $(\operatorname{adj} X)X = (\det X)I$, we have

$$(\operatorname{adj}(\operatorname{adj} X))(\det X)I = (\det X)^{n-1} X. \tag{4.30}$$

Then since $F[x_{11}, x_{12}, x_{13}, \ldots, x_{nn}]$ is a domain and $\det X$ is not the zero polynomial, we have

$$(\operatorname{adj}(\operatorname{adj} X)) = (\det X)^{n-2} X.$$

Using the evaluation homomorphism to replace the indeterminates x_{ij} with entries of A, we have

$$(\operatorname{adj}(\operatorname{adj} A)) = (\det A)^{n-2} A. \qquad \square$$

We apply this lemma to the matrix M that we were discussing earlier. Since M is a 3×3 matrix, we have $n = 3$. By Lemma 4.24, we have $\operatorname{adj}\operatorname{adj} M = (\det M)M$. Since $\det M = -2R$, we have $\operatorname{adj}\operatorname{adj} M = (\det M)M = -2RM$. We matching the diagonal entries in the three equal matrices $(\det M)M$, $-2RM$, and $\operatorname{adj}\operatorname{adj} M$ as given in (4.23):

$$\begin{aligned}
(\det M)(2C_4) &= -2R(2C_4) = -4C_4R = D_yD_z - E_x^2 \\
(\det M)(2C_5) &= -2R(2C_5) = -4C_5R = D_xD_z - E_y^2 \\
(\det M)(2C_0) &= -2R(2C_0) = -4C_0R = D_xD_y - E_z^2.
\end{aligned} \tag{4.31}$$

Matching the rest of the entries in the three equal matrices, we have

$$\begin{aligned}
(\det M)(C_3) &= -2R(C_3) = -2C_3R = E_xE_y + D_zE_z \\
(\det M)(C_1) &= -2R(C_1) = -2C_1R = E_xE_z + D_yE_y \\
(\det M)(C_2) &= -2R(C_2) = -2C_2R = E_yE_z + D_xE_x.
\end{aligned} \tag{4.32}$$

We put the three equations in (4.31) into a lemma.

Lemma 4.33. *With the definitions for the Es and the Ds in* (4.8)–(4.13) *and the* *definition for R in* (4.2) *, the following equations are true.*

$$E_x^2 = D_y D_z + 4C_4 R$$
$$E_y^2 = D_x D_z + 4C_5 R \qquad (4.34)$$
$$E_z^2 = D_x D_y + 4C_0 R.$$

Proof. One way to show this is to expand the above expressions using the definitions in (4.8)–(4.16).

$$E_x^2 = C_1^2 C_3^2 - 4C_1C_2C_3C_4 + 4C_2^2C_4^2$$
$$D_yD_z = C_1^2C_3^2 - 4C_0C_3^2C_4 - 4C_1^2C_4C_5 + 16C_0C_4^2C_5$$
$$4C_4R = -4C_1C_2C_3C_4 + 4C_0C_3^2C_4 + 4C_2^2C_4^2 + 4C_1^2C_4C_5 - 16C_0C_4^2C_5$$

From these, we see that $E_x^2 = D_yD_z + 4C_4R$. Because of the symmetry diagram in Figure 4.1, it suffices to prove only one of the equations.

Alternatively, these equations can be shown to come directly from the diagonal of the $\operatorname{adj}\operatorname{adj} M$ in (4.31) as it has been shown earlier. □

4.1 When Is f A Square

When $\operatorname{char} F \neq 2$, $R = 0$ and $D_x = D_y = D_z = 0$, then f is a square times a constant. Towards proving this, we first prove a lemma showing the relationship between f, L_x, D_y, D_z, and E_x.

Lemma 4.35. *With the definitions in* (4.8)–(4.16)*, we have the following.*

$$4C_4 f = L_x^2 - D_y z^2 - D_z y^2 - 2E_x yz$$
$$4C_5 f = L_y^2 - D_x z^2 - D_z x^2 - 2E_y xz \qquad (4.36)$$
$$4C_0 f = L_z^2 - D_y x^2 - D_x y^2 - 2E_z xy$$

Proof. We will expand L_z^2 and $-D_yx^2 - D_xy^2 - 2E_zxy$ in the third equation $4C_0f = L_z^2 - D_yx^2 - D_xy^2 - 2E_zxy$ using the definitions in (4.8)–(4.16).

$$L_z^2 = C_1^2x^2 + 2C_1C_2xy + C_2^2y^2 + 4C_0C_1xz + 4C_0C_2yz + 4C_0^2z^2 \tag{4.37}$$

$$\begin{aligned} -D_yx^2 - D_xy^2 - 2E_zxy = (-C_1^2 + 4C_0C_4)x^2 - (C_2^2 - 4C_0C_5)y^2 \\ - 2(C_1C_2 - 2C_0C_3)xy. \end{aligned} \tag{4.38}$$

Adding these two equations, we get

$$\begin{aligned} L_z^2 - D_yx^2 - D_xy^2 - 2E_zxy &= 4C_0C_1xz + 4C_0C_2yz + 4C_0^2z^2 + 4C_0C_4x^2 \\ &\quad + 4C_0C_5y^2 + 4C_0C_3xy \\ &= 4C_0(C_1xz + C_2yz + C_0z^2 + C_4x^2 + C_5y^2 \\ &\quad + C_3xy) \\ &= 4C_0f. \end{aligned} \tag{4.39}$$

Similarly, using the symmetry diagram in Figure 4.1, we get the other two equations.

□

Equations (4.36) leads to factorizations of f over the base field.

Theorem 4.40. *If* $\operatorname{char} F \neq 2$ *and* $R = D_x = D_y = D_z = 0$*, then* f *factors over the base field* F *as a square times a constant.*

Proof. Plugging $R = D_x = D_y = D_z = 0$ into (4.34) in Lemma 4.33, we have $E_x = E_y = E_z = 0$. If $C_0 \neq 0$, then equation (4.36) says $f = \dfrac{L_z^2}{4C_0}$. Similarly, if $C_4 \neq 0$, then $f = \dfrac{L_x^2}{4C_4}$. And if $C_5 \neq 0$, then $f = \dfrac{L_y^2}{4C_5}$. So if one of the C_0, C_4, or C_5 is nonzero, then by Lemma 4.35, f is a square times a constant.

If $C_0 = C_4 = C_5 = 0$, then by the definitions for D_x, D_y, and D_z in

(4.8)–(4.10), we have $C_1 = C_2 = C_3 = 0$ and $f = 0$, which is a square times a constant. $\square$

It is straightforward to show that the if-condition above is in fact an if-and-only-if condition.

Theorem 4.41. *If* $\operatorname{char} F \neq 2$*, then* $f = A(a_0z+a_1x+a_2y)^2$ *for some* $A, a_0, a_1, a_2 \in F$ *if and only if* $R = D_x = D_y = D_z = 0$.

Proof. ($\Leftarrow$) This direction has been proven in Theorem 4.40.

($\Rightarrow$) Since f factors, Lemma 4.6 tells us that $R = 0$. Expanding $A(a_0z + a_1x + a_2y)^2$, and matching coefficients with f, we have

$$\begin{aligned} C_0 &= Aa_0^2 \\ C_1 &= 2Aa_1a_0 \\ C_2 &= 2Aa_2a_0 \\ C_3 &= 2Aa_1a_2 \\ C_4 &= Aa_1^2 \\ C_5 &= Aa_2^2. \end{aligned} \tag{4.42}$$

Plugging these into the definitions for D_x, D_y, and D_z in (4.8)–(4.10), we have

$$\begin{aligned} D_x &= C_2^2 - 4C_0C_5 = (2Aa_2a_0)^2 - 4(Aa_0^2)(Aa_2^2) = 0 \\ D_y &= C_1^2 - 4C_0C_4 = (2Aa_1a_0)^2 - 4(Aa_0^2)(Aa_1^2) = 0 \\ D_z &= C_3^2 - 4C_4C_5 = (2Aa_1a_2)^2 - 4(Aa_1^2)(Aa_2^2) = 0. \end{aligned}$$

$\square$

4.2 When Is f Reducible But Not A Square

If $f(x,y,z) = g(x,y,z)h(x,y,z)$, then setting $y = 0$, we have $f(x,0,z) = g(x,0,z)h(x,0,z)$. So a factorization for the trivariate case gives a factorization for the bivariate case. While the converse does not hold, a factorization for the bivariate case might give us some information about the factorization for the trivariate case. We take f as in (4.1) and set $y = 0$. Then we get the polynomial $C_0z^2 + C_1xz + C_4x^2$ whose discriminant is $D_y = C_1^2 - 4C_0C_4$. The following factorizations for $f(x,0,z) = C_0z^2 + C_1xz + C_4x^2$, $f(0,y,z) = C_5y^2 + C_2yz + C_0z^2$, and $f(x,y,0) = C_4x^2 + C_3xy + C_5y^2$ (which correspond to the sides of Figure 4.1) leads to the factorizations of f that are not a square.

Theorem 4.43. *Suppose $C_0z^2 + C_1xz + C_4x^2 = (a_0z + a_1x)(b_0z + b_1x)$. Then*

$$D_yf = f_1f_2 + Ry^2 \tag{4.44}$$

where $f_1 = d_y(a_1x + a_0z) + (a_1C_2 - a_0C_3)y$, $f_2 = d_y(b_1x + b_0z) - (b_1C_2 - b_0C_3)y$, and $d_y = a_1b_0 - a_0b_1$. Similarly, suppose $C_5y^2 + C_2yz + C_0z^2 = (a_2y + a_0z)(b_2y + b_0z)$. Then

$$D_xf = g_1g_2 + Rx^2 \tag{4.45}$$

where $g_1 = d_x(a_0z + a_2y) + (a_0C_2 - a_2C_1)x$, $g_2 = d_x(b_0z + b_2y) - (b_0C_3 - b_2C_1)x$, and $d_x = a_0b_2 - a_2b_0$. And similarly, suppose $C_4x^2 + C_3xy + C_5y^2 = (a_1x + a_2y)(b_1x + b_2y)$. Then

$$D_zf = h_1h_2 + Rz^2 \tag{4.46}$$

where $h_1 = d_z(a_2y + a_1x) + (a_2C_1 - a_1C_2)z$, $h_2 = d_z(b_2y + b_1x) - (b_2C_1 - b_1C_2)z$, and $d_z = a_2b_1 - a_1b_2$.

Proof. Matching coefficients in $C_0z^2 + C_1xz + C_4x^2 = (a_0z + a_1x)(b_0z + b_1x)$, we get

$$C_4 = a_1b_1$$
$$C_1 = a_1b_0 + a_0b_1$$
$$C_0 = a_0b_0.$$

Plugging these into (4.1) and (4.9), we find

$$\begin{aligned}
D_yf &= (-4a_0a_1b_0b_1 + (a_1b_0 + a_0b_1)^2) \\
&\quad \times (a_1b_1x^2 + C_3xy + C_5y^2 + (a_1b_0 + a_0b_1)xz + C_2yz + a_0b_0z^2) \\
&= (a_1^3b_0^2b_1 - 2a_0a_1^2b_0b_1^2 + a_0^2a_1b_1^3)x^2 \\
&\quad + (a_1^2b_0^2C_5 - 2a_0a_1b_0b_1C_5 + a_0^2b_1^2C_5)y^2 \\
&\quad + (a_1^2b_0^2C_2 - 2a_0a_1b_0b_1C_2 + a_0^2b_1^2C_2)yz \\
&\quad + (a_0a_1^2b_0^3 - 2a_0^2a_1b_0^2b_1 + a_0^3b_0b_1^2)z^2 \\
&\quad + (a_1^2b_0^2C_3 - 2a_0a_1b_0b_1C_3 + a_0^2b_1^2C_3)xy \\
&\quad + (a_1^3b_0^3 - a_0a_1^2b_0^2b_1 - a_0^2a_1b_0b_1^2 + a_0^3b_1^3)xz.
\end{aligned} \tag{4.47}$$

Plugging the same set of substitutions for C_i into f_1, f_2, and R, we find

$$f_1 = (a_1^2b_0 - a_0a_1b_1)x + (a_1C_2 - a_0C_3)y + (a_0a_1b_0 - a_0^2b_1)z \tag{4.48}$$

$$f_2 = (a_1b_0b_1 - a_0b_1^2)x + (-b_1C_2 + b_0C_3)y + (a_1b_0^2 - a_0b_0b_1)z \tag{4.49}$$

$$
\begin{aligned}
f_1 f_2 = {} & (a_1^3 b_0^2 b_1 - 2a_0 a_1^2 b_0 b_1^2 + a_0^2 a_1 b_1^3)x^2 \\
& + (-a_1 b_1 C_2^2 + a_1 b_0 C_2 C_3 + a_0 b_1 C_2 C_3 - a_0 b_0 C_3^2)y^2 \\
& + (a_1^2 b_0^2 C_2 - 2a_0 a_1 b_0 b_1 C_2 + a_0^2 b_1^2 C_2)yz \\
& + (a_0 a_1^2 b_0^3 - 2a_0^2 a_1 b_0^2 b_1 + a_0^3 b_0 b_1^2)z^2 \\
& + (a_1^2 b_0^2 C_3 - 2a_0 a_1 b_0 b_1 C_3 + a_0^2 b_1^2 C_3)xy \\
& + (a_1^3 b_0^3 - a_0 a_1^2 b_0^2 b_1 - a_0^2 a_1 b_0 b_1^2 + a_0^3 b_1^3)xz
\end{aligned}
\tag{4.50}
$$

$$
\begin{aligned}
R = {} & a_1 b_1 C_2^2 - a_1 b_0 C_2 C_3 - a_0 b_1 C_2 C_3 + a_0 b_0 C_3^2 + a_1^2 b_0^2 C_5 - 2a_0 a_1 b_0 b_1 C_5 \\
& + a_0^2 b_1^2 C_5.
\end{aligned}
\tag{4.51}
$$

From these equations, we can almost see that $D_y f = f_1 f_2 + Ry^2$. Almost all the terms of $D_y f$ appear in $f_1 f_2$. Only the y^2-term requires a bit of thoughts. The four terms in the coefficient of y^2 in $f_1 f_2$ cancels with the first four terms in the coefficient of y^2 in Ry^2. The remaining three terms in the coefficient of y^2 in Ry^2 gives the coefficients of y^2 in $D_y f$. Similarly, by rotating the triangle diagram in Figure 4.1, we get the other two assertions. □

An immediate consequence of Lemma 4.43 is the following.

