HILBERT SPACE APPLICATIONS

IN INTEGRAL EQUATIONS

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Misha M. Avetisyan

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The thesis of Misha M. Avetisyan is approved.

Borislava Gutarts

Derek Chang

Michael Hoffman, Committee Chair

Grant Fraser, Department Chair

California State University, Los Angeles

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ABSTRACT

Hilbert Space Applications in Integral Equations

By

Misha M. Avetisyan

The purpose of this paper is to describe some applications of the theory of Hilbert spaces to integral equations. The main goal is to illustrate possible applications of techniques developed in theory and to include the standard classification of the important integral equations (Volterra, Fredholm, Integro-Differential, Singular and Abel's integral equations) and their solvability. The most available methods of the subject are abstract and most of them are based on comprehensive theories such as topological methods of functional analysis. This paper uses some recent developments in the solution of integral equations without using the extensive mathematical theory and leaves out abstract and comprehensive methods. The main goal of this paper is to provide both a systematic exposition of the basic ideas and results of Hilbert space theory and functional analysis, and an introduction to various methods of solution of integral equations. Several examples with solutions are introduced in each section.

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TABLE OF CONTENTS

CHAPTER 1

Historical Introduction

We began with a historical survey that the name integral equation [10], [13], [18] for any equation in which the unknown function $\phi(x)$ is under an integral sign was introduced by du Bois-Reymond in 1888. In 1782 Laplace used the integral transform

$$f(x) = \int_0^\infty e^{-xs}\phi(s)ds \tag{1.1}$$

to solve differential equations. In 1822 Fourier found the formulas

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin xs\, \phi(s)ds, \tag{1.2}$$

$$\phi(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin xs\, f(x)dx \tag{1.3}$$

and

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos xs\, \phi(s)ds \tag{1.4}$$

$$\phi(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos xs\, f(x)dx \tag{1.5}$$

The integral equations (1.2) and (1.4) can be solved in terms of the known function $f(x)$ by (1.3) and (1.5).

In 1826 Abel solved the integral equation, having the form

$$f(x) = \int_a^x (x-s)^{-\propto} \phi(s) ds, \tag{1.6}$$

where $f(x)$ is a continuous function satisfying $f(a) = 0$, and $0 < \propto < 1$.

An integral equation of the type

$$\phi(x) = f(x) + \lambda \int_0^x k(x-s)\phi(s) ds \tag{1.7}$$

in which the unknown function $\phi(x)$ occurs outside and within the integral and the variable x appears as upper limit of the integral, was obtained by Poisson in 1826 the theory of magnetism. He solved it by expanding $\phi(s)$ in powers of the parameter λ without proving the convergence of this series. In 1837 Liouville proved the convergence of such a series.

The determination of the function Ψ having values over certain boundary surface S and satisfying Laplace's equation $\nabla^2 \Psi = 0$ within the region enclosed by S was shown by Neumann in 1870 to be equivalent to the solution of an integral equation. This is similar to the procedure used by Poisson and Liouville, and leads to good method of successive approximations.

In 1896 Volterra gave the first general solution of the class of linear integral equations bearing his name with the variable x appearing as the upper limit of the integral.

A more general class of linear integral equations having the form

$$\phi(x) = f(x) + \int_a^b K(x,s)\phi(s) ds \tag{1.8}$$

which includes Volterra's class of integral equations as the special case given by $K(x,s) = 0$ for $s > x$, was first discussed by Fredholm in 1900. He employed a similar

approach to that introduced by Volterra in 1884. In this method the Fredholm equation (1.8) is regarded as the limiting form as $n \to \infty$ of a set of n linear algebraic equations

$$\Phi(x_r) = f(x_r) + \sum_{s=1}^{n} K(x_r, x_s)\phi(x_s)\delta_n \quad (r = 1, \dots \dots, n) \tag{1.9}$$

where $\delta_n = (b-a)/n$ and $x_r = a + r\delta_n$. The solution of these equations can be obtained and Fredholm verified by direct substitution in the integral equation (1.8) that his limiting formula for $n \to \infty$ gave the true solution.

In the remaining chapters of this paper we shall be giving a discussion of the general theory of linear integral equations as developed by Volterra, Fredholm, Hilbert, and Schmidt. For this purpose we need to introduce the concept of a Hilbert Space [10], [17], [22]. This is a good generalization of ordinary three-dimensional Euclidian space to a linear vector space of infinite dimensions which, for the subject of integral equations, is chosen to be the complete linear space composed of square integrable functions having a distance property defined in terms of an inner product. The whole theory was initiated by the work of D. Hilbert (1912) on integral equations.

To explain the meanings of these terms we consider Euclidian space first then the Hilbert space of sequences and function space.

CHAPTER 2

Hilbert Space

2.1 Euclidian Space

In three-dimensional Euclidian space each point is represented by a coordinates (x_1, x_2, x_3) forming the components of the position vector $\boldsymbol{x}$. The vector $\lambda\boldsymbol{x}$ has coordinates $(\lambda x_1, \lambda x_2, \lambda x_3)$ and the vector sum $\boldsymbol{x} + \mathbf{y}$ of two vectors $\boldsymbol{x}, \mathbf{y}$ has coordinates $(x_1 + y_1, x_2 + y_2, x_3 + y_3)$. These obey the properties of a linear vector space.

The scalar product of two vectors $\boldsymbol{x}, \mathbf{y}$ is defined as

$$(\boldsymbol{x}, \mathbf{y}) = x_1 y_1 + x_2 y_2 + x_3 y_3. \tag{2.1.1}$$

The vectors are orthogonal, at right angles, if $(\boldsymbol{x}, \mathbf{y}) = 0$.

We have

$$(\boldsymbol{x}, \boldsymbol{x}) = x_1^2 + x_2^2 + x_3^2 \geq 0,$$

where $(\boldsymbol{x}, \boldsymbol{x}) = 0$ if and only if $\boldsymbol{x}$ is the zero vector $\mathbf{0}$ with coordinates $(0,0,0)$.

The magnitude or norm $\|\boldsymbol{x}\|$ of a vector $\boldsymbol{x}$ is given by

$$\|\boldsymbol{x}\| = \sqrt{(\boldsymbol{x}, \boldsymbol{x})} = \sqrt{x_1^2 + x_2^2 + x_3^2} < \infty. \tag{2.1.2}$$

The vector is said to be normalized if $\|\boldsymbol{x}\| = 1$.

The distance between two points specified by the vectors $\boldsymbol{x}, \mathbf{y}$ is given by $\|\boldsymbol{x} - \boldsymbol{y}\|$. Since the length of a side of a triangle is less than, or equal to the sum of the lengths of the other two sides, we have the triangle inequality.

$$\|\boldsymbol{x} - \boldsymbol{y}\| \leq \|\mathbf{x}\| + \|\boldsymbol{y}\|. \tag{2.1.3}$$

Suppose that $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ are linearly independent vectors so that $\mu\,\boldsymbol{x} + \gamma\,\boldsymbol{y} + \beta\boldsymbol{z}$ is not the zero vector $\mathbf{0}$ except when $\mu = \gamma = \beta = 0$. We can use them to construct an

orthogonal and normalized, i.e. orthonormal set of three vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$. This vector is normalized, $\boldsymbol{e_1} = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$ then

$$\boldsymbol{y_1} = \boldsymbol{y} - (\boldsymbol{y}, \boldsymbol{e_1})\boldsymbol{e_1}$$

is orthogonal to $\boldsymbol{e_1}$ and $\boldsymbol{e_2} = \boldsymbol{y_1}/\|\boldsymbol{y_1}\|$ is normalized. Further

$$\boldsymbol{z_1} = \boldsymbol{z} - (\boldsymbol{z}, \boldsymbol{e_1})\boldsymbol{e_1} - (\boldsymbol{z}, \boldsymbol{e_2})\boldsymbol{e_2}$$

is orthogonal to $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$ while $\boldsymbol{e_3} = \boldsymbol{z_1}/\|\boldsymbol{z_1}\|$ is also normalized.

The three orthogonal unit vectors e_1 e_2 e_3 are said to form a basis since any vector **a** of the three-dimensional space can be expressed as the linear combination

$$\boldsymbol{a} = (\boldsymbol{a}, \boldsymbol{e_1})\boldsymbol{e_1} + (\boldsymbol{a}, \boldsymbol{e_2})\boldsymbol{e_2} + (\boldsymbol{a}, \boldsymbol{e_3})\boldsymbol{e_3}.$$

The foregoing vector algebra can be extended to an n-dimensional space whose points are specified by an ordered set of n complex numbers $(x_1, x_2, \ldots\ldots, x_n)$ denoted by the vector $\boldsymbol{x}$. The inner product of two vectors $\boldsymbol{x}, \mathbf{y}$ is now defined as

$$(\boldsymbol{x}, \mathbf{y}) = \sum_{\boldsymbol{r=1}}^{\boldsymbol{n}} \boldsymbol{x_r}, \overline{\boldsymbol{y_r}} = \overline{(\boldsymbol{y}, \mathbf{x})} \tag{2.1.4}$$

while the norm of the vector $\boldsymbol{x}$ is given by

$$\|\boldsymbol{x}\| = \sqrt{(\boldsymbol{x}, \boldsymbol{x})} = \sqrt{\sum_{r=1}^{n} |x_r|^2}. \tag{2.1.5}$$

An orthonormal set of n vectors $\boldsymbol{e_1}, \boldsymbol{e_2}, \ldots, \boldsymbol{e_n}$ form a basis of the n-dimensional space, and an arbitrary vector **a** in the space can be expressed as the linear combination

$$\boldsymbol{a} = \sum_{r=1}^{n} (\boldsymbol{a}, \boldsymbol{e_r})\boldsymbol{e_r}$$

where the $a_r = (\boldsymbol{a}, \boldsymbol{e_r})(\mathrm{r} = 1, \ldots, \mathrm{n})$ are the components of the vector **a** with respect to the basis vectors. The only vector which is orthogonal to every vector of the basis is the

zero vectors **0** and so the orthonormal set $\boldsymbol{e_1}, \boldsymbol{e_2}, \ldots, \boldsymbol{e_n}$ is said to span the n-dimensional space. If we take $\boldsymbol{e_1} = (1,0,0,\ldots,0)$, $\boldsymbol{e_2} = (0,1,0,\ldots,0), \ldots, \boldsymbol{e_n} = (0,0,0,\ldots,1)$ we see that **a** is the vector specified by $(a_1, a_2, \ldots \ldots, a_n)$, where $a_r = (\boldsymbol{a}, \boldsymbol{e_r})$**.**

2.2 Hilbert Space of Sequences

By a natural generalization of a finite dimensional space, we can consider an infinite dimensional space [10], [13] and [14] whose points are represented by vectors $\boldsymbol{x}$ having components, or coordinates, given by the infinite sequence of complex numbers $\{x_r\} = (x_1, x_2, \ldots x_r, \ldots)$ satisfying

$$\sum_{r=1}^{\infty} |x_r|^2 < \infty. \tag{2.2.1}$$

The scalar product or inner product of two vectors $\boldsymbol{x}$ and $\mathbf{y}$ given by

$$(\boldsymbol{x}, \mathbf{y}) = \sum_{\boldsymbol{r=1}}^{\infty} \boldsymbol{x_r}, \overline{\boldsymbol{y_r}} = \overline{(\boldsymbol{y}, \mathbf{x})}. \tag{2.2.2}$$

Then we have $0 \leq (\boldsymbol{x}, \mathbf{x}) < \infty$**,** where $(\boldsymbol{x}, \mathbf{x}) = 0$ if and only if $\boldsymbol{x}$ is the zero vector **0** whose components all vanish.

Also we define the norm $\|\boldsymbol{x}\|$ of a vector $\boldsymbol{x}$ by the formula

$$\|\boldsymbol{x}\| = \sqrt{(\boldsymbol{x}, \boldsymbol{x})} = \sqrt{\sum_{r=1}^{\infty} |x_r|^2} < \infty. \tag{2.2.3}$$

Thus $\|\boldsymbol{x}\| = 0$ if and only if $\boldsymbol{x} = 0$.

Further we let $\lambda\boldsymbol{x}$ be the vector with components $\{\lambda x_r\}$ so that $\|\lambda\boldsymbol{x}\| = |\lambda|\|\boldsymbol{x}\|$, and let the sum $\boldsymbol{x} + \boldsymbol{y}$ of two vectors $\boldsymbol{x}, \boldsymbol{y}$ be the vector having components $\{x_r + y_r\}$ as in the case of a finite dimensional space.