Theorem 4.52. *If $R = 0$, and one of D_x, D_y, or D_z is nonzero, then f factors over a quadratic extension of F as in Lemma 4.43, and f is not a square.*

Proof. If $D_y \neq 0$, then by Theorem 3.1, $C_0 z^2 + C_1 xz + C_4 x^2 = (a_0 z + a_1 x)(b_0 z + b_1 x)$ for some a_0, a_1, b_0, and b_1 in $F\left[\sqrt{D_y}\right]$. The a_i and b_i can then be used to construct f_1 and f_2 as in Lemma 4.43. Since $R = 0$ and $D_y \neq 0$, we have $f = \dfrac{f_1 f_2}{D_y}$ by Lemma 4.43. Similarly, if D_x or D_z is nonzero, then f factors over $F\left[\sqrt{D_x}\right]$ or $F\left[\sqrt{D_z}\right]$. □

We combine Lemma 4.6, Theorem 4.40, and Theorem 4.52 to get an if-and-only-if condition for f to factor over F or over an extension of F.

Theorem 4.53. *Suppose* char $F \neq 2$. *Then f is reducible over F or over an extension of F if and only if $R = 0$.*

Proof. ($\Rightarrow$) This direction was proven in Lemma 4.6.

($\Leftarrow$) Assume $R = 0$. If $D_x = D_y = D_z = 0$, then by Theorem 4.40, f factors over the base field F as a square times a constant. If one of D_x, D_y, or D_z is nonzero, then by Theorem 4.52, f factors over a quadratic extension of F as in Lemma 4.43. □

When does f factor over the base field F without the need to go to an extension of F? Towards answering this question, we start with the following lemma.

Lemma 4.54. *If $R = 0$ and one of the followings holds*

(1) $0 \neq D_z$ *is a square in* F*, or*

(2) $0 \neq D_y$ *is a square in* F*, or*

(3) $0 \neq D_x$ *is a square in* F*,*

then f factors over F.

Proof. Suppose D_z is a square in F. Then $C_4x^2+C_3xy+C_5y^2 = (a_1x+a_2y)(b_1x+b_2y)$ for some $a_i, b_i \in F$. Then the following expressions, as defined in Theorem 4.43, $d_z = a_2b_1 - a_1b_2$ is in F, and $h_1 = d_z(a_2y + a_1x) + (a_2C_1 - a_1C_2)z$ and $h_2 = d_z(b_2y + b_1x) - (b_2C_1 - b_1C_2)z$ are in $F[x, y]$. By (4.46), $f = \dfrac{h_1h_2}{D_z}$ factors over F.

Similarly, if D_y is a square in F, then $f = \dfrac{f_1f_2}{D_y}$ factors over F where f_1 and f_2 are as defined in Theorem 4.43.

Similarly, if D_x is a square in F, then $f = \dfrac{g_1g_2}{D_x}$ factors over F where g_1 and g_2 are as defined in Theorem 4.43. □

We now give an if-and-only-if condition for f to factor over the base field F.

Theorem 4.55. *If* $\operatorname{char} F \neq 2$*, then* f *factors over* F *if and only if* $R = 0$ *and* D_x*,* D_y*, and* D_z *are squares in* F*.*

Proof. ($\Leftarrow$) Assume $R = 0$ and D_x, D_y, and D_z are squares in F. If $D_x = D_y = D_z = 0$, then by Theorem 4.40, f factors over F as a square times a constant. If one of the D_x, D_y, or D_z is nonzero, then by Lemma 4.54, f factors over F.

($\Rightarrow$) Suppose f factors over F as $f = (a_0z + a_1x + a_2y)(b_0z + b_1x + b_2y)$ for some $a_i, b_i \in F$. Then the coefficents of f are given by (4.7) Plugging these coefficients into $D_z = C_3^2 - 4C_4C_5$, we have

$$\begin{aligned} D_z &= (a_2b_1 + a_1b_2)^2 - 4(a_1b_1)(a_2b_2) \\ &= a_2^2b_1^2 + 2a_1a_2b_1b_2 + a_1^2b_2^2 - 4a_1a_2b_1b_2 \\ &= a_2^2b_1^2 - 2a_1a_2b_1b_2 + a_1^2b_2^2 \\ &= (a_2b_1 - a_1b_2)^2 \end{aligned} \tag{4.56}$$

and $a_2b_1 - a_1b_2$ is in F since $a_i, b_i \in F$. Thus, D_z is a square in F. Similarly, D_y and D_x are squares in F. $\square$

Example 4.57. Let F be a field. Let $f = x^2-6xy-2y^2-20xz-6yz+z^2 \in F[x, y, z]$. We will use (4.44) in Lemma 4.43 to obtain a factorization of f. The other two equations (4.45) and (4.46) in the same lemma gives factorizations of f that are essentially the same as the one obtained from (4.44) except for order and a constant factor, because $F[x, y, z]$ is a UFD. Matching coefficients of f with (4.1), we get

$$\begin{aligned} C_0 &= 1 \\ C_1 &= -20 \\ C_2 &= -6 \\ C_3 &= -6 \\ C_4 &= 1 \\ C_5 &= -2. \end{aligned} \tag{4.58}$$

We calculate R, D_y, D_x, and D_z.

$$\begin{aligned} R &= -C_1C_2C_3 + C_0C_3^2 + C_2^2C_4 + C_1^2C_5 - 4C_0C_4C_5 \\ &= 720 + 36 + 36 - 800 + 8 \\ &= 0 \\ D_y &= C_1^2 - 4C_0C_4 = 400 - 4 = 396 = 6^2 \cdot 11 \\ D_x &= C_2^2 - 4C_0C_5 = 36 + 8 = 44 = 2^2 \cdot 11 \\ D_z &= C_3^2 - 4C_4C_5 = 36 + 8 = 2^2 \cdot 11. \end{aligned} \tag{4.59}$$

Consider the polynomial $C_0z^2 + C_1xz + C_4x^2 = z^2 - 20xz + x^2$. By Theorem 3.1, we have

$$
\begin{aligned}
C_0 z^2 + C_1 xz + C_4 x^2 &= z^2 - 20xz + x^2 \\
&= \frac{1}{4C_0}\left(2C_0 z + \left(C_1 - \sqrt{C_1^2 - 4C_0C_4}\right)x\right) \\
&\quad \times \left(2C_0 z + \left(C_1 + \sqrt{C_1^2 - 4C_0C_4}\right)x\right) \\
&= \frac{1}{4}\left(2z + \left(-20 - 6\sqrt{11}\right)x\right) \\
&\quad \times \left(2z + \left(-20 + 6\sqrt{11}\right)x\right) \\
&= \left(z + \left(-10 - 3\sqrt{11}\right)x\right) \\
&\quad \times \left(z + \left(-10 + 3\sqrt{11}\right)x\right).
\end{aligned}
\tag{4.60}
$$

We set $a_0 = 1$, $a_1 = -10 - 3\sqrt{11}$, $b_0 = 1$, and $b_1 = -10 + 3\sqrt{11}$. Then $d_y = a_1 b_0 - a_0 b_1 = -6\sqrt{11}$,

$$
\begin{aligned}
f_1 &= -6\sqrt{11}\left(\left(-10 - 3\sqrt{11}\right)x + z\right) \\
&\quad + \left(\left(-10 - 3\sqrt{11}\right)(-6) + 6\right)y \\
&= 6\left[\left(33 + 10\sqrt{11}\right)x + \left(11 + 3\sqrt{11}\right)y - \sqrt{11}z\right]
\end{aligned}
\tag{4.61}
$$

$$
\begin{aligned}
f_2 &= -6\sqrt{11}\left(\left(-10 + 3\sqrt{11}\right)x + z\right) \\
&\quad - \left(\left(-10 + 3\sqrt{11}\right)(-6) + 6\right)y \\
&= 6\left[\left(-33 + 10\sqrt{11}\right)x + \left(-11 + 3\sqrt{11}\right)y - \sqrt{11}z\right]
\end{aligned}
\tag{4.62}
$$

By (4.44), we have

$$
\begin{aligned}
f &= \frac{1}{6^2 \cdot 11}\left(6\left[\left(33 + 10\sqrt{11}\right)x + \left(11 + 3\sqrt{11}\right)y - \sqrt{11}z\right]\right) \\
&\quad \times \left(6\left[\left(-33 + 10\sqrt{11}\right)x + \left(-11 + 3\sqrt{11}\right)y - \sqrt{11}z\right]\right) \\
&= \left[\left(10 + 3\sqrt{11}\right)x + \left(3 + \sqrt{11}\right)y - z\right] \\
&\quad \times \left[\left(10 - 3\sqrt{11}\right)x + \left(3 - \sqrt{11}\right)y - z\right].
\end{aligned}
\tag{4.63}
$$

We will show that f is a difference of squares when $R = 0$. We first rewrite f by completing the square, treating f as a polynomial in x:

$$\begin{aligned} f &= C_0z^2 + C_1xz + C_2yz + C_3xy + C_4x^2 + C_5y^2 \\ &= C_4x^2 + (C_1z + C_3y)x + (C_0z^2 + C_5y^2 + C_2yz). \end{aligned} \tag{4.64}$$

With L_x as defined in (4.14), we have

$$\begin{aligned} 4C_4f &= (2C_4x + C_1z + C_3y)^2 - ((C_1z + C_3y)^2 - 4C_4(C_0z^2 + C_5y^2 + C_2yz)) \\ &= L_x^2 - (C_1^2z^2 + 2C_1C_3yz + C_3^2y^2 - 4C_0C_4z^2 - 4C_4C_5y^2 - 4C_4C_2yz) \\ &= L_x^2 - ((C_1^2 - 4C_0C_4)z^2 + (C_3^2 - 4C_4C_5)y^2 + 2(C_1C_3 - 2C_2C_4)yz) \\ &= L_x^2 - (D_yz^2 + 2E_xyz + D_zy^2). \end{aligned} \tag{4.65}$$

This equation is exactly the same as (4.36). Then we rewrite the second term $-(D_yz^2 + 2E_xyz + D_zy^2)$ by completing the square, treating it as a polynomial in y:

$$\begin{aligned} D_z(4C_4f - L_x^2) &= -(D_z^2y^2 + 2D_zE_xyz + D_yD_zz^2) \\ &= -((D_zy + E_xz)^2 - E_x^2z^2 + D_yD_zz^2) \\ &= -((D_zy + E_xz)^2 + (D_yD_z - E_x^2)z^2) \\ &= -((D_zy + E_xz)^2 - 4C_4Rz^2) \quad \text{by (4.34)}. \end{aligned} \tag{4.66}$$

From this, we can solve for $4D_zC_4f$:

$$4D_zC_4f = D_zL_x^2 - (D_zy + E_xz)^2 + 4C_4Rz^2. \tag{4.67}$$

Similarly, and by symmetry using Figure 4.1, we have five other equations:

$$4D_zC_5f = D_zL_y^2 - (D_zx + E_yz)^2 + 4C_5Rz^2. \tag{4.68}$$

$$4D_xC_0f = D_xL_z^2 - (D_xy + E_zx)^2 + 4C_0Rx^2. \tag{4.69}$$

$$4D_xC_5f = D_xL_y^2 - (D_xz + E_yx)^2 + 4C_5Rx^2. \tag{4.70}$$

$$4D_yC_0f = D_yL_z^2 - (D_yx + E_zy)^2 + 4C_0Ry^2. \tag{4.71}$$

$$4D_yC_4f = D_yL_x^2 - (D_yz + E_xy)^2 + 4C_4Ry^2. \tag{4.72}$$

From (4.67), if $R = 0$ and $D_zC_4 \neq 0$, then f is a difference of squares

$$f = \frac{1}{4D_zC_4}\left(D_zL_x^2 - (D_zy + E_xz)^2\right). \tag{4.73}$$

Similarly, if $R = 0$ and $D_zC_5 \neq 0$, then f is a difference of squares

$$f = \frac{1}{4D_zC_5}\left(D_zL_y^2 - (D_zx + E_yz)^2\right). \tag{4.74}$$

Similarly, if $R = 0$ and $D_xC_0 \neq 0$, then f is a difference of squares

$$f = \frac{1}{4D_xC_0}\left(D_xL_z^2 - (D_xy + E_zx)^2\right). \tag{4.75}$$

Similarly, if $R = 0$ and $D_xC_5 \neq 0$, then f is a difference of squares

$$f = \frac{1}{4D_xC_5}\left(D_xL_y^2 - (D_xz + E_yx)^2\right). \tag{4.76}$$

Similarly, if $R = 0$ and $D_yC_0 \neq 0$, then f is a difference of squares

$$f = \frac{1}{4D_yC_0}\left(D_yL_z^2 - (D_yx + E_zy)^2\right). \tag{4.77}$$

Similarly, if $R = 0$ and $D_yC_4 \neq 0$, then f is a difference of squares

$$f = \frac{1}{4D_yC_4}\left(D_yL_x^2 - (D_yz + E_xy)^2\right). \tag{4.78}$$

Otherwise if we are not in any one of the cases above, then suppose $R = 0$ and

$$\begin{gathered} C_4D_z = C_5D_z = 0 \\ C_0D_x = C_5D_x = 0 \\ C_0D_y = C_4D_y = 0. \end{gathered} \tag{4.79}$$

If all the discriminants D_z, D_y, and D_x are zero, then by Theorem 4.41, we are done.