Now, by the Cauchy inequality

$$\sum_{r=1}^{\infty}|x_r||y_r| \leq \sqrt{\left(\sum_{r=1}^{\infty}|x_r|^2\right)\left(\sum_{r=1}^{\infty}|y_r|^2\right)} \tag{2.2.4}$$

and the inequality

$$\left|\sum_{r=1}^{\infty}x_r\bar{y}_r\right| \leq \sum_{r=1}^{\infty}|x_r|\,|y_r|, \tag{2.2.5}$$

we obtain Schwarz's inequality

$$|(\boldsymbol{x},\boldsymbol{y})| \leq \|\boldsymbol{x}\|\|\boldsymbol{y}\|. \tag{2.2.6}$$

This is the generalization to infinite sequences of the corresponding result in three-dimensional Euclidian space which follows from $(\boldsymbol{x},\boldsymbol{y}) = \|\boldsymbol{x}\|\|\boldsymbol{y}\|\cos\alpha$ where α is the angle between the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$.

Hence

$$\begin{aligned}\|\boldsymbol{x}+\boldsymbol{y}\|^2 &= \sum_{r=1}^{\infty}|x_r+y_r|^2\\ &= \sum_{r=1}^{\infty}|x_r|^2+\sum_{r=1}^{\infty}|y_r|^2+\sum_{r=1}^{\infty}(x_r\bar{y}_r+\bar{x}_ry_r)\\ &\leq (\|\boldsymbol{x}\|+\|\boldsymbol{y}\|)^2<\infty.\end{aligned}$$

This shows that the sum $\boldsymbol{x}+\boldsymbol{y}$ satisfies the condition

$$\sum_{r=1}^{\infty}|x_r+y_r|^2<\infty \tag{2.2.7}$$

and also yields the triangle inequality

$$\|\boldsymbol{x}+\boldsymbol{y}\| \leq \|\boldsymbol{x}\|+\|\boldsymbol{y}\| \tag{2.2.8}$$

which can be rewritten in the form (2.1.3) by reversing the sign of **y**, as

$$\|\boldsymbol{x}-\boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|.$$

The real number

$$d(\boldsymbol{x},\boldsymbol{y}) = \|\boldsymbol{x}-\boldsymbol{y}\| = \sqrt{\sum_{r=1}^{\infty} |x_r - y_r|^2} \tag{2.2.9}$$

represents the distance between two points characterized by vectors **x** and **y**. Clearly $\|\boldsymbol{x}\|$ is the distance of the point $\boldsymbol{x}$ from the origin given by the zero vector $\mathbf{0}$.

A sequence of vectors $\{\boldsymbol{x}_n\}$ converges strongly or "in norm" to a limit vector $\boldsymbol{x}$ if, given any $\varepsilon > 0$, there exists N such that for $n > N$ we have $\|\boldsymbol{x}_n - \boldsymbol{x}\| < \varepsilon$. Strong convergence is denoted by $\boldsymbol{x}_n \to \boldsymbol{x}$.

If $\boldsymbol{x}_n \to \boldsymbol{x}$ we have, using the triangle inequality,

$$\begin{aligned}\|\boldsymbol{x}_n - \boldsymbol{x}_m\| &= \|\boldsymbol{x}_n - \boldsymbol{x} + \boldsymbol{x} - \boldsymbol{x}_m\| \\ &\leq \|\boldsymbol{x}_n - \boldsymbol{x}\| + \|\boldsymbol{x}_m - \boldsymbol{x}\| < \varepsilon \quad \text{for large } n, m.\end{aligned}$$

A sequence $\{\boldsymbol{x}_n\}$ satisfying $\|\boldsymbol{x}_n - \boldsymbol{x}_m\| < \varepsilon$ for large n and m is known as a Cauchy sequence. A sequence of points $\{\boldsymbol{x}_n\}$ in a Hilbert space $\boldsymbol{H}$ is said to converge weakly to a point $\boldsymbol{x}$ in $\boldsymbol{H}$ if

$$(\boldsymbol{x}_n, \boldsymbol{y}) \to (\boldsymbol{x}, \boldsymbol{y})$$

for all $\boldsymbol{y}$ in $\boldsymbol{H}$. Here, $(\cdot,\cdot)$ is understood to be the inner product on the Hilbert space.

Weak convergence is in contrast to strong convergence or convergence in the norm, which is defined by $\|\boldsymbol{x}_n - \boldsymbol{x}\| \to 0$ where $\|\boldsymbol{x}\| = \sqrt{(\boldsymbol{x},\boldsymbol{x})}$ is the norm of $\boldsymbol{x}$. The notation of weak convergence defines a topology on $\boldsymbol{H}$ and this is called the weak topology on $\boldsymbol{H}$. In other words, the weak topology is the topology generated by the

bounded functions on $\boldsymbol{H}$. It follows from Schwarz inequality that the weak topology is weaker than the norm topology. Therefore convergence in norm implies weak convergence while the converse is not true in general. However, if for each $\boldsymbol{y}$

$$(\boldsymbol{x}_n, \boldsymbol{y}) \to (\boldsymbol{x}, \boldsymbol{y}) \text{ and } \|\boldsymbol{x}_n\| \to \boldsymbol{x}, \text{ then we have } \|\boldsymbol{x}_n - \boldsymbol{x}\| \to 0 \text{ as } n \to \infty.$$

On the level of operators, a bounded operator T is also continuous in the weak topology. If $\boldsymbol{x}_n \to \boldsymbol{x}$ weakly, then for all $\boldsymbol{y}$

$$(\boldsymbol{T}\boldsymbol{x}_n, \boldsymbol{y}) = (\boldsymbol{x}_n, \boldsymbol{T}^*\boldsymbol{y}) \to (\boldsymbol{x}, \boldsymbol{T}^*\boldsymbol{y}) = (\boldsymbol{T}\boldsymbol{x}, \boldsymbol{y}).$$

Now we consider a sequence e_n which was constructed to be orthonormal, that is,

$$(e_n, e_m) = \delta_{mn}.$$

Where δ_{mn} equals one if $m = n$ and zero otherwise. We claim that if the sequence is infinite, then it converges weakly to zero. A simple proof is as follows. For $x \in \boldsymbol{H}$, we have

$$\sum_n |(e_n, x)|^2 \leq \|x\|^2 \qquad (Bessel's\ inequality),$$

where equality holds when $\{e_n\}$ is a Hilbert space basis. Therefore

$$|(e_n, x)|^2 \to 0 \ \ i.e. (e_n, x) \to 0.$$

Uniform Bounded Theorem (UBT) for Hilbert space: If $\{y_n\}_{n=1}^{\infty}$ is a sequence of vectors in $\boldsymbol{H}$, such that the numerical sequence $t_n = (x, y_n)$ is bounded for each x in $\boldsymbol{H}$. The sequence $\{y_n\}_{n=1}^{\infty}$ is bounded in norm.

UBT for Hilbert space: If there are for each x in $\boldsymbol{H}$ constants M_x such that

$$|(x, y_n)| \leq M_x \ , \ \forall n$$

then there is a single constant B such that

$$\|y_n\| \leq B \qquad all\ n.$$

Corollary: weakly convergent sequence is bounded.

If $y_n \overset{w}{\to} y$ then $(x, y_n) \to (x, y) \quad \forall x,$ this is a number and any convergent sequence of numbers is bounded. So, the sequence $\{(x, y_n)\}$ is bounded in $\boldsymbol{F}$ for each x. By UBT the sequence $\{y_n\}$ is bounded in norm.

Now we demonstrate that every Cauchy sequence has a limit vector $\boldsymbol{x}$ in the space. Suppose that $\boldsymbol{e_1} = (1,0,0, \ldots),\ \boldsymbol{e_2} = (0,1,0, \ldots)$, ... are vectors in the infinite dimensional space. They form a basis which spans the space, and the arbitrary vector $\boldsymbol{a} = (a_1, a_2, \ldots a_r \ldots)$ can be expressed as

$$\boldsymbol{a} = \sum_{r=1}^{\infty} a_r \boldsymbol{e}_r$$

where $a_r = (\boldsymbol{a}, \boldsymbol{e_r})$ and $\boldsymbol{e}_r$ is the rth unit vector satisfying $\|\boldsymbol{e}_r\| = 1$.

Then we have, using Schwarz's inequality,

$$\begin{aligned} |(\boldsymbol{x}_n, \boldsymbol{e}_r) - (\boldsymbol{x}_m, \boldsymbol{e}_r)| &= |(\boldsymbol{x}_n - \boldsymbol{x}_m, \boldsymbol{e}_r)| \\ &\leq \|\boldsymbol{x}_n - \boldsymbol{x}_m\| < \varepsilon \end{aligned}$$

for large n and m. It follows that the sequence of numbers $(\boldsymbol{x}_n, \boldsymbol{e}_r) = {x_r}^{(n)}$ is a Cauchy sequence and approaches a limiting value $x_r (r = 1,2, \ldots)$ as $n \to \infty$. For large n, m we have

$$\|\boldsymbol{x}_n - \boldsymbol{x}_m\| = \sqrt{\sum_{r=1}^{\infty} |{x_r}^{(n)} - {x_r}^{(m)}|^2} < \varepsilon$$

and so for every k

$$\sqrt{\sum_{r=1}^{k} |{x_r}^{(n)} - {x_r}^{(m)}|^2} < \varepsilon.$$

Hence, in the limit as $m \to \infty$ we obtain

$$\sqrt{\sum_{r=1}^{k} |x_r{}^{(n)} - x_r|^2} < \varepsilon$$

and since this is true for every k we get

$$\|\boldsymbol{x}_n - \boldsymbol{x}\| = \sqrt{\sum_{r=1}^{\infty} |x_r{}^{(n)} - x_r|^2} < \varepsilon.$$

But

$$\sqrt{\sum_{r=1}^{\infty} |x_r|^2} = \|\boldsymbol{x}\| = \|(\boldsymbol{x} - \boldsymbol{x_n}) + \boldsymbol{x_n}\|$$

$$\leq \|\boldsymbol{x} - \boldsymbol{x_n}\| + \|\boldsymbol{x_n}\|$$

$$< \varepsilon + \|\boldsymbol{x_n}\|$$

and so we have

$$\sum_{r=1}^{\infty} |x_r|^2 < \infty.$$

Thus $\boldsymbol{x} = (x_1, x_2, \ldots, x_r \ldots)$ belongs to the space and $\boldsymbol{x_n} \to \boldsymbol{x}$ as $n \to \infty$. A space in which $\boldsymbol{x_n} \to \boldsymbol{x}$ when $\{\boldsymbol{x_n}\}$ is a Cauchy sequence is called complete. The space of sequences described above is an example of a Hilbert space.

2.3 Function Space

We consider the function space [10], [13] composed of all continuous complex functions $f(x)$ of a real variable x, defined in the interval $a \leq x \leq b$, which are square integrable and thus satisfy the condition

$$\int_a^b |f(x)|^2 dx < \infty. \tag{2.3.1}$$

The inner product of two functions $f(x)$ and $g(x)$ given by

$$(f,g) = \int_a^b f(x)\overline{g(x)}dx, \tag{2.3.2}$$

we define the norm of the function $f(x)$ as

$$\|f\| = \sqrt{(f,f)}. \tag{2.3.3}$$

Next we establish the important inequality named after Schwarz.

We have

$$\int_a^b \left| f(x) - \frac{(f,g)}{(g,g)} g(x) \right|^2 dx \geq 0 \tag{2.3.4}$$

so that

$$(f,f) - 2\frac{|(f,g)|^2}{(g,g)} + \frac{|(f,g)|^2}{(g,g)} \geq 0$$

i.e.

$$(f,f)(g,g) \geq |(f,g)|^2. \tag{2.3.5}$$

Hence

$$\|f\|\|g\| \geq |(f,g)| \tag{2.3.6}$$

this is Schwarz's inequality for square integrable functions.

Also

$$(\|f\| + \|g\|)^2 = \|f\|^2 + \|g\|^2 + 2\|f\|\|g\|$$

$$\geq (f,f) + (g,g) + 2|(f,g)|$$

by Schwarz's inequality. But

$$2|(f,g)| \geq (f,g) + \overline{(f,g)}$$

and so

$$(\|f\| + \|g\|)^2 \geq (f,f) + (g,g) + (f,g) + (g,f)$$

$$= (f+g, f+g).$$

Hence we have

$$\|f\| + \|g\| \geq \|f+g\| \tag{2.3.7}$$

which is the triangle inequality for functions, known as Minkowski's inequality.