The remaining case is that one of the discriminants is nonzero. So, WLOG, suppose $D_z \neq 0$. Then by (4.79), we have $C_4 = C_5 = 0$, which implies $D_z = C_3^2$. By Lemma 4.54, $f = \dfrac{h_1 h_2}{D_z}$ where h_1 and h_2 are as defined in Lemma 4.43 and are in $F[x, y, z]$. Since f factors over F, f can be readily rewritten as a difference of squares with coefficients in F:

$$\begin{aligned} f &= \frac{h_1 h_2}{D_z} \\ &= \frac{1}{4D_z}\left((h_1 + h_2)^2 - (h_1 - h_2)^2\right). \end{aligned} \tag{4.80}$$

Similarly, if D_y or D_x is nonzero, then f can be factored as a difference of squares with coefficients in F.

From (4.73), f can be readily factored as a difference of squares:

Theorem 4.81. *Suppose* char $F \neq 2$. *If* $C_4 D_z \neq 0$, *and* $R = 0$, *then*

$$f = \frac{1}{4C_4 D_z}\left(L_x\sqrt{D_z} + (D_z y + E_x z)\right)\left(L_x\sqrt{D_z} - (D_z y + E_x z)\right). \tag{4.82}$$

Similarly, if $C_5 D_z \neq 0$, *and* $R = 0$, *then*

$$f = \frac{1}{4C_5 D_z}\left(L_y\sqrt{D_z} + (D_z x + E_y z)\right)\left(L_y\sqrt{D_z} - (D_z x + E_y z)\right). \tag{4.83}$$

Similarly, if $C_0 D_x \neq 0$, *and* $R = 0$, *then*

$$f = \frac{1}{4C_0 D_x}\left(L_z\sqrt{D_x} + (D_x y + E_z x)\right)\left(L_z\sqrt{D_x} - (D_x y + E_z x)\right). \tag{4.84}$$

Similarly, if $C_5 D_x \neq 0$, *and* $R = 0$, *then*

$$f = \frac{1}{4C_5 D_x}\left(L_y\sqrt{D_x} + (D_x z + E_y x)\right)\left(L_y\sqrt{D_x} - (D_x z + E_y x)\right). \tag{4.85}$$

Similarly, if $C_0 D_y \neq 0$, and $R = 0$, then

$$f = \frac{1}{4C_0 D_y}\left(L_z\sqrt{D_y} + (D_y x + E_z y)\right)\left(L_z\sqrt{D_y} - (D_y x + E_z y)\right). \tag{4.86}$$

Similarly, if $C_4 D_y \neq 0$, and $R = 0$, then

$$f = \frac{1}{4C_4 D_y}\left(L_x\sqrt{D_y} + (D_y z + E_x y)\right)\left(L_x\sqrt{D_y} - (D_y z + E_x y)\right). \tag{4.87}$$

One might be curious about whether $R = 0$ and $D_x = D_y = 0$ implies $D_z = 0$. This is false because of the following counterexample.

Counterexample 4.88. Let $f = x^2 + y^2$. Matching coefficients, we find $C_0 = C_1 = C_2 = C_3 = 0$ and $C_4 = C_5 = 1$. Then $R = -C_1C_2C_3 + C_0C_3^2 + C_2^2C_4 + C_1^2C_5 - 4C_0C_4C_5 = 0$, $D_x = C_2^2 - 4C_0C_5 = 0$ and $D_y = C_1^2 - 4C_0C_4 = 0$, but $D_z = C_3 - 4C_4C_5 = -4 \neq 0$.

However, the statement is almost true because if we add $C_0 \neq 0$ to the assumption, then we will be able to prove $D_z = 0$.

Theorem 4.89. *If $C_0 \neq 0$ and $R = D_x = D_y = 0$, then $D_z = 0$.*

Proof. Plugging $R = D_x = D_y = 0$ into (4.34), we have $E_z = 0$. From the definition of D_x, D_y, and E_z, we have

$$\begin{aligned} D_x = 0 &\implies C_2^2 = 4C_0C_5 \\ D_y = 0 &\implies C_1^2 = 4C_0C_4 \\ E_z = 0 &\implies C_1C_2 = 2C_0C_3. \end{aligned} \tag{4.90}$$

We use these to calculate the following:

$$\begin{aligned} 4C_0^2 D_z &= 4C_0^2(C_3^2 - 4C_4C_5) \\ &= (2C_0C_3)^2 - (4C_0C_4)(4C_0C_5) \\ &= (C_1C_2)^2 - C_1^2C_2^2 \\ &= 0. \end{aligned} \tag{4.91}$$

Since $C_0 \neq 0$, we have $D_z = 0$. $\square$

Let A and B be elements of a field F. How does a general symmetric degree 2 trivariate polynomial $f = A(x^2+y^2+z^2)+B(xy+xz+yz)$ factor when it does? We will show either $f = 4A^2(x+y+z)^2$, or $f = A(x+\omega z+\omega^2 y)(x+\omega^2 z+\omega y)$ where $\omega = e^{\frac{2\pi i}{3}}$ is a root of x^2+x+1.

Example 4.92. With f defined as in (4.1) and $f = A(x^2+y^2+z^2)+B(xy+xz+yz)$, matching coefficients, we have $C_4 = C_5 = C_0 = A$ and $C_3 = C_1 = C_2 = B$. Plugging these values into R, we find $R = -(2A-B)^2(A+B)$. Suppose f is reducible over an extension of F. Then $B = 2A$ or $B = -A$.

Case (1): If $B = 2A$, then

$$f = A(x^2+y^2+z^2) + 2A(xy+xz+yz) = A(x+y+z)^2.$$

Case (2): If $B = -A$, then

$$f = A(x^2+y^2+z^2) - A(xy+xz+yz). \tag{4.93}$$

Matching coefficients, we find $C_0 = C_4 = C_5 = A$, $C_1 = C_2 = C_3 = -A$. Then $D_x = D_y = D_z = A^2 - 4A^2 = -3A^2$. If $A = 0$, then $D_x = D_y = D_z = 0$ and $f = 0$. If $A \neq 0$, then $D_x = D_y = D_z = -3A^2 \neq 0$. By Lemma 4.43, $D_y f = D_x f =$

$D_z f = -3A^2 f = f_1 f_2 = g_1 g_2 = h_1 h_2$. Then $C_0 z^2 + C_1 xz + C_4 x^2$ factors as in (4.44).

$C_0 z^2 + C_1 xz + C_4 x^2 = Az^2 - Axz + Ax^2 = (Ax + A\omega z)(x + \omega^2 z) = (a_0 z + a_1 x)(b_0 z + b_1 x)$.

Then $a_0 = A\omega$, $a_1 = A$, $b_0 = \omega^2$, and $b_1 = 1$. Then $d_y = A\omega^2 - A\omega$, $d_y \omega^2 = A(\omega - 1)$,

$$\begin{aligned} f_1 &= d_y(Ax + A\omega z) + (-A^2 + A^2\omega)y \\ &= d_y A(x + \omega z) + A \cdot \underbrace{A(\omega - 1)}_{d_y \omega^2} y \\ &= A d_y (x + \omega z + \omega^2 y) \end{aligned} \tag{4.94}$$

$$\begin{aligned} f_2 &= d_y(x + \omega^2 z) - (-A + A\omega^2)y \\ &= d_y(x + \omega^2 z) - \underbrace{A(\omega - 1)}_{d_y \omega^2}(\omega + 1)y \\ &= d_y(x + \omega^2 z - (1 + \omega^2)y) \\ &= d_y(x + \omega^2 z + \omega y). \end{aligned} \tag{4.95}$$

Then

$$\begin{aligned} -3A^2 f &= f_1 f_2 \\ &= A d_y^2 (x + \omega z + \omega^2 y)(x + \omega^2 z + \omega y). \end{aligned} \tag{4.96}$$

Since $A \neq 0$, we have

$$\begin{aligned} f &= \frac{d_y^2}{-3A}(x + \omega z + \omega^2 y)(x + \omega^2 z + \omega y) \\ &= A(x + \omega z + \omega^2 y)(x + \omega^2 z + \omega y) \end{aligned} \tag{4.97}$$

since $d_y^2 = (A(\omega^2 - \omega))^2 = A^2(\omega - 2 + \omega^2) = -3A^2$.

We will show that the field extensions obtained by adjoining the square root of $D_x \neq 0$, $D_y \neq 0$, or $D_z \neq 0$ are in fact the same extension. Towards proving this, we first prove some similar assertions in a more general context.

Theorem 4.98. *Let $\alpha, \beta \in F$. If $F\left(\sqrt{\alpha}\right) = F\left(\sqrt{\beta}\right)$, then $\alpha\beta = s^2$ for some $s \in F$.*

Proof. (1) If $F\left(\sqrt{\alpha}\right) = F\left(\sqrt{\beta}\right) = F$, then $\alpha = s^2$ and $\beta = t^2$ for some $s, t \in F$. Then $\alpha\beta = s^2t^2 = (st)^2$.

(2) If $F\left(\sqrt{\alpha}\right) = F\left(\sqrt{\beta}\right) \neq F$, then $\sqrt{\alpha} = a + b\sqrt{\beta}$ for some $a, b \in F$. Then $\alpha = a^2 + 2ab\sqrt{\beta} + b^2\beta$. This can be rewritten as $0 = (-\alpha + a^2 + b^2\beta) + 2ab\sqrt{\beta}$. Since $\sqrt{\beta} \notin F$, we have

$$-\alpha + a^2 + b^2\beta = 0 \tag{4.99}$$

and $2ab = 0$. The latter implies $a = 0$ or $b = 0$. If $a = 0$, then (4.99) implies $\alpha = b^2\beta$. Then $\alpha\beta = b^2\beta^2 = (b\beta)^2$.

If $b = 0$, then then (4.99) implies $\alpha = a^2$. Then $F\left(\sqrt{\alpha}\right) = F(a) = F$ which contradicts our assumption. Therefore $F\left(\sqrt{\alpha}\right) \neq F$. □

Theorem 4.100. *If $\alpha\beta = s^2 \neq 0$ for some $s \in F$, then $F\left(\sqrt{\alpha}\right) = F\left(\sqrt{\beta}\right)$.*

Proof. From $\alpha\beta = s^2$, we have $\sqrt{\alpha}\sqrt{\beta} = s$. Since $s \neq 0$, we have $\beta \neq 0$ and $\sqrt{\beta} \neq 0$. Since $\sqrt{\alpha}\sqrt{\beta} = s$, we have $\sqrt{\alpha} = \dfrac{s}{\sqrt{\beta}} \in F\left(\sqrt{\beta}\right)$. So $F\left(\sqrt{\alpha}\right) \subseteq F\left(\sqrt{\beta}\right)$. Similarly, $F\left(\sqrt{\alpha}\right) \supseteq F\left(\sqrt{\beta}\right)$. □

We are now ready to prove that the field extensions obtained by joining either D_x, D_y, or D_z are in fact the same.

Theorem 4.101. *If $R = 0$, $D_x \neq 0$, and $D_y \neq 0$, then $F\left(\sqrt{D_x}\right) = F\left(\sqrt{D_y}\right)$.*

Proof. Plugging $R = 0$ into $E_z^2 = D_xD_y + 4C_0R$, we find $D_xD_y = E_z^2 \neq 0$ since D_x and D_y are nonzero. By (4.13), we have $E_z \in F$ since $C_1, C_2, C_0, C_3 \in F$. By Theorem 4.100, we have $F\left(\sqrt{D_x}\right) = F\left(\sqrt{D_y}\right)$. □

CHAPTER 5

Trivariate Homogeneous Degree 3 Polynomials

We do not have complete answer to the question of reducibility of trivariate homogeneous degree 3 polynomials. We only have partial results. We will show that for a trivariate homogeneous degree 3 polynomial f over a field F, the rank of a 3×3 matrix M, derived from the coefficients of f, gives us some information about the reducibility of f over F or over an extension of F. Specifically,

(1) If f factors, then M has rank 1 or 0.

(2) If f factors completely over F or over an extension of F, then M is the zero matrix.

(3) If $\operatorname{rank} M = 1$, then we have a candidate factor for f that has to be checked using long division.

(4) If f factors and $M \neq 0$, then f can be factored over the coefficient field F.

Later on, we will extend M to a 9×3 matrix V by appending six rows to M. All of the results mentioned above will continue to hold when M is replaced by V. For example, if f factors, then V has rank 1 or 0. We will show an example where f does not factor and M has rank 1 but V has rank 3. This shows that V has an advantage in telling when f is irreducible. We will show an example where $V = 0$ and f factors over F, but f does not factor completely. One of the unanswered questions is: When V has rank 1 or 0, does this imply that f is reducible?

Towards showing the mentioned results, we start by defining f and then show some consequences when f is reducible over F or over an extension of F. Let F be a field, and let $C_0, C_1, C_2, \ldots, C_9 \in F$, and let

$$f(x,y,z) = C_0z^3 + C_1xz^2 + C_2yz^2 + C_4x^2z + C_3xyz + C_5y^2z + C_6x^2y + C_7xy^2 + C_8x^3 + C_9y^3 \tag{5.1}$$

with f not equal to the zero polynomial. Suppose

$$f(x,y,z) = (a_0z + a_1x + a_2y)(b_0z^2 + b_1xz + b_2yz + b_4x^2 + b_3xy + b_5y^2) \tag{5.2}$$

for some a_i and b_i in an extension of F, where not all the a_i are zero and not all the b_i are zero.

Consider the change of variable in x, y, and z with new variables u, v, and w where

$$x = -ua_0 + wa_2 \qquad y = -va_0 - wa_1 \qquad z = ua_1 + va_2.$$

If you make this change in $f(x,y,z)$ in (5.2), then the first factor in that equation becomes zero:

$$\begin{aligned} &a_0z + a_1x + a_2y \\ &= a_0(ua_1 + va_2) + a_1(-ua_0 + wa_2) + a_2(-va_0 - wa_1) \\ &= ua_0a_1 + va_0a_2 - ua_0a_1 + wa_1a_2 - va_0a_2 - wa_1a_2 \\ &= 0. \end{aligned} \tag{5.3}$$

Hence making this change in $f(x,y,z)$, we get 0.

So $f(\underbrace{-ua_0 + wa_2}_{x}, \underbrace{-va_0 - wa_1}_{y}, \underbrace{ua_1 + va_2}_{z}) = 0$. Plugging these underbraced quantities into (5.1) (using a computer algebra system), we find

$$
\begin{aligned}
&f(-ua_0 + wa_2, -va_0 - wa_1, ua_1 + va_2) \\
&= (K_{w^3})w^3 + (K_{uw^2})uw^2 + (K_{vw^2})vw^2 + (K_{u^2w})u^2w + (K_{uvw})uvw \\
&\quad + (K_{v^2w})v^2w + (K_{u^3})u^3 + (K_{u^2v})u^2v + (K_{uv^2})uv^2 + (K_{v^3})v^3 \\
&= 0
\end{aligned}
\tag{5.4}
$$

where

$$
\begin{aligned}
K_{w^3} &= -a_1a_2^2C_6 + a_1^2a_2C_7 + a_2^3C_8 - a_1^3C_9 \\
K_{uw^2} &= -a_1^2a_2C_3 + a_1a_2^2C_4 + a_1^3C_5 + 2a_0a_1a_2C_6 - a_0a_1^2C_7 - 3a_0a_2^2C_8 \\
K_{u^2w} &= a_1^2a_2C_1 - a_1^3C_2 + a_0a_1^2C_3 - 2a_0a_1a_2C_4 - a_0^2a_1C_6 + 3a_0^2a_2C_8 \\
K_{u^3} &= a_1^3C_0 - a_0a_1^2C_1 + a_0^2a_1C_4 - a_0^3C_8 \\
K_{vw^2} &= -a_1a_2^2C_3 + a_2^3C_4 + a_1^2a_2C_5 - a_0a_2^2C_6 + 2a_0a_1a_2C_7 - 3a_0a_1^2C_9 \\
K_{v^2w} &= a_2^3C_1 - a_1a_2^2C_2 - a_0a_2^2C_3 + 2a_0a_1a_2C_5 + a_0^2a_2C_7 - 3a_0^2a_1C_9 \\
K_{v^3} &= a_2^3C_0 - a_0a_2^2C_2 + a_0^2a_2C_5 - a_0^3C_9 \\
K_{u^2v} &= 3a_1^2a_2C_0 - 2a_0a_1a_2C_1 - a_0a_1^2C_2 + a_0^2a_1C_3 + a_0^2a_2C_4 - a_0^3C_6 \\
K_{uv^2} &= 3a_1a_2^2C_0 - a_0a_2^2C_1 - 2a_0a_1a_2C_2 + a_0^2a_2C_3 + a_0^2a_1C_5 - a_0^3C_7 \\
K_{uvw} &= a_1a_2^2C_1 - a_1^2a_2C_2 - a_0a_2^2C_4 + a_0a_1^2C_5 + a_0^2a_2C_6 - a_0^2a_1C_7.
\end{aligned}
\tag{5.5}
$$

Since $f(ua_1 + va_2, -ua_0 + wa_2, -va_0 - wa_1) = 0 \in F[u, v, w]$, all the ten Ks must be zero.

Thus we have the following theorem.

Theorem 5.6. *If $a_0z + a_1x + a_2y$ divides f, then all the ten Ks in* (5.5) *must be zero.*

(Curiously, K_{uvw} is irrelevant in the argument for the converse.)

Theorem 5.7. *If* $a_0z + a_1x + a_2y \neq 0$ *and all the ten* K*s in* (5.5) *are zero, then* $a_0z + a_1x + a_2y$ *divides* f.

Proof. To facilitate our discussion, see Figure 5.1 for a symmetry diagram for the ten Ks in (5.5).

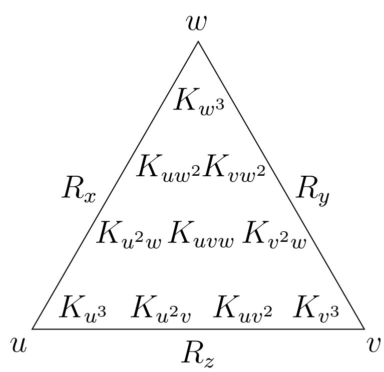

Figure 5.1: Symmetry diagram for the ten Ks.

We can write a_0f, a_1f, and a_2f each as a multiple of $a_0z + a_1x + a_2y$ plus a remainder as follows:

$$\begin{aligned} a_0^3 f &= (a_0z + a_1x + a_2y)Q_z + R_z \\ a_1^3 f &= (a_0z + a_1x + a_2y)Q_x + R_x \\ a_2^3 f &= (a_0z + a_1x + a_2y)Q_y + R_y \end{aligned} \tag{5.8}$$

where

$$
\begin{aligned}
Q_z &= (a_1^2C_0 - a_0a_1C_1 + a_0^2C_4)x^2 + (a_2^2C_0 - a_0a_2C_2 + a_0^2C_5)y^2 \\
&\quad + (-a_0a_2C_0 + a_0^2C_2)yz + a_0^2C_0z^2 \\
&\quad + (2a_1a_2C_0 - a_0a_2C_1 - a_0a_1C_2 + a_0^2C_3)xy + (-a_0a_1C_0 + a_0^2C_1)xz \\
R_z &= -(K_{u^3})x^3 - (K_{u^2v})x^2y - (K_{uv^2})xy^2 - (K_{v^3})y^3 \\
Q_x &= a_1^2C_8x^2 + (a_1^2C_6 - a_1a_2C_8)xy + (-a_1a_2C_6 + a_1^2C_7 + a_2^2C_8)y^2 \\
&\quad + (a_1^2C_4 - a_0a_1C_8)xz + (a_1^2C_3 - a_1a_2C_4 - a_0a_1C_6 + 2a_0a_2C_8)yz \\
&\quad + (a_1^2C_1 - a_0a_1C_4 + a_0^2C_8)z^2 \\
R_x &= -(K_{w^3})y^3 + (K_{uw^2})y^2z - (K_{u^2w})yz^2 + (K_{u^3})z^3 \\
Q_y &= (a_2^2C_6 - a_1a_2C_7 + a_1^2C_9)x^2 + (a_2^2C_7 - a_1a_2C_9)xy \\
&\quad + a_2^2C_9y^2 + (a_2^2C_3 - a_1a_2C_5 - a_0a_2C_7 + 2a_0a_1C_9)xz \\
&\quad + (a_2^2C_5 - a_0a_2C_9)yz + (a_2^2C_2 - a_0a_2C_5 + a_0^2C_9)z^2 \\
R_y &= (K_{w^3})x^3 + (K_{vw^2})x^2z + (K_{v^2w})xz^2 + (K_{v^3})z^3.
\end{aligned} \tag{5.9}
$$

Since all the ten Ks are zero, we have $R_z = R_x = R_y = 0$. Since $a_0z + a_1x + a_2y \neq 0$, we have $a_0 \neq 0$, $a_1 \neq 0$, or $a_2 \neq 0$. Then one of the three equations in (5.8) implies that $a_0z + a_1x + a_2y$ divides f. $\square$

Notice that Theorem 5.7 is true if

$$
\begin{aligned}
&a_0 \neq 0 \text{ and } K_{u^3} = K_{u^2v} = K_{uv^2} = K_{v^3} = 0, \text{ or if} \\
&a_1 \neq 0 \text{ and } K_{w^3} = K_{uw^2} = K_{u^2w} = K_{u^3} = 0, \text{ or if} \\
&a_2 \neq 0 \text{ and } K_{w^3} = K_{vw^2} = K_{v^2w} = K_{v^3} = 0.
\end{aligned}
$$

We combine Theorem 5.6 and Theorem 5.7 into one theorem below.

Theorem 5.10. *Suppose $l = a_0 z + a_1 x + a_2 y \neq 0$. Then l divides f if and only if all of the ten Ks in* (5.5) *are zero.*

This theorem will lead to the test, that we mentioned at the beginning, for the reducibility of f when the rank of M is 1. First we need to define the matrix M. Towards this goal, we recall that when f factors as in (5.2), we have $a_0 \neq 0$, $a_1 \neq 0$, or $a_2 \neq 0$. If $a_1 \neq 0$, then (5.8) implies that $R_x = 0$ and (5.9) implies that $K_{w^3} = K_{uw^2} = K_{u^2w} = K_{u^3} = 0$. We calculate the resultant (defined in the Introduction chapter) of K_{u^2w} and K_{u^3}, treating them as polynomials in a_0 and assuming that their leading coefficients are nonzero. In order for $K_{u^2w} = K_{u^3} = 0$ to hold, this resultant must be zero. Since this resultant contains a factor of $a_1^6 \neq 0$, the other part of the resultant without the factor of a_1^6 must be zero. We call this other part R_{xz}:

$$R_{xz} = \frac{1}{a_1^6} \operatorname{Res}_{a_0} \left(K_{u^2w}, K_{u^3} \right).$$

Then we use polynomial long division to divide R_{xz} by K_{w^3}, treating them as polynomials in a_2 and assuming that their leading coefficients are nonzero. Then $R_{xz} = K_{w^3} Q + R$ for some quotient Q and some remainder R. The remainder R contains a factor of a_1^2. Since $R_{xz} = 0$ and $K_{w^3} = 0$, we have $R = 0$. Since $a_1 \neq 0$, the other part of R without the factor of a_1^2 must be zero. We call this other part R_{xzy} and it looks like the following:

$$R_{xzy} = (-m_{32}) a_2 + (m_{33}) a_1$$

where

$$
\begin{aligned}
m_{32} = &-C_1^2C_3C_4C_6 + 4C_0C_3C_4^2C_6 + C_1^3C_6^2 - 4C_0C_1C_4C_6^2 \\
&+ C_1^2C_4^2C_7 - 4C_0C_4^3C_7 + C_1^2C_3^2C_8 - C_1C_2C_3C_4C_8 \\
&- 3C_0C_3^2C_4C_8 + C_2^2C_4^2C_8 + 2C_1^2C_2C_6C_8 \\
&- 3C_0C_1C_3C_6C_8 - 6C_0C_2C_4C_6C_8 + 9C_0^2C_6^2C_8 \\
&- 4C_1^3C_7C_8 + 18C_0C_1C_4C_7C_8 - 3C_1C_2^2C_8^2 \\
&+ 9C_0C_2C_3C_8^2 - 27C_0^2C_7C_8^2 \qquad (5.11) \\
m_{33} = &-C_1C_2C_3C_4C_6 + C_0C_3^2C_4C_6 + C_2^2C_4^2C_6 + C_1^2C_2C_6^2 \\
&- C_0C_1C_3C_6^2 - 2C_0C_2C_4C_6^2 + C_0^2C_6^3 + C_1C_2C_3^2C_8 \\
&- C_0C_3^3C_8 - C_2^2C_3C_4C_8 - 2C_1C_2^2C_6C_8 + 3C_0C_2C_3C_6C_8 \\
&+ C_2^3C_8^2 + C_1^2C_4^2C_9 - 4C_0C_4^3C_9 - 4C_1^3C_8C_9 \\
&+ 18C_0C_1C_4C_8C_9 - 27C_0^2C_8^2C_9.
\end{aligned}
$$

The first quantity m_{32} is the entries of M in row 3, column 2. The second quantity m_{33} is the entries of M in row 3, column 3.