Two functions $f(x)$ and $g(x)$ belonging to the function space are said to be orthogonal if

$$(f,g) = \int_a^b f(x)\overline{g(x)}\,dx = 0 \tag{2.3.1.1}$$

and the function $f(x)$ normalized if

$$\|f\| = 1. \tag{2.3.1.2}$$

We consider a set of sectionally continuous complex functions $\phi_1(x), \phi_2(x), \ldots, \phi_r(x), \ldots$ satisfying the orthonormality condition

$$(\phi_r, \phi_s) = \int_a^b \phi_r(x)\overline{\phi_s(x)}dx = \delta_{rs} \tag{2.3.1.3}$$

where δ_{rs} is the Kronecker delta symbol

$$\delta_{rs} = \begin{cases} 1 & (r = s) \\ 0 & (r \neq s). \end{cases} \tag{2.3.1.4}$$

Such a set of functions is called orthonormal.

An orthonormal system of functions [6], [13] is said to form a basis or a complete system if and only if the sole function which is orthogonal to every number $\phi_r(x)$ of the system is the null function which vanishes throughout the interval $a \leq x \leq b$ except at a

finite number of points. We note here that a complete system of functions should not be confused with a complete space.

CHAPTER 3

Abstract Hilbert Space H

3.1 Properties of Hilbert Space

We shall conclude this chapter by abstracting the common axioms that all Hilbert spaces must satisfy [6], [10], [13], [14].

A Hilbert space $\boldsymbol{H}$ is a linear vector space possessing a distance function or metric which is given by an inner product and which is complete with sequent to that metric.

A linear vector space points or vectors, forming an Abelian group and permitting multiplication by the field of complex numbers λ over $\boldsymbol{C}$.

An Abelian group has an internal law of composition satisfying the commutative law

$$f + g = g + f \tag{3.1.1}$$

and the associative law

$$f + (g + h) = (f + g) + h, \tag{3.1.2}$$

having a zero element 0 such that

$$0 + f = f + 0 = f, \tag{3.1.3}$$

and an inverse element $-f$ corresponding to each element f of the set such that

$$f + (-f) = (-f) + f = 0. \tag{3.1.4}$$

The multiplication by the field of complex numbers satisfies

$$1 \cdot f = f, \tag{3.1.5}$$

$$0 \cdot f = 0, \tag{3.1.6}$$

$$(\lambda\mu)f = \lambda(\mu f) \tag{3.1.7}$$

and satisfies the distributive law with respect to the elements f, g

$$\lambda(f + g) = \lambda f + \lambda g \tag{3.1.8}$$

and the distributive law with respect to the numbers λ, μ

$$(\lambda + \mu)f = \lambda f + \mu f. \tag{3.1.9}$$

The inner product of two elements f, g is a complex number denoted by (f, g) satisfying

$$(f, g) = \overline{(g, f)}, \tag{3.1.10}$$

$$(\lambda f, g) = \lambda(f, g), \tag{3.1.11}$$

$$(f_1 + f_2, g) = (f_1, g) + (f_2, g). \tag{3.1.12}$$

It follows at once that $(f, f) = \overline{(f, f)}$ so that (f, f) is a real number. We shall assume that

$$(f, f) \geq 0 \tag{3.1.13}$$

and further that $(f, f) = 0$ if and only if $f = 0$.

Then we have also that

$$(f, \lambda g) = \overline{(\lambda g, f)} = \bar{\lambda}\,\overline{(g, f)} = \bar{\lambda}(f, g) \tag{3.1.14}$$

$$(f, g_1 + g_2) = \overline{(g_1 + g_2, f)} = \overline{(g_1, f)} + \overline{(g_2, f)} = (f, g_1) + (f, g_2).$$

(3.1.15)

The norm of the element f is denoted as

$$\|f\| = \sqrt{(f, f)}\,. \tag{3.1.16}$$

The norm $\|f\| = 0$ if and only if $f = 0$ and $\|\lambda f\| = |\lambda|\|f\|$.

Now $(f + \lambda g, f + \lambda g) \geq 0$, and so taking $\lambda = -(f, g)/(g, g)$ we obtain Schwarz's inequality

$$\|f\|\|g\| \geq |(f, g)| \tag{3.1.17}$$

and we have the triangle inequality

$$\|f\| + \|g\| \geq \|f + g\|. \tag{3.1.18}$$

We now define a distance function $d(f, g)$ in terms of the norm according to the formula

$$d(f, g) = \|f - g\|. \tag{3.1.19}$$

This satisfies the conditions required of a distance between two points f and g, namely

(i) $d(f, g) = d(g, f)$,

(ii) $d(f, g) \geq 0$,

(iii) $d(f, g) = 0$ if and only if $f = g$,

(iv) $d(f, g) \leq d(f, h) + d(h, g)$. (3.1.20)

This last condition follows from the triangle inequality for the norm:

$$\|f - g\| = \|(f - h) + (h - g)\| \leq \|f - h\| + \|h - g\|.$$

We say that a sequence of elements $\{f_n\}$ converges strongly to a limit element f if, given any $\varepsilon > 0$, there exists a N such that for $n > N$ we have $\|f_n - f\| < \varepsilon$. Strong convergence is denoted by $f_n \to f$.

If $f_n \to f$ we have $\|f_n - f_m\| = \|(f_n - f) + (f - f_m)\|$

$$\leq \|f_n - f\| + \|f_m - f\| < \varepsilon$$

for sufficiently large n, m. A sequence $\{f_n\}$ of elements satisfying $\|f_n - f_m\| < \varepsilon$ for large n, m is known as a Cauchy sequence.

A Hilbert space is complete, that is every Cauchy sequence converges to a limit vector in the space.

The theory of Hilbert spaces initiated, by David Hilbert (1862-1943), in his 1912 work on quadratic forms in infinitely many variables which he applied to the theory of integral equations. John von Neumann first formulated an axiomatic theory of Hilbert spaces and developed the modern theory of operators on Hilbert spaces. Necessary condition for a normed space is to be an inner product space.

A complete inner product space is called a Hilbert space [6], [10], [13]. Since **C** is complete, it is a Hilbert space. $\boldsymbol{C}^N$ is a Hilbert space. l^2 is a Hilbert space.
Every inner product space is a normed space and the convergence defined by the norm.

Definition 3.1.1: (Strong Convergence). A sequence $\{x_n\}$ of vectors in an inner product space E *is* called strongly convergent to a vector $x \in E$ if

$$\|x_n - x\| \to 0 \quad \text{as} \quad n \to \infty.$$

Definition 3.1.2: (Weak Convergence). A sequence $\{x_n\}$ of vectors in an inner product space E is called weakly convergent to a vector $x \in E$ if $(x_n, y) \to (x, y)$ as $n \to \infty$, for every $y \in E$.