Using similar technique, but performing the previously described operations with respect to a different variable at each stage, we get the other entries of M:

Define

$$
M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \tag{5.12}
$$

where

$$
\begin{aligned}
m_{11} = {} & C_4C_5^2C_6^2 - C_3C_4C_5C_6C_7 + C_4^2C_5C_7^2 + C_0C_6^2C_7^2 \\
& - C_3C_5^2C_6C_8 + C_3^2C_5C_7C_8 - 2C_4C_5^2C_7C_8 - 4C_0C_7^3C_8 \\
& + C_5^3C_8^2 + C_3^2C_4C_6C_9 - 2C_4^2C_5C_6C_9 - 4C_0C_6^3C_9 \\
& - C_3C_4^2C_7C_9 - C_3^3C_8C_9 + 3C_3C_4C_5C_8C_9 + 18C_0C_6C_7C_8C_9 \\
& + C_4^3C_9^2 - 27C_0C_8^2C_9^2 \\
m_{12} = {} & -C_3C_4C_6C_7^2 + C_1C_6^2C_7^2 + C_4^2C_7^3 + C_5^2C_6^2C_8 \\
& - C_3C_5C_6C_7C_8 + C_3^2C_7^2C_8 + 2C_4C_5C_7^2C_8 - 4C_1C_7^3C_8 \\
& - 3C_5^2C_7C_8^2 + 4C_3C_4C_6^2C_9 - 4C_1C_6^3C_9 - 4C_4^2C_6C_7C_9 \\
& - 3C_3^2C_6C_8C_9 - 6C_4C_5C_6C_8C_9 - 3C_3C_4C_7C_8C_9 \\
& + 18C_1C_6C_7C_8C_9 + 9C_3C_5C_8^2C_9 + 9C_4^2C_8C_9^2 \\
& - 27C_1C_8^2C_9^2 \\
m_{13} = {} & C_5^2C_6^3 - C_3C_5C_6^2C_7 + C_2C_6^2C_7^2 - 4C_5^2C_6C_7C_8 \\
& + 4C_3C_5C_7^2C_8 - 4C_2C_7^3C_8 + C_3^2C_6^2C_9 + 2C_4C_5C_6^2C_9 \\
& - 4C_2C_6^3C_9 - C_3C_4C_6C_7C_9 + C_4^2C_7^2C_9 - 3C_3C_5C_6C_8C_9 \\
& - 3C_3^2C_7C_8C_9 - 6C_4C_5C_7C_8C_9 + 18C_2C_6C_7C_8C_9 \\
& + 9C_5^2C_8^2C_9 - 3C_4^2C_6C_9^2 + 9C_3C_4C_8C_9^2 - 27C_2C_8^2C_9^2
\end{aligned}
\tag{5.13}
$$

$$
\begin{aligned}
m_{21} = &-C_1C_2C_3C_5^2 + C_0C_3^2C_5^2 + C_2^2C_4C_5^2 + C_1^2C_5^3 \\
&- 4C_0C_4C_5^3 - C_0C_2C_3C_5C_7 + 2C_0C_1C_5^2C_7 + C_0C_2^2C_7^2 \\
&- 3C_0^2C_5C_7^2 + 4C_1C_2^2C_3C_9 - 3C_0C_2C_3^2C_9 - 4C_2^3C_4C_9 \\
&- 4C_1^2C_2C_5C_9 - 3C_0C_1C_3C_5C_9 + 18C_0C_2C_4C_5C_9 \\
&- 6C_0C_1C_2C_7C_9 + 9C_0^2C_3C_7C_9 + 9C_0C_1^2C_9^2 - 27C_0^2C_4C_9^2 \\
m_{22} = &-C_1C_2C_3C_5C_7 + C_0C_3^2C_5C_7 + C_1^2C_5^2C_7 + C_1C_2^2C_7^2 \\
&- C_0C_2C_3C_7^2 - 2C_0C_1C_5C_7^2 + C_0^2C_7^3 + C_2^2C_5^2C_8 \\
&- 4C_0C_5^3C_8 + C_1C_2C_3^2C_9 - C_0C_3^3C_9 - C_1^2C_3C_5C_9 \\
&- 2C_1^2C_2C_7C_9 + 3C_0C_1C_3C_7C_9 - 4C_2^3C_8C_9 \\
&+ 18C_0C_2C_5C_8C_9 + C_1^3C_9^2 - 27C_0^2C_8C_9^2 \\
m_{23} = &\, C_2^2C_5^2C_6 - 4C_0C_5^3C_6 - C_2^2C_3C_5C_7 + 4C_0C_3C_5^2C_7 \\
&+ C_2^3C_7^2 - 4C_0C_2C_5C_7^2 + C_2^2C_3^2C_9 - C_1C_2C_3C_5C_9 \\
&- 3C_0C_3^2C_5C_9 + C_1^2C_5^2C_9 - 4C_2^3C_6C_9 + 18C_0C_2C_5C_6C_9 \\
&+ 2C_1C_2^2C_7C_9 - 3C_0C_2C_3C_7C_9 - 6C_0C_1C_5C_7C_9 \\
&+ 9C_0^2C_7^2C_9 - 3C_1^2C_2C_9^2 + 9C_0C_1C_3C_9^2 - 27C_0^2C_6C_9^2
\end{aligned}
\tag{5.14}
$$

$$
\begin{aligned}
m_{31} = &-C_1C_2C_3C_4^2 + C_0C_3^2C_4^2 + C_2^2C_4^3 + C_1^2C_4^2C_5 \\
&- 4C_0C_4^3C_5 - C_0C_1C_3C_4C_6 + 2C_0C_2C_4^2C_6 + C_0C_1^2C_6^2 \\
&- 3C_0^2C_4C_6^2 + 4C_1^2C_2C_3C_8 - 3C_0C_1C_3^2C_8 \\
&- 4C_1C_2^2C_4C_8 - 3C_0C_2C_3C_4C_8 - 4C_1^3C_5C_8 \\
&+ 18C_0C_1C_4C_5C_8 - 6C_0C_1C_2C_6C_8 + 9C_0^2C_3C_6C_8 \\
&+ 9C_0C_2^2C_8^2 - 27C_0^2C_5C_8^2 \\
m_{32} = &-C_1^2C_3C_4C_6 + 4C_0C_3C_4^2C_6 + C_1^3C_6^2 - 4C_0C_1C_4C_6^2 \\
&+ C_1^2C_4^2C_7 - 4C_0C_4^3C_7 + C_1^2C_3^2C_8 - C_1C_2C_3C_4C_8 \\
&- 3C_0C_3^2C_4C_8 + C_2^2C_4^2C_8 + 2C_1^2C_2C_6C_8 \\
&- 3C_0C_1C_3C_6C_8 - 6C_0C_2C_4C_6C_8 + 9C_0^2C_6^2C_8 \\
&- 4C_1^3C_7C_8 + 18C_0C_1C_4C_7C_8 - 3C_1C_2^2C_8^2 \\
&+ 9C_0C_2C_3C_8^2 - 27C_0^2C_7C_8^2 \\
m_{33} = &-C_1C_2C_3C_4C_6 + C_0C_3^2C_4C_6 + C_2^2C_4^2C_6 + C_1^2C_2C_6^2 \\
&- C_0C_1C_3C_6^2 - 2C_0C_2C_4C_6^2 + C_0^2C_6^3 + C_1C_2C_3^2C_8 \\
&- C_0C_3^3C_8 - C_2^2C_3C_4C_8 - 2C_1C_2^2C_6C_8 + 3C_0C_2C_3C_6C_8 \\
&+ C_2^3C_8^2 + C_1^2C_4^2C_9 - 4C_0C_4^3C_9 - 4C_1^3C_8C_9 \\
&+ 18C_0C_1C_4C_8C_9 - 27C_0^2C_8^2C_9.
\end{aligned}
\tag{5.15}
$$

Let's see what happens when f factors as in (5.2). Matching the coefficients in (5.2), and we find

$$\begin{aligned}
C_0 &= a_0b_0 \\
C_1 &= a_1b_0 + a_0b_1 \\
C_2 &= a_2b_0 + a_0b_2 \\
C_3 &= a_2b_1 + a_1b_2 + a_0b_3 \\
C_4 &= a_1b_1 + a_0b_4 \\
C_5 &= a_2b_2 + a_0b_5 \\
C_6 &= a_1b_3 + a_2b_4 \\
C_7 &= a_2b_3 + a_1b_5 \\
C_8 &= a_1b_4 \\
C_9 &= a_2b_5.
\end{aligned} \tag{5.16}$$

With the definitions for the entries m_{ij} of the matrix M, one could use (5.16) to substitute the Cs in the definitions of m_{ij} with the as and the bs in (5.16), but this is unwieldy and tedious by hand. However, using a computer algebra system, this can easily be done. After the substitutions are made and the entries factored, one will find that

$$M = \begin{bmatrix} a_0d_z^2E & a_1d_z^2E & a_2d_z^2E \\ a_0d_x^2E & a_1d_x^2E & a_2d_x^2E \\ a_0d_y^2E & a_1d_y^2E & a_2d_y^2E \end{bmatrix} \tag{5.17}$$

where

$$\begin{aligned} d_z &= -a_1a_2b_3 + a_2^2b_4 + a_1^2b_5 \\ d_x &= a_2^2b_0 - a_0a_2b_2 + a_0^2b_5 \\ d_y &= a_1^2b_0 - a_0a_1b_1 + a_0^2b_4 \\ E &= -b_1b_2b_3 + b_0b_3^2 + b_2^2b_4 + b_1^2b_5 - 4b_0b_4b_5. \end{aligned} \tag{5.18}$$

From this, we get the following three theorems.

Theorem 5.19. *If f factors over F or over an extension of F, then* $\operatorname{rank} M = 0$ *or* 1.

Proof. Since f factors over F or over an extension of F, M has the form in (5.17) which has rank 1 or 0. □

Theorem 5.20. *If f factors completely over F or over an extension of F, then M is the zero matrix.*

Proof. Since f factors, f can be written as in (5.2). Then M has the form in (5.17). Since f factors completely, the second factor $b_0z^2 + b_1xz + b_2yz + b_4x^2 + b_3xy + b_5y^2$ in (5.2) is reducible. By Lemma 4.6, we have $E = 0$ and M is the zero matrix. □

Theorem 5.21. *If f factors over F or over an extension of F and $M = 0$, then one or both of the following holds:*

(1) *$E = 0$ and f factors completely over F or over an extension of F, or*

(2) $d_z = d_x = d_y = 0$.

Proof. Again, since f factors, f can be written as in (5.2) and M has the form in (5.17) and at least one of the a_0, a_1, and a_2 is nonzero. If $a_0 \neq 0$, then since $M = 0$, the first column of M as given in (5.17) says that $E = 0$ or $d_z = d_x = d_y = 0$. Similarly, if $a_1 \neq 0$ or if $a_2 \neq 0$, then $E = 0$ or $d_z = d_x = d_y = 0$. □

We now describe the test for the reducibility of f when the rank of M is 1. Recall equation (5.17). It says that if M has rank 1 and f factors over F or over an extension of F, then one of the rows in (5.12) is nonzero and that row has entries that are essentially a_0, a_1, and a_2 up to a constant multiple.

Thus if M has rank 1 and we do not know whether f is reducible, we could choose a nonzero row of M and use the entries of that row as a_0, a_1, and a_2. If $a_0 \neq 0$, then we can use the first equation in (5.8) to determine whether f factors. We plug the values of a_0, a_1, and a_2 into R_z as in (5.9) which in turn plugs the values of a_0, a_1, and a_2 into the definitions for K_{u^3}, K_{u^2v}, K_{uv^2}, and K_{v^3} as in (5.5). By the first equation in (5.8), f factors if and only if $R_z = 0$. Similarly, if $a_1 \neq 0$, then f factors if and only if $R_x = 0$. And similarly, if $a_2 \neq 0$, then f factors if and only if $R_y = 0$.

Knowing only rank $M = 1$ is not sufficient for f to factor. We give an example.

Example 5.22. Let $f = x^2y + xy^2 + x^2z + y^2z + xz^2 + yz^2$. Then calculating M, we find

$$M = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

Since M has rank 1 and all the entries are equal, we can apply our test with $a_0 = a_1 = a_2 = 1$. We plug these values into the coefficients of R_x. These coefficients are K_{w^3}, K_{uw^2}, K_{u^2w}, and K_{u^3} as in (5.5), and we find

$$K_{w^3} = 0 \qquad K_{uw^2} = 3 \qquad K_{u^2w} = -3 \qquad K_{u^3} = 0$$

$$\begin{aligned} R_x &= -(K_{w^3})y^3 + (K_{uw^2})y^2z - (K_{u^2w})yz^2 + (K_{u^3})z^3 \\ &= 3y^2z + 3yz^2. \end{aligned}$$

Since $R_x \neq 0$, equation (5.8) implies that the only candidate $x + y + z$ that might be a factor of f (up to a constant multiple) is in fact not a factor of f. Hence f is irreducible over any field.

If f is reducible and $M \neq 0$, then f can be factored over the base field F. We prove this in the following theorem.

Theorem 5.23. *Let f be as given in* (5.1)*. If $f = (a_0z + a_1x + a_2y)(b_0z^2 + b_1xz + b_2yz + b_4x^2 + b_3xy + b_5y^2)$ for some a_i and b_j in an extension of F and $M \neq 0$, where M is as given in* (5.12)*, then f can be factored over the base field F.*

Proof. The entries m_{ij} of M as defined in (5.13) consist of the Cs which are in the coefficient field F. So $m_{ij} \in F$ for all i and j. Since f factors, M has the form in (5.17) where each row of M is a multiple of (a_0, a_1, a_2). So the rank of M must be 0 or 1. Since $M \neq 0$, we have $\operatorname{rank} M \neq 0$. So M as in (5.17) must have rank 1 and it must have a nonzero row. We can choose that nonzero row to replace a_0, a_1, and a_2 in a factorization of f. Since every row of M is in F, we have $a_i \in F$. We can divide f by $a_0z + a_1x + a_2y$ using polynomial long division to find the bs. Since $f \in F[x, y, z]$ and $a_0, a_1, a_2 \in F$, the polynomial long division gives $b_j \in F$ for all j. $\square$

Suppose f factors as in (5.2) and let

$$A(x, y, z) = a_1x + a_2y + a_0z$$

$$B(x, y, z) = b_0z^2 + b_1xz + b_2yz + b_4x^2 + b_3xy + b_5y^2.$$

Consider the change of variable in x, y, and z with new variables u, v, and w where

$$x = -ua_0 + wa_2 \qquad y = -va_0 - wa_1 \qquad z = ua_1 + va_2.$$

Suppose $a_1x + a_2y + a_0z$ divides $b_0z^2 + b_1xz + b_2yz + b_4x^2 + b_3xy + b_5y^2$. Then since $A(-ua_0 + wa_2, -va_0 - wa_1, ua_1 + va_2) = 0$ as was shown in (5.3), and since $A(-ua_0+wa_2, -va_0-wa_1, ua_1+va_2)$ divides $B(-ua_0+wa_2, -va_0-wa_1, ua_1+va_2)$, we have

$$B(-ua_0 + wa_2, -va_0 - wa_1, ua_1 + va_2) = 0.$$

Calculating $B(-ua_0 + wa_2, -va_0 - wa_1, ua_1 + va_2)$, we find

$$\begin{aligned}
&B(-ua_0 + wa_2, -va_0 - wa_1, ua_1 + va_2)\\
&= d_yu^2 + d_xv^2 + d_{xz}vw + d_zw^2 + d_{xy}uv - d_{yz}uw\\
&= 0
\end{aligned}$$

where d_x, d_y, and d_z are as defined in (5.18), and

$$\begin{aligned}
d_{xz} &= a_2^2b_1 - a_1a_2b_2 - a_0a_2b_3 + 2a_0a_1b_5\\
d_{xy} &= 2a_1a_2b_0 - a_0a_2b_1 - a_0a_1b_2 + a_0^2b_3\\
d_{yz} &= -a_1a_2b_1 + a_1^2b_2 - a_0a_1b_3 + 2a_0a_2b_4.
\end{aligned} \tag{5.24}$$

Thus if f factors as in (5.2) and $a_1x + a_2y + a_0z$ divides $b_0z^2 + b_1xz + b_2yz + b_4x^2 + b_3xy + b_5y^2$, then $d_x = d_y = d_z = d_{xy} = d_{xz} = d_{yz} = 0$.