Theorem 3.1.3: A strong convergent sequence is weakly convergent.

Proof. Suppose the sequence $\|x_n - x\| \to 0$ as $n \to \infty$. By Schwarz's inequality

$$|(x_n - x, y)| \leq \|x_n - x\|\|y\| \to 0 \quad \text{as } n \to \infty, \text{ and}$$

$(x_n - x, y) \to 0 \quad \text{as} \quad n \to \infty,$ for every $y \in E$.

3.2 Properties of Orthonormal Systems

Using the principle of mathematical induction, we can generalize the Pythagorean formula [6] as follows:

Theorem 3.2.1: (Pythagorean Formula). If $x_1, \dots, x_n$ are orthogonal vectors in an inner product space, then

$$\left\| \sum_{k=1}^{n} x_k \right\|^2 = \sum_{k=1}^{n} \|x_k\|^2. \tag{3.2.1}$$

Proof. If $x_1 \perp x_2$, then $\|x_1 + x_2\|^2 = \|x_1\|^2 + \|x_2\|^2$. Thus the theorem is true for $n = 2$. Assume now that the (3.2.1) holds for $n - 1$, that is

$$\left\| \sum_{k=1}^{n-1} x_k \right\|^2 = \sum_{k=1}^{n-1} \|x_k\|^2 .$$

Set $x = \sum_{k=1}^{n-1} x_k$ and $y = x_n$. Clearly $x \perp y$. Thus

$$\left\| \sum_{k=1}^{n} x_k \right\|^2 = \|x + y\|^2 = \|x\|^2 + \|y\|^2 = \sum_{k=1}^{n-1} \|x_k\|^2 + \|x_n\|^2 = \sum_{k=1}^{n} \|x_k\|^2.$$

This proves the theorem.

Theorem 3.2.2: (Bessel's Equality and Inequality). Let $x_1, \dots, x_n$ be an orthonormal set of vectors in an inner product space E. Then, for every $x \in E$, we have

$$\left\| x - \sum_{k=1}^{n} (x, x_k) x_k \right\|^2 = \|x\|^2 - \sum_{k=1}^{n} |(x, x_k)|^2 \tag{3.2.2}$$

and

$$\sum_{k=1}^{n} |(x, x_k)|^2 \leq \|x\|^2. \tag{3.2.3}$$

Proof. In view of the Pythagorean formula (3.2.1), we have

$$\left\| \sum_{k=1}^{n} \alpha_k x_k \right\|^2 = \sum_{k=1}^{n} \|\alpha_k x_k\|^2 = \sum_{k=1}^{n} |\alpha_k|^2$$

for arbitrary complex numbers $\alpha_1, \dots, \alpha_n$. Hence

$$\left\| x - \sum_{k=1}^{n} \alpha_k x_k \right\|^2 = (x - \sum_{k=1}^{n} \alpha_k x_k, x - \sum_{k=1}^{n} \alpha_k x_k)$$

$$= \|x\|^2 - \left(x, \sum_{k=1}^{n} \alpha_k x_k \right) - (\sum_{k=1}^{n} \alpha_k x_k, x) + \sum_{k=1}^{n} |\alpha_k|^2 \|x_k\|^2$$

$$= \|x\|^2 - \sum_{k=1}^{n} \overline{\alpha_k}(x, x_k) - \sum_{k=1}^{n} \alpha_k \overline{(x, x_k)} + \sum_{k=1}^{n} \alpha_k \overline{\alpha_k}$$

$$= \|x\|^2 - \sum_{k=1}^{n} |(x, x_k)|^2 + \sum_{k=1}^{n} |(x, x_k) - \alpha_k|^2. \tag{3.2.4}$$

In particular, if $\alpha_k = (x, x_k)$, this results yields (3.2.2). From (3.2.2) it follows that

$$0 \leq \|x\|^2 - \sum_{k=1}^{n} |(x, x_k)|^2,$$

which gives (3.2.3). The proof is complete.

Remarks. 1. Note that expression (3.2.4) is minimized by taking $\alpha_k = (x, x_k)$. This choice of α_k minimizes $\|x - \sum_{k=1}^{n} \alpha_k x_k\|$ and thus it provides the best approximation of x by a linear combination of vectors $\{x_1, \dots, x_n\}$.

2. If $\{x_n\}$ is an infinite orthonormal sequence of vectors in an inner product space E, then from (3.2.3), by letting $n \to \infty$, we obtain

$$\sum_{k=1}^{\infty} |(x, x_k)|^2 \leq \|x\|^2. \tag{3.2.5}$$

This shows that the series $\sum_{k=1}^{\infty} |(x, x_k)|^2$ converges for every $x \in E$. In other words, the sequence $\{(x, x_n)\}$ is an element of l^2. We can say that an orthonormal sequence in E induces a mapping from E into l^2. The expansion

$$x \sim \sum_{n=1}^{\infty} (x, x_n)\, x_n \tag{3.2.6}$$

is called a generalized Fourier series of x with respect to the orthonormal sequence $\{x_n\}$. This set of coefficients gives the best approximation. In general we do not know whether the series in (3.2.6) is convergent. The next theorem shows, the completeness of E ensures the convergence.

Theorem 3.2.3: Let $\{x_n\}$ be an orthonormal sequence in a Hilbert space $\boldsymbol{H}$ and let $\{\alpha_n\}$ be a sequence of complex numbers. Then the series $\sum_{n=1}^{\infty} \alpha_n x_n$ converges if and only if $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$, and in that case

$$\left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\| = \sqrt{\sum_{n=1}^{\infty} |\alpha_n|^2}. \tag{3.2.7}$$

Proof. For every $m > k > 0$, we have

$$\left\| \sum_{n=k}^{m} \alpha_n x_n \right\| = \sqrt{\sum_{n=k}^{m} |\alpha_n|^2}, \tag{3.2.8}$$

If $\sum_{n=1}^{\infty}|\alpha_n|^2 < \infty$, then, by (3.2.8) the sequence $s_m = \sum_{n=1}^{m} \alpha_n x_n$ is a Cauchy sequence. This implies convergence of the series $\sum_{n=1}^{\infty} \alpha_n x_n$, because of the completeness of $\boldsymbol{H}$.