Conversely, suppose f factors as in (5.2) and $d_x = d_y = d_z = d_{xy} = d_{xz} = d_{yz} = 0$. We first re-write $b_0z^2 + b_1xz + b_2yz + b_4x^2 + b_3xy + b_5y^2$ in the following three ways:

$$\begin{aligned}
a_0^2B &= (a_1x + a_2y + a_0z)q_z + (d_yx^2 + d_{xy}xy + d_xy^2)\\
a_1^2B &= (a_1x + a_2y + a_0z)q_x + (d_zy^2 + d_{yz}yz + d_yz^2)\\
a_2^2B &= (a_1x + a_2y + a_0z)q_y + (d_zx^2 + d_{xz}xz + d_xz^2)
\end{aligned} \tag{5.25}$$

where

$$q_z = (-a_1b_0 + a_0b_1)x + (-a_2b_0 + a_0b_2)y + a_0b_0z$$

$$q_x = a_1b_4x + (a_1b_3 - a_2b_4)y + (a_1b_1 - a_0b_4)z$$

$$q_y = (a_2b_3 - a_1b_5)x + a_2b_5y + (a_2b_2 - a_0b_5)z.$$

Since $d_x = d_y = d_z = d_{xy} = d_{xz} = d_{yz} = 0$, equation (5.25) implies that $a_1x+a_2y+a_0z$ divides $b_0z^2 + b_1xz + b_2yz + b_4x^2 + b_3xy + b_5y^2$.

We record these results in the following theorem.

Theorem 5.26. *Suppose f factors as in* (5.2)*. Then $a_1x + a_2y + a_0z$ divides $b_0z^2 + b_1xz + b_2yz + b_4x^2 + b_3xy + b_5y^2$ if and only if $d_x = d_y = d_z = d_{xy} = d_{xz} = d_{yz} = 0$.*

The three newly introduced quantities d_{xy}, d_{xz}, and d_{yz} will allow us to extend the 3×3 matrix M to a 9×3 matrix V such that all the previous theorems that contain M will continue to hold when M is replaced by V. We now describe how the matrix M is extended. Recall the matrix Mas given in (5.17). It says that if f factors as in (5.2), then the entries of M factors as shown in (5.17). For example, row 2 column 1 of M factors into $a_0d_x^2E$ when f factors as in (5.2). We note that the d_x as given in (5.18) contains two as and one b as factors in each one of its terms. And the E as given in (5.18) contains three bs as factors in each one of its terms. So $a_0d_x^2E$ contains five as and five bs as factors in each one of its terms. Looking at (5.14), we see that m_{21} the entries of M in row 2 column 2 has five Cs in each one of its terms. When f factors as in (5.2), the relationship between the Cs and the as and the bs are as given in (5.16). In (5.16), each one of the Cs contains one a and one b in each one of its term. Since m_{21} as in (5.14) contains five Cs in each one of its terms, m_{21} must contain five a and five bs in each one of its terms. Since $m_{21} = a_0d_x^2E$ when f

factors as in (5.2), the five as and the five bs in m_{21} must match the number of as and the number of bs in $a_0 d_x^2 E$, and they do because our previous counting indicated so. Since $a_0 d_x^2 E$ came from m_{21} as defined in (5.14), one might speculate whether $a_0 d_{xy}^2 E$ can be written in terms of the Cs as $a_0 d_x^2 E$ can be written terms of the Cs as given by m_{21} in (5.14). Using a computer algebra system, one can show that $a_0 d_{xy}^2 E$ cannot be written in terms of the Cs, but $a_0 d_x d_{xy} E$ can.

We now describe how to write $a_0 d_x d_{xy} E$ in terms of the Cs. (The process is also used to show that $a_0 d_{xy}^2 E$ cannot be written in terms of the Cs.) First, we note that $a_0 d_x d_{xy} E$ contains five as and five bs as factors in each one of its terms. Since each Cs as given in (5.16) contains one a and one b, our expression for $a_0 d_x d_{xy} E$ in terms of the Cs must contain five Cs in each one of its terms. Let X, Y, and Z be indeterminates

$$
\begin{aligned}
\bar{f} &= f(Xx, Yy, Zz) \\
&= (C_0 Z^3)z^3 + (C_1 X Z^2)xz^2 + (C_2 Y Z^2)yz^2 + (C_4 X^2 Z)x^2 z + (C_3 XYZ)xyz \\
&\quad + (C_5 Y^2 Z)y^2 z + (C_6 X^2 Y)x^2 y + (C_7 X Y^2)xy^2 + (C_8 X^3)x^3 + (C_9 Y^3)y^3 \\
&= [(a_0 Z)z + (a_1 X)x + (a_2 Y)y] \\
&\quad \times \left[(b_0 Z^2)z^2 + (b_1 XZ)xz + (b_2 YZ)yz + (b_4 X^2)x^2 + (b_3 XY)xy + (b_5 Y^2)y^2\right].
\end{aligned}
$$

Let

$$
\begin{aligned}
&\bar{C}_0 = C_0 Z^3 && \bar{C}_1 = C_1 X Z^2 && \bar{C}_2 = C_2 Y Z^2 \\
&\bar{C}_4 = C_4 X^2 Z && \bar{C}_3 = C_3 X Y Z && \bar{C}_5 = C_5 Y^2 Z \\
&\bar{C}_6 = C_6 X^2 Y && \bar{C}_7 = C_7 X Y^2 && \bar{C}_8 = C_8 X^3 \\
&\bar{C}_9 = C_9 Y^3 && && \qquad (5.27) \\
&\bar{a}_0 = a_0 Z && \bar{a}_1 = a_1 X && \bar{a}_2 = a_2 Y \\
&\bar{b}_0 = b_0 Z^2 && \bar{b}_1 = b_1 X Z && \bar{b}_2 = b_2 Y Z \\
&\bar{b}_3 = b_3 X Y && \bar{b}_4 = b_4 Y^2 && \bar{b}_5 = b_5 Y^2.
\end{aligned}
$$

We note that the $\bar{C}_i$, $\bar{a}_i$, and $\bar{b}_i$ are in $F[X, Y, Z]$. Let $\bar{d}_x$ be the same as the definition for d_x in (5.18) but with bars for a_i and b_i, i.e.,

$$\bar{d}_x = \bar{a}_2^2 \bar{b}_0 - \bar{a}_0 \bar{a}_2 \bar{b}_2 + \bar{a}_0^2 \bar{b}_5.$$

Similarly, we define $\bar{E}$ using the definition in (5.18) but with bars. Let

$$\bar{E} = -\bar{b}_1 \bar{b}_2 \bar{b}_3 + \bar{b}_0 \bar{b}_3^2 + \bar{b}_2^2 \bar{b}_4 + \bar{b}_1^2 \bar{b}_5 - 4 \bar{b}_0 \bar{b}_4 \bar{b}_5.$$

Let's calculate $\bar{a}_0\bar{d}_x^2\bar{E}$. But first we calculate $\bar{d}_x^2$ and $\bar{E}$.

$$\begin{aligned}
\bar{d}_x &= \bar{a}_2^2\bar{b}_0 - \bar{a}_0\bar{a}_2\bar{b}_2 + \bar{a}_0^2\bar{b}_5 \\
&= a_2^2Y^2b_0Z^2 - a_0Za_2Yb_2YZ + a_0^2Z^2b_5Y^2 \\
&= Y^2Z^2(a_2^2b_0 - a_0a_2b_2 + a_0^2b_5) \\
&= Y^2Z^2d_x \\
\bar{E} &= -b_1XZb_2YZb_3XY + b_0Z^2b_3^2X^2Y^2 + b_2^2Y^2Z^2b_4X^2 + b_1^2X^2Z^2b_5Y^2 \\
&\quad - 4b_0Z^2b_4X^2B_5Y^2 \\
&= X^2Y^2Z^2(-b_1b_2b_3 + b_0b_3^2 + b_2^2b_4 + b_1^2b_5 - 4b_0b_4b_5) \\
&= X^2Y^2Z^2E.
\end{aligned}$$

Then

$$\begin{aligned}
\bar{a}_0\bar{d}_x^2\bar{E} &= (a_0Z)(Y^4Z^4d_x^2)(X^2Y^2Z^2E) \\
&= a_0d_x^2E(X^2Y^6Z^7).
\end{aligned}$$

We note that this is a polynomial in $F[X, Y, Z]$ and it has total degree 15. We want to write this in terms of the $\bar{C}$s. Since each $\bar{C}_i$ as in (5.27) has total-degree 3, we will need five $\bar{C}$s in each one of our terms, and the five $\bar{C}$s must have multi-degree $X^2Y^6Z^7$. One can use a computer algebra system to look for all possibilities for the five $\bar{C}$s, and one will find that these possibilities are precisely the monomials in the definition for m_{21} as in (5.14) but with a bar over each C_i. For example,

$$\begin{aligned}
\bar{C}_1\bar{C}_2\bar{C}_3\bar{C}_5^2 &= (C_1XZ^2)(C_2YZ^2)(C_3XYZ)(C_5Y^2Z)^2 = C_1C_2C_3C_5^2X^2Y^6Z^7 \\
\bar{C}_0\bar{C}_3^2\bar{C}_5^2 &= (C_0Z^3)(C_3XYZ)^2(C_5Y^2Z)^2 = C_0C_3^2C_5^2X^2Y^6Z^7 \\
&\vdots \\
\bar{C}_0^2\bar{C}_4\bar{C}_9^2 &= (C_0Z^3)^2(C_4X^2Z)(C_9Y^3)^2 = C_0^2C_4C_9^2X^2Y^6Z^7.
\end{aligned} \tag{5.28}$$

Using a computer algebra system, one can look for linear combination of the monomials without the bars such that the linear combination equals $a_0 d_x^2 E$, and it will show that the only linear combination is the one that is given in the definition of m_{21} as in (5.14).

Using similar technique, one can show that $a_0 d_{xy}^2 E$ cannot be written in terms of the Cs, but the following 18 quantities can be written in terms of the Cs, and these expressions in the Cs are used to defined the entries in the six rows that are appended to M to form V:

$$a_i d_x d_{xy} E$$

$$a_i d_x d_{xz} E$$

$$a_i d_y d_{xy} E$$

$$a_i d_y d_{yz} E$$

$$a_i d_z d_{xz} E$$

$$a_i d_z d_{yz} E.$$

We now extend the 3×3 matrix M as defined in (5.12), and call the extended 9×3 matrix V: Let

$$V = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \\ v_{41} & v_{42} & v_{43} \\ v_{51} & v_{52} & v_{53} \\ \vdots & \vdots & \vdots \\ v_{91} & v_{92} & v_{93} \end{bmatrix} \tag{5.29}$$

where

$$
\begin{aligned}
v_{41} = {} & C_1C_5^2C_6^2 - C_1C_3C_5C_6C_7 - C_2C_4C_5C_6C_7 + C_0C_5C_6^2C_7 \\
& + 2C_1C_4C_5C_7^2 + C_0C_3C_6C_7^2 - 2C_0C_4C_7^3 - C_2C_5^2C_6C_8 \\
& + 2C_2C_3C_5C_7C_8 - 2C_1C_5^2C_7C_8 - 6C_0C_5C_7^2C_8 + C_1C_3^2C_6C_9 \\
& + 2C_2C_3C_4C_6C_9 - 4C_1C_4C_5C_6C_9 - 6C_0C_3C_6^2C_9 - 2C_1C_3C_4C_7C_9 \\
& - C_2C_4^2C_7C_9 + 9C_0C_4C_6C_7C_9 - 3C_2C_3^2C_8C_9 + 3C_1C_3C_5C_8C_9 \\
& + 3C_2C_4C_5C_8C_9 + 9C_0C_5C_6C_8C_9 + 9C_0C_3C_7C_8C_9 + 3C_1C_4^2C_9^2 \\
& - 27C_0C_4C_8C_9^2
\end{aligned}
\tag{5.30}
$$

$$
\begin{aligned}
v_{42} = {} & -C_3C_4C_5C_6C_7 + C_1C_5C_6^2C_7 + 2C_4^2C_5C_7^2 \\
& - C_2C_4C_6C_7^2 + C_0C_6^2C_7^2 + C_3C_5^2C_6C_8 - C_2C_5C_6C_7C_8 \\
& + 2C_2C_3C_7^2C_8 - 4C_1C_5C_7^2C_8 - 4C_0C_7^3C_8 - 2C_5^3C_8^2 \\
& + 2C_3^2C_4C_6C_9 - 4C_4^2C_5C_6C_9 - 2C_1C_3C_6^2C_9 + 4C_2C_4C_6^2C_9 \\
& - 4C_0C_6^3C_9 - 3C_3C_4^2C_7C_9 + C_1C_4C_6C_7C_9 - C_3^3C_8C_9 \\
& + 3C_3C_4C_5C_8C_9 - 6C_2C_3C_6C_8C_9 + 3C_1C_5C_6C_8C_9 \\
& + 6C_1C_3C_7C_8C_9 - 3C_2C_4C_7C_8C_9 + 18C_0C_6C_7C_8C_9 \\
& + 9C_2C_5C_8^2C_9 + 4C_4^3C_9^2 - 9C_1C_4C_8C_9^2 - 27C_0C_8^2C_9^2
\end{aligned}
\tag{5.31}
$$

$$
\begin{aligned}
v_{43} = {} & C_3C_5^2C_6^2 - C_3^2C_5C_6C_7 - 2C_4C_5^2C_6C_7 + 2C_3C_4C_5C_7^2 \\
& + C_2C_3C_6C_7^2 - 2C_2C_4C_7^3 - 2C_5^3C_6C_8 + 2C_3C_5^2C_7C_8 \\
& - 2C_2C_5C_7^2C_8 + C_3^3C_6C_9 - 4C_2C_3C_6^2C_9 + 2C_1C_5C_6^2C_9 \\
& - 2C_3^2C_4C_7C_9 - 2C_4^2C_5C_7C_9 - C_1C_3C_6C_7C_9 + 8C_2C_4C_6C_7C_9 \\
& + 2C_1C_4C_7^2C_9 - 3C_3^2C_5C_8C_9 + 6C_4C_5^2C_8C_9 + 6C_2C_5C_6C_8C_9 \\
& + 3C_2C_3C_7C_8C_9 - 6C_1C_5C_7C_8C_9 + 3C_3C_4^2C_9^2 - 6C_1C_4C_6C_9^2 \\
& + 9C_1C_3C_8C_9^2 - 18C_2C_4C_8C_9^2
\end{aligned} \tag{5.32}
$$

$$
\begin{aligned}
v_{51} = {} & 2C_2C_4C_5C_6^2 - 2C_0C_5C_6^3 - C_2C_3C_4C_6C_7 - C_1C_4C_5C_6C_7 \\
& + C_0C_3C_6^2C_7 + C_2C_4^2C_7^2 + C_0C_4C_6C_7^2 - 2C_2C_3C_5C_6C_8 \\
& - C_1C_5^2C_6C_8 + C_2C_3^2C_7C_8 + 2C_1C_3C_5C_7C_8 \\
& - 4C_2C_4C_5C_7C_8 + 9C_0C_5C_6C_7C_8 - 6C_0C_3C_7^2C_8 \\
& + 3C_2C_5^2C_8^2 + 2C_1C_3C_4C_6C_9 - 2C_2C_4^2C_6C_9 \\
& - 6C_0C_4C_6^2C_9 - C_1C_4^2C_7C_9 - 3C_1C_3^2C_8C_9 \\
& + 3C_2C_3C_4C_8C_9 + 3C_1C_4C_5C_8C_9 + 9C_0C_3C_6C_8C_9 \\
& + 9C_0C_4C_7C_8C_9 - 27C_0C_5C_8^2C_9
\end{aligned} \tag{5.33}
$$

$$
\begin{aligned}
v_{52} = {} & 2C_3C_4C_5C_6^2 - 2C_1C_5C_6^3 - C_3^2C_4C_6C_7 - 2C_4^2C_5C_6C_7 \\
& + C_1C_3C_6^2C_7 + C_3C_4^2C_7^2 - 2C_3^2C_5C_6C_8 - 2C_4C_5^2C_6C_8 \\
& + 2C_2C_5C_6^2C_8 + C_3^3C_7C_8 - C_2C_3C_6C_7C_8 + 8C_1C_5C_6C_7C_8 \\
& - 4C_1C_3C_7^2C_8 + 2C_2C_4C_7^2C_8 + 3C_3C_5^2C_8^2 \\
& - 6C_2C_5C_7C_8^2 + 2C_3C_4^2C_6C_9 - 2C_1C_4C_6^2C_9 - 2C_4^3C_7C_9 \\
& - 3C_3^2C_4C_8C_9 + 6C_4^2C_5C_8C_9 + 3C_1C_3C_6C_8C_9 \\
& - 6C_2C_4C_6C_8C_9 + 6C_1C_4C_7C_8C_9 + 9C_2C_3C_8^2C_9 \\
& - 18C_1C_5C_8^2C_9
\end{aligned} \tag{5.34}
$$

$$
\begin{aligned}
v_{53} = {} & 2C_4C_5^2C_6^2 - C_3C_4C_5C_6C_7 - C_1C_5C_6^2C_7 + C_2C_4C_6C_7^2 \\
& + C_0C_6^2C_7^2 - 3C_3C_5^2C_6C_8 + 2C_3^2C_5C_7C_8 - 4C_4C_5^2C_7C_8 \\
& + C_2C_5C_6C_7C_8 - 2C_2C_3C_7^2C_8 + 4C_1C_5C_7^2C_8 - 4C_0C_7^3C_8 \\
& + 4C_5^3C_8^2 + 2C_1C_3C_6^2C_9 - 4C_2C_4C_6^2C_9 - 4C_0C_6^3C_9 \\
& + C_3C_4^2C_7C_9 - C_1C_4C_6C_7C_9 - C_3^3C_8C_9 + 3C_3C_4C_5C_8C_9 \\
& + 6C_2C_3C_6C_8C_9 - 3C_1C_5C_6C_8C_9 - 6C_1C_3C_7C_8C_9 \\
& + 3C_2C_4C_7C_8C_9 + 18C_0C_6C_7C_8C_9 - 9C_2C_5C_8^2C_9 \\
& - 2C_4^3C_9^2 + 9C_1C_4C_8C_9^2 - 27C_0C_8^2C_9^2
\end{aligned} \tag{5.35}
$$

$$
\begin{aligned}
v_{61} = &-C_1C_2C_5^2C_6 + 2C_0C_3C_5^2C_6 - C_1C_2C_3C_5C_7 + C_2^2C_4C_5C_7 \\
&+ 2C_1^2C_5^2C_7 - 4C_0C_4C_5^2C_7 - C_0C_2C_5C_6C_7 + C_0C_2C_3C_7^2 \\
&- 2C_0^2C_7^3 + C_2^2C_5^2C_8 - 4C_0C_5^3C_8 + 2C_1C_2C_3^2C_9 \\
&- C_0C_3^3C_9 - 2C_2^2C_3C_4C_9 - 3C_1^2C_3C_5C_9 + C_1C_2C_4C_5C_9 \\
&+ 6C_0C_3C_4C_5C_9 + 4C_1C_2^2C_6C_9 - 6C_0C_2C_3C_6C_9 \\
&- 3C_0C_1C_5C_6C_9 - 4C_1^2C_2C_7C_9 + 3C_0C_1C_3C_7C_9 \\
&+ 3C_0C_2C_4C_7C_9 + 9C_0^2C_6C_7C_9 - 4C_2^3C_8C_9 \\
&+ 18C_0C_2C_5C_8C_9 + 4C_1^3C_9^2 - 9C_0C_1C_4C_9^2 - 27C_0^2C_8C_9^2
\end{aligned}
\tag{5.36}
$$

$$
\begin{aligned}
v_{62} = &-C_2C_3C_4C_5C_7 + 2C_1C_4C_5^2C_7 - C_1C_2C_5C_6C_7 \\
&+ 2C_0C_3C_5C_6C_7 + C_2^2C_4C_7^2 - 2C_0C_4C_5C_7^2 - C_0C_2C_6C_7^2 \\
&+ C_2C_3C_5^2C_8 - 2C_1C_5^3C_8 + C_2^2C_5C_7C_8 - 6C_0C_5^2C_7C_8 \\
&+ C_2C_3^2C_4C_9 - 2C_1C_3C_4C_5C_9 + 2C_1C_2C_3C_6C_9 \\
&- 3C_0C_3^2C_6C_9 - C_1^2C_5C_6C_9 - 4C_1C_2C_4C_7C_9 \\
&+ 3C_0C_3C_4C_7C_9 + 3C_0C_1C_6C_7C_9 - 6C_2^2C_3C_8C_9 \\
&+ 9C_1C_2C_5C_8C_9 + 9C_0C_3C_5C_8C_9 + 9C_0C_2C_7C_8C_9 \\
&+ 3C_1^2C_4C_9^2 - 27C_0C_1C_8C_9^2
\end{aligned}
\tag{5.37}
$$

$$
\begin{aligned}
v_{63} = {} & C_2C_3C_5^2C_6 - 2C_1C_5^3C_6 - C_2C_3^2C_5C_7 + 2C_1C_3C_5^2C_7 \\
& - 2C_0C_5^2C_6C_7 + C_2^2C_3C_7^2 - 2C_1C_2C_5C_7^2 + 2C_0C_3C_5C_7^2 \\
& - 2C_0C_2C_7^3 + C_2C_3^3C_9 - 2C_1C_3^2C_5C_9 - C_2C_3C_4C_5C_9 \\
& + 2C_1C_4C_5^2C_9 - 4C_2^2C_3C_6C_9 + 8C_1C_2C_5C_6C_9 \\
& + 3C_0C_3C_5C_6C_9 - 3C_0C_3^2C_7C_9 + 2C_2^2C_4C_7C_9 \\
& - 2C_1^2C_5C_7C_9 - 6C_0C_4C_5C_7C_9 + 6C_0C_2C_6C_7C_9 \\
& + 6C_0C_1C_7^2C_9 + 3C_1^2C_3C_9^2 - 6C_1C_2C_4C_9^2 \\
& + 9C_0C_3C_4C_9^2 - 18C_0C_1C_6C_9^2
\end{aligned}
\tag{5.38}
$$

$$
\begin{aligned}
v_{71} = {} & -C_1C_2C_3^2C_5 + C_0C_3^3C_5 + C_2^2C_3C_4C_5 + C_1^2C_3C_5^2 \\
& - 4C_0C_3C_4C_5^2 - C_0C_2C_3C_5C_6 + 2C_0C_1C_5^2C_6 \\
& + 2C_1C_2^2C_3C_7 - 2C_0C_2C_3^2C_7 - 2C_2^3C_4C_7 - 2C_1^2C_2C_5C_7 \\
& + 8C_0C_2C_4C_5C_7 + 2C_0C_2^2C_6C_7 - 6C_0^2C_5C_6C_7 \\
& - 2C_0C_1C_2C_7^2 + 3C_0^2C_3C_7^2 + 2C_1^2C_2C_3C_9 \\
& - 3C_0C_1C_3^2C_9 - 2C_1C_2^2C_4C_9 + 3C_0C_2C_3C_4C_9 \\
& - 2C_1^3C_5C_9 + 6C_0C_1C_4C_5C_9 - 6C_0C_1C_2C_6C_9 \\
& + 9C_0^2C_3C_6C_9 + 6C_0C_1^2C_7C_9 - 18C_0^2C_4C_7C_9
\end{aligned}
\tag{5.39}
$$

$$
\begin{aligned}
v_{72} = &-C_1C_2C_3C_5C_6 + C_0C_3^2C_5C_6 + C_1^2C_5^2C_6 - C_1C_2C_4C_5C_7 \\
&+ 2C_0C_3C_4C_5C_7 + 2C_1C_2^2C_6C_7 - 2C_0C_2C_3C_6C_7 \\
&- 4C_0C_1C_5C_6C_7 - C_0C_2C_4C_7^2 + 3C_0^2C_6C_7^2 + C_2^2C_3C_5C_8 \\
&+ C_1C_2C_5^2C_8 - 6C_0C_3C_5^2C_8 - 2C_2^3C_7C_8 + 9C_0C_2C_5C_7C_8 \\
&+ 2C_1C_2C_3C_4C_9 - 3C_0C_3^2C_4C_9 - C_1^2C_4C_5C_9 \\
&- 2C_1^2C_2C_6C_9 + 3C_0C_1C_3C_6C_9 + 3C_0C_1C_4C_7C_9 \\
&- 6C_1C_2^2C_8C_9 + 9C_0C_2C_3C_8C_9 + 9C_0C_1C_5C_8C_9 \\
&- 27C_0^2C_7C_8C_9
\end{aligned}
\tag{5.40}
$$

$$
\begin{aligned}
v_{73} = &\ C_1C_2C_5^2C_6 - 2C_0C_3C_5^2C_6 - C_1C_2C_3C_5C_7 + 2C_0C_3^2C_5C_7 \\
&- C_2^2C_4C_5C_7 + 4C_0C_4C_5^2C_7 + C_0C_2C_5C_6C_7 + 2C_1C_2^2C_7^2 \\
&- 3C_0C_2C_3C_7^2 - 4C_0C_1C_5C_7^2 + 4C_0^2C_7^3 + C_2^2C_5^2C_8 \\
&- 4C_0C_5^3C_8 - C_0C_3^3C_9 + 2C_2^2C_3C_4C_9 + C_1^2C_3C_5C_9 \\
&- C_1C_2C_4C_5C_9 - 6C_0C_3C_4C_5C_9 - 4C_1C_2^2C_6C_9 \\
&+ 6C_0C_2C_3C_6C_9 + 3C_0C_1C_5C_6C_9 + 3C_0C_1C_3C_7C_9 \\
&- 3C_0C_2C_4C_7C_9 - 9C_0^2C_6C_7C_9 - 4C_2^3C_8C_9 \\
&+ 18C_0C_2C_5C_8C_9 - 2C_1^3C_9^2 + 9C_0C_1C_4C_9^2 - 27C_0^2C_8C_9^2
\end{aligned}
\tag{5.41}
$$