Conversely, if the series $\sum_{n=1}^{\infty} \alpha_n x_n$ converges, then the same formula (3.2.8) implies the convergence of $\sum_{n=1}^{\infty}|\alpha_n|^2$, because the sequence of numbers

$\sigma_m = \sum_{n=1}^{m}|\alpha_n|^2$ is a Cauchy sequence in $\boldsymbol{R}$.

To obtain (3.2.7) it is enough to take $k = 1$ and let $m \to \infty$ in (3.2.8).

Definition 3.2.1: (Complete Sequence). An orthonormal sequence $\{x_n\}$ in a Hilbert space $\boldsymbol{H}$ is said to be complete if for every $x \in \boldsymbol{H}$ we have

$$x = \sum_{n=1}^{\infty} (x, x_n) x_n. \tag{3.2.9}$$

Since the right side of (3.2.8) is a series, the equality means

$$\lim_{n\to\infty} \left\| x - \sum_{k=1}^{n} (x, x_k) x_k \right\| = 0,$$

where $\|\cdot\|$ is the norm in $\boldsymbol{H}$. For example, if $\boldsymbol{H} = \boldsymbol{L^2}([-\pi, \pi])$ and $\{f_n\}$ is an orthonormal sequence in $\boldsymbol{H}$, then by

$$f = \sum_{n=1}^{\infty} (f, f_n) f_n\,,$$

we mean

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \left| f(t) - \sum_{k=1}^{n} \alpha_k f_k(t) \right|^2 dt = 0, \qquad \alpha_k = \int_{-\pi}^{\pi} f(t)\overline{f_k(t)}dt.$$

This, in general, does not imply the pointwise convergence: $f(x) = \sum_{n=1}^{\infty}(f, f_n) f_n(x)$.

The following theorem gives important characterization of complete orthonormal sequences [6], [10], [17].

Theorem 3.2.4: An orthonormal sequence $\{x_n\}$ in a Hilbert space $\boldsymbol{H}$ is complete if and only if the condition $(x, x_n) = 0$ for all $n \in N$ implies $x = 0$.

Proof. Suppose $\{x_n\}$ is complete in $\boldsymbol{H}$. Then every $x \in \boldsymbol{H}$ has the representation

$$x = \sum_{n=1}^{\infty} (x, x_n) x_n.$$

Thus, if $(x, x_n) = 0$ for every $n \in N$, then $x = 0$.

Conversely, suppose the condition $(x, x_n) = 0$ for all n implies $x = 0$. Let x be an element of $\boldsymbol{H}$. Define

$$y = \sum_{n=1}^{\infty} (x, x_n)\, x_n.$$

The sum y exists in $\boldsymbol{H}$ by (3.2.5) and Theorem 3.2.3. Since, for every $\in N$,

$$(x - y, x_n) = (x, x_n) - \left(\sum_{k=1}^{\infty} (x, x_k) x_k, x_n \right)$$

$$= (x, x_n) - \sum_{k=1}^{\infty} (x, x_k)(x_k, x_n) = (x, x_n) - (x, x_n) = 0,$$

we have $x - y = 0$, and hence

$$x = \sum_{n=1}^{\infty} (x, x_n)\, x_n.$$

CHAPTER 4

Linear Operators in Hilbert Spaces

In an integral equation the unknown function occurs under an integral sign and thus, if the functions involved belong to a Hilbert space, it is clear that we have to deal with integral operators acting on a Hilbert space of functions.

Linear operators (or transformations) [6], [10], [17] on a normed vector space are widely used to represent physical quantities. The most important operators include differential, integral, and matrix operators. The major objective of this chapter is to introduce linear operators in Hilbert spaces with some attention to different kinds of operators and their basic properties. Although linear mappings may be from a vector space E_1 into a vector space E_2, we are interested mostly in the case when $E_1 = E_2 = E$, where E is a Hilbert space or a subspace of a Hilbert space. However, in order to avoid unduly difficult concepts, we shall suppose that our functions and kernels are square integrable without usually specifying the sense in which the integrals are to be performed.

It is useful to place linear integral operators in a more general context, so this chapter will conclude with an introduction to the theory of linear operators in an abstract Hilbert space.

4.1 Linear Integral Operators

We consider the linear integral operator $\boldsymbol{K}$ given by

$$\boldsymbol{K} = \int_a^b K(x,s)ds \tag{4.1.1}$$

where $K(x,s)$ is a square integrable kernel. This is abbreviation for

$$(\boldsymbol{K}\phi)(x) = \int_a^b K(x,s)\phi(s)ds, \tag{4.1.2}$$

where $\phi(s)$ is a square integrable function, in the symbolic form

$$\Psi = \boldsymbol{K}\phi. \tag{4.1.3}$$

The operator $\boldsymbol{K}$ is linear since

$$\boldsymbol{K}(\lambda_1\phi_1 + \lambda_2\phi_2) = \lambda_1\boldsymbol{K}\phi_1 + \lambda_2\boldsymbol{K}\phi_2 \tag{4.1.4}$$

where λ_1, λ_2 are constants and ϕ_1,ϕ_2 are square integrable functions.

If

$$\boldsymbol{L} = \int_a^b L(x,s)ds \tag{4.1.5}$$

is a second integral operator we have

$$\chi = \boldsymbol{L}\Psi = \boldsymbol{L}(\boldsymbol{K}\phi) \tag{4.1.6}$$

where

$$\chi(x) = \int_a^b L(x,t)dt \int_a^b K(t,s)\ \phi(s)ds$$

$$= \int_a^b P(x,s)\phi(s)ds \tag{4.1.7}$$

and

$$P(x,s) = \int_a^b L(x,t)K(t,s)dt, \tag{4.1.8}$$

that is

$$\chi = \boldsymbol{P}\phi \tag{4.1.9}$$

where $\boldsymbol{P} = \boldsymbol{LK}$ is the integral operator with kernel $P(x,s)$.

Integral operators satisfy the associative law

$$\boldsymbol{M}(\boldsymbol{LK}) = (\boldsymbol{ML})\boldsymbol{K} \tag{4.1.10}$$

and the distributive laws

$$\boldsymbol{M}(\boldsymbol{L} + \boldsymbol{K}) = \boldsymbol{ML} + \boldsymbol{MK} \tag{4.1.11}$$

$$(\boldsymbol{L}+\boldsymbol{K})\boldsymbol{M}=\boldsymbol{L}\boldsymbol{M}+\boldsymbol{K}\boldsymbol{M} \tag{4.1.12}$$

but, in general, linear operators and integral operators in particular, do not satisfy the commutative law.