$$
\begin{aligned}
v_{81} = &-C_1C_2C_3C_4C_6 + 2C_2^2C_4^2C_6 + C_1^2C_4C_5C_6 - 4C_0C_4^2C_5C_6 \\
&+ C_0C_1C_3C_6^2 - 2C_0^2C_6^3 - C_1C_2C_4^2C_7 + 2C_0C_3C_4^2C_7 \\
&- C_0C_1C_4C_6C_7 + 2C_1C_2C_3^2C_8 - C_0C_3^3C_8 - 3C_2^2C_3C_4C_8 \\
&- 2C_1^2C_3C_5C_8 + C_1C_2C_4C_5C_8 + 6C_0C_3C_4C_5C_8 \\
&- 4C_1C_2^2C_6C_8 + 3C_0C_2C_3C_6C_8 + 3C_0C_1C_5C_6C_8 \\
&+ 4C_1^2C_2C_7C_8 - 6C_0C_1C_3C_7C_8 - 3C_0C_2C_4C_7C_8 \\
&+ 9C_0^2C_6C_7C_8 + 4C_2^3C_8^2 - 9C_0C_2C_5C_8^2 + C_1^2C_4^2C_9 \\
&- 4C_0C_4^3C_9 - 4C_1^3C_8C_9 + 18C_0C_1C_4C_8C_9 - 27C_0^2C_8^2C_9
\end{aligned} \tag{5.42}
$$

$$
\begin{aligned}
v_{82} = &-C_1C_3^2C_4C_6 + 2C_2C_3C_4^2C_6 + C_1^2C_3C_6^2 - 2C_1C_2C_4C_6^2 \\
&+ 2C_0C_3C_4C_6^2 - 2C_0C_1C_6^3 + C_1C_3C_4^2C_7 - 2C_2C_4^3C_7 \\
&- 2C_0C_4^2C_6C_7 + C_1C_3^3C_8 - 2C_2C_3^2C_4C_8 - C_1C_3C_4C_5C_8 \\
&+ 2C_2C_4^2C_5C_8 - 3C_0C_3^2C_6C_8 - 2C_2^2C_4C_6C_8 \\
&+ 2C_1^2C_5C_6C_8 - 6C_0C_4C_5C_6C_8 + 6C_0C_2C_6^2C_8 \\
&- 4C_1^2C_3C_7C_8 + 8C_1C_2C_4C_7C_8 + 3C_0C_3C_4C_7C_8 \\
&+ 6C_0C_1C_6C_7C_8 + 3C_2^2C_3C_8^2 - 6C_1C_2C_5C_8^2 \\
&+ 9C_0C_3C_5C_8^2 - 18C_0C_2C_7C_8^2
\end{aligned} \tag{5.43}
$$

$$
\begin{aligned}
v_{83} = &-C_1C_3C_4C_5C_6 + 2C_2C_4^2C_5C_6 + C_1^2C_5C_6^2 - 2C_0C_4C_5C_6^2 \\
&- C_1C_2C_4C_6C_7 + 2C_0C_3C_4C_6C_7 - C_0C_1C_6^2C_7 + C_1C_3^2C_5C_8 \\
&- 2C_2C_3C_4C_5C_8 - 4C_1C_2C_5C_6C_8 + 3C_0C_3C_5C_6C_8 \\
&+ 2C_1C_2C_3C_7C_8 - 3C_0C_3^2C_7C_8 - C_2^2C_4C_7C_8 \\
&+ 3C_0C_2C_6C_7C_8 + 3C_2^2C_5C_8^2 + C_1C_3C_4^2C_9 - 2C_2C_4^3C_9 \\
&+ C_1^2C_4C_6C_9 - 6C_0C_4^2C_6C_9 - 6C_1^2C_3C_8C_9 \\
&+ 9C_1C_2C_4C_8C_9 + 9C_0C_3C_4C_8C_9 + 9C_0C_1C_6C_8C_9 \\
&- 27C_0C_2C_8^2C_9
\end{aligned}
\tag{5.44}
$$

$$
\begin{aligned}
v_{91} = &-C_1C_2C_3^2C_4 + C_0C_3^3C_4 + C_2^2C_3C_4^2 + C_1^2C_3C_4C_5 \\
&- 4C_0C_3C_4^2C_5 + 2C_1^2C_2C_3C_6 - 2C_0C_1C_3^2C_6 \\
&- 2C_1C_2^2C_4C_6 - 2C_1^3C_5C_6 + 8C_0C_1C_4C_5C_6 \\
&- 2C_0C_1C_2C_6^2 + 3C_0^2C_3C_6^2 - C_0C_1C_3C_4C_7 \\
&+ 2C_0C_2C_4^2C_7 + 2C_0C_1^2C_6C_7 - 6C_0^2C_4C_6C_7 \\
&+ 2C_1C_2^2C_3C_8 - 3C_0C_2C_3^2C_8 - 2C_2^3C_4C_8 - 2C_1^2C_2C_5C_8 \\
&+ 3C_0C_1C_3C_5C_8 + 6C_0C_2C_4C_5C_8 + 6C_0C_2^2C_6C_8 \\
&- 18C_0^2C_5C_6C_8 - 6C_0C_1C_2C_7C_8 + 9C_0^2C_3C_7C_8
\end{aligned}
\tag{5.45}
$$

$$\begin{aligned}
v_{92} = &-C_1C_2C_3C_4C_6 + 2C_0C_3^2C_4C_6 - C_1^2C_4C_5C_6 + 4C_0C_4^2C_5C_6 \\
&+ 2C_1^2C_2C_6^2 - 3C_0C_1C_3C_6^2 - 4C_0C_2C_4C_6^2 + 4C_0^2C_6^3 \\
&+ C_1C_2C_4^2C_7 - 2C_0C_3C_4^2C_7 + C_0C_1C_4C_6C_7 - C_0C_3^3C_8 \\
&+ C_2^2C_3C_4C_8 + 2C_1^2C_3C_5C_8 - C_1C_2C_4C_5C_8 \\
&- 6C_0C_3C_4C_5C_8 + 3C_0C_2C_3C_6C_8 - 3C_0C_1C_5C_6C_8 \\
&- 4C_1^2C_2C_7C_8 + 6C_0C_1C_3C_7C_8 + 3C_0C_2C_4C_7C_8 \\
&- 9C_0^2C_6C_7C_8 - 2C_2^3C_8^2 + 9C_0C_2C_5C_8^2 + C_1^2C_4^2C_9 \\
&- 4C_0C_4^3C_9 - 4C_1^3C_8C_9 + 18C_0C_1C_4C_8C_9 - 27C_0^2C_8^2C_9
\end{aligned} \tag{5.46}$$

$$\begin{aligned}
v_{93} = &-C_1C_2C_4C_5C_6 + 2C_0C_3C_4C_5C_6 - C_0C_1C_5C_6^2 - C_1C_2C_3C_4C_7 \\
&+ C_0C_3^2C_4C_7 + C_2^2C_4^2C_7 + 2C_1^2C_2C_6C_7 - 2C_0C_1C_3C_6C_7 \\
&- 4C_0C_2C_4C_6C_7 + 3C_0^2C_6^2C_7 + 2C_1C_2C_3C_5C_8 \\
&- 3C_0C_3^2C_5C_8 - C_2^2C_4C_5C_8 + 3C_0C_2C_5C_6C_8 \\
&- 2C_1C_2^2C_7C_8 + 3C_0C_2C_3C_7C_8 + C_1^2C_3C_4C_9 + C_1C_2C_4^2C_9 \\
&- 6C_0C_3C_4^2C_9 - 2C_1^3C_6C_9 + 9C_0C_1C_4C_6C_9 \\
&- 6C_1^2C_2C_8C_9 + 9C_0C_1C_3C_8C_9 + 9C_0C_2C_4C_8C_9 \\
&- 27C_0^2C_6C_8C_9.
\end{aligned} \tag{5.47}$$

With these definitions for the entries of V, if f factors as in (5.2), then each row of V is a multiple of (a_0, a_1, a_2):

$$\begin{aligned}
v_{11} &= a_0 d_z^2 E & v_{42} &= a_1 d_z^2 E & v_{43} &= a_2 d_z^2 E \\
v_{21} &= a_0 d_x^2 E & v_{42} &= a_1 d_x^2 E & v_{43} &= a_2 d_x^2 E \\
v_{31} &= a_0 d_y^2 E & v_{42} &= a_1 d_y^2 E & v_{43} &= a_2 d_y^2 E \\
v_{41} &= a_0 d_z d_{xz} E & v_{42} &= a_1 d_z d_{xz} E & v_{43} &= a_2 d_z d_{xz} E \\
v_{51} &= a_0 d_z d_{yz} E & v_{52} &= a_1 d_z d_{yz} E & v_{53} &= a_2 d_z d_{yz} E \\
v_{61} &= a_0 d_x d_{xz} E & v_{62} &= a_1 d_x d_{xz} E & v_{63} &= a_2 d_x d_{xz} E \qquad (5.48) \\
v_{71} &= a_0 d_x d_{xy} E & v_{72} &= a_1 d_x d_{xy} E & v_{73} &= a_2 d_x d_{xy} E \\
v_{81} &= a_0 d_y d_{yz} E & v_{82} &= a_1 d_y d_{yz} E & v_{83} &= a_2 d_y d_{yz} E \\
v_{91} &= a_0 d_y d_{xy} E & v_{92} &= a_1 d_y d_{xy} E & v_{93} &= a_2 d_y d_{xy} E.
\end{aligned}$$

(We did not define the v's this way, because if f does not factor, then the v's may not factor this way.) So if f factors, then V has the form in (5.48) which has rank 1 or 0. We record this in the following theorem.

Theorem 5.49. *If f as in* (5.1) *factors, then V as defined in* (5.29) *has rank* 1 *or* 0.

Example 5.50. Recall Example 5.22 where $f = x^2y + xy^2 + x^2z + y^2z + xz^2 + yz^2$

does not factor and M has rank 1. For this polynomial, we have

$$V = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & -4 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \\ -4 & 2 & 2 \\ 2 & -4 & 2 \\ -4 & 2 & 2. \end{bmatrix}$$

The rank of V is greater than 1. By Theorem 5.49, f does not factor. This shows that V has an advantage over M in telling when f is irreducible.

By Theorem 4.53, we have that $b_0z^2 + b_1xz + b_2yz + b_4x^2 + b_3xy + b_5y^2$ as in (5.2) is reducible if and only if E as given in (5.18) is zero. Since each entry of V in (5.48) contains E as a factor, if f factors completely, then $V = 0$. We record this in the following theorem.

Theorem 5.51. *If f as given in* (5.1) *factors completely, then V as given in* (5.29) *is the zero matrix.*

If f is reducible and $V \neq 0$, then f factors over the coefficient field. We state this in the following theorem.

Theorem 5.52. *Let f be as given in* (5.1). *If $f = (a_0z + a_1x + a_2y)(b_0z^2 + b_1xz + b_2yz + b_4x^2 + b_3xy + b_5y^2)$ and $V \neq 0$, where V is as given in* (5.29), *then $a_i, b_j \in F$ for all i, j.*

This result comes from the same proof as that of Theorem 5.23, but with M replaced by V and m_{ij} replaced by v_{ij}.

Suppose we do not know whether f is reducible. And suppose that the rank of V is 1 (where V is as defined in (5.29)). Then there is a test for the reducibility of f, and we describe it here. Since V has rank 1, V has a nonzero row that is a multiple of (a_0, a_1, a_2), and we can use this row or a constant multiple of it as a_0, a_1, and a_2 since the factorization is only unique up to a constant multiple. If $a_0 \neq 0$, then (5.8) implies that if $R_z = 0$, then f factors. To check if R_z as given in (5.9) is zero, we calculate its four coefficients K_{u^3}, K_{u^2v}, K_{uv^2}, and K_{v^3} as given in (5.5) and see if they are all zero. If these four Ks are zero, then f factors. Similarly, if a_1 or a_2 is nonzero, then we calculate R_x or R_y as given in (5.9), which in turn, leads to the calculation of their corresponding four Ks as defined in (5.5), and if they are zero, then f factors.

We give an example of this test for the reducibility of f when f has rank 1:

Example 5.53. Let $f = 2x^3 - 3x^2y + 3xy^2 - y^3 + x^2z - 6xyz + 5y^2z - xz^2 - 7yz^2 + 3z^3$. Matching coefficients with (5.1) and calculating V using a computer algebra system, we find

$$V = \begin{bmatrix} 108 & 72 & -36 \\ 192 & 128 & -64 \\ 4332 & 2888 & -1444 \\ 144 & 96 & -48 \\ -360 & -240 & 120 \\ 192 & 128 & -64 \\ -192 & -128 & 64 \\ -2280 & -1520 & 760 \\ -912 & -608 & 304. \end{bmatrix}$$

Every row in this matrix is a multiple of $(3, 2, -1)$. So V has rank 1 and we can choose $a_0 = 3$, $a_1 = 2$, and $a_2 = -1$ as a candidate for the factorization in (5.2). To check if f really have a linear factor with these coefficients, we pick a nonzero value among the three coefficients. Suppose we pick a_2. Then (5.8) tells us that if $R_y = 0$, then $a_2^3 f = (a_0 z + a_1 x + a_2 y)Q_y$ where R_y and Q_y are given in (5.9). The coefficients of R_y, as given in (5.9), are K_{w^3}, K_{vw^2}, K_{v^2w}, and K_{v^3} which are defined in (5.5). Using a computer algebra system, we find that these four Ks are zero. So $R_y = 0$ and

$$\begin{aligned} a_2^3 f &= (a_0 z + a_1 x + a_2 y)Q_y \\ &= (3z + 2x - y)(-x^2 + xy - y^2 + xz + 2yz - z^2). \end{aligned}$$

Since we have chosen $a_2 = -1$, we have

$$f = (3z + 2x - y)(x^2 - xy + y^2 - xz - 2yz + z^2).$$

If f is reducible and $V = 0$, does this imply that f is completely reducible? We give a counterexample for this.

Counterexample 5.54. Let $f = z(xy + z^2)$. We note that f is not completely reducible. Matching coefficients with (5.1) and (5.2), we find $a_0 = 1$, $a_1 = a_2 = 0$, $b_0 = b_3 = 1$, and $b_1 = b_2 = b_4 = b_5 = 0$. Plugging these into (5.18), we find $d_x = d_y = d_z = 0$. Since each entry of V in (5.48) contains either d_x, d_y, or d_z as a factor, we have $V = 0$.

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