Using the associative law we see that

$$\boldsymbol{K}^m\boldsymbol{K}^n=\boldsymbol{K}^{m+n},\ (\boldsymbol{K}^m)^n=\boldsymbol{K}^{mn} \qquad (m,n\geq 1) \tag{4.1.13}$$

where

$$\boldsymbol{K}^n=\int_a^b K_n(x,s)ds \tag{4.1.14}$$

and $K_n(x,s)$ is the iterated kernel.

4.1.1 Norm of an Integral Operator

If $K(x,s)$ is a square integrable kernel its norm is defined by

$$\|\boldsymbol{K}\|_2=[\int_a^b\int_a^b|K(x,s)|^2dxds]^{1/2}. \tag{4.1.1.1}$$

Then if $\phi(s)$ is a square integrable function and $\Psi(x)$ is given by (4.1.2), using Schwarz's inequality, we have

$$|\Psi(x)|^2\leq\int_a^b|K(x,s)|^2ds\int_a^b|\phi(s)|^2\,ds$$

which yields

$$\int_a^b|\Psi(x)|^2dx\leq\int_a^b\int_a^b|K(x,s)|^2\,dxds\int_a^b|\phi(s)|^2\,ds$$

so that

$$\|(K\phi)\|=\ \|\Psi\|\leq\|\boldsymbol{K}\|_2\|\phi\|<\infty, \tag{4.1.1.2}$$

and thus $\Psi(x)$ is square integrable. So that $\|\boldsymbol{K}\|\leq\|\boldsymbol{K}\|_2$ where $\|\boldsymbol{K}\|$ is operator norm and $\|\boldsymbol{K}\|_2$ is Hilbert-Schmidt norm in $\boldsymbol{L}^2$of the operator $\boldsymbol{K}$.

When $\|\boldsymbol{K}\|_2 = 0$ then $\|\Psi\| = 0$ so $\|(K\phi)\| = 0$ for all φ so that $\boldsymbol{K}\phi$ vanishes 'almost everywhere', for all square integrable functions ϕ, and $\boldsymbol{K}$ is called a null operator.

Also, if $L(x,t)$ and $K(t,s)$ are square integrable kernels, then $P(x,s)$ given by (4.1.8) is square integrable since, by Schwarz's inequality,

$$|P(x,s)|^2 \leq \int_a^b |L(x,t)|^2 dt \int_a^b |K(t,s)|^2\, dt$$

so that

$$\int_a^b \int_a^b |P(x,s)|^2\, dxds \leq \int_a^b \int_a^b |L(x,t)|^2 dxdt \int_a^b \int_a^b |K(t,s)|^2\, dtds.$$

For left hand side we have

$$\int_{s=a}^b \int_{x=a}^b |P(x,s)|^2\, dxds = \int_a^b (\int_a^b |P(x,s)|^2\, dx) ds$$

$$\leq \int_{s=a}^b (\int_{x=a}^b (\int_{t=a}^b |L(x,t)|^2 dt)(\int_{t=a}^b |K(t,x)|^2\, dt) dx) ds$$

$$= \int_{s=a}^b \left((\int_{t=a}^b |K(t,s)|^2 dt) \int_{x=a}^b (\int_{t=a}^b |L(x,t)|^2 dt) dx \right) ds$$

$$= \left(\int_{x=a}^b \int_{t=a}^b |L(x,t)|^2\, dtdx \right) \left(\int_{s=a}^b \int_{t=a}^b |K(t,s)|^2\, dtds \right)$$

$$= \left(\int_a^b \int_a^b |L(x,t)|^2 dtdx \right) \left(\int_a^b \int_a^b |K(t,s)|^2\, dtds \right),$$

giving

$$\|\boldsymbol{LK}\|_2 \leq \|\boldsymbol{L}\|_2 \|\boldsymbol{K}\|_2 < \infty. \qquad (4.1.1.3)$$

Putting $\boldsymbol{L} = \boldsymbol{K}$ we see that $\|\boldsymbol{K}^2\|_2 \leq \|\boldsymbol{K}\|_2^2$ and in general

$$\|\boldsymbol{K}^n\|_2 \leq \|\boldsymbol{K}\|_2^n \quad (n = 1,2,3, \dots) \qquad (4.1.1.4)$$

4.1.2 Hermitian Adjoint

The Hermitian adjoint of a kernel $K(x,s)$ which is defined to be the kernel

$$K^*(x,s) = \overline{K(s,x)}. \tag{4.1.2.1}$$

We see that

$$(\lambda \boldsymbol{K})^* = \bar{\lambda}\boldsymbol{K}^*, \qquad \boldsymbol{K}^{**} = \boldsymbol{K}. \tag{4.1.2.2}$$

Also

$$\begin{aligned}(LK)^*(x,s) &= \overline{LK(s,x)} \\ &= \int_a^b \overline{L(s,t)}\,\overline{K(t,x)}dt \\ &= \int_a^b K^*(x,t)L^*(t,s)dt \\ &= K^*L^*(x,s)\end{aligned}$$

and so

$$(\boldsymbol{LK})^* = \boldsymbol{K}^*\boldsymbol{L}^*. \tag{4.1.2.3}$$

Further, using the definition of the inner product, we have

$$\begin{aligned}(\boldsymbol{K}\phi, \Psi) &= \int_a^b (\boldsymbol{K}\phi)(x)\,\overline{\Psi(x)}dx \\ &= \int_a^b \left(\int_a^b K(x,t)\,\phi(\mathrm{t})\mathrm{dt}\right)\overline{\Psi(x)}dx \\ &= \int_{x=a}^b \int_{t=a}^b K(x,t)\,\phi(\mathrm{t})\overline{\Psi(x)}dtdx \\ &= \int_{t=a}^b \int_{x=a}^b K(x,t)\phi(\mathrm{t})\overline{\Psi(x)}dxdt \\ &= \int_{t=a}^b \overline{\int_a^b \overline{K(x,t)}\,\overline{\phi(\mathrm{t})}\Psi(x)dx}\;\;dt\end{aligned}$$

$$= \int_{t=a}^{b} \overline{\overline{\phi(t)}} \overline{\left(\int_a^b \overline{K(x,t)}\, \Psi(x) dx \right)} dt$$

$$= \int_{t=a}^{b} \phi(t) \overline{\left(\int_a^b K^*(t,x) \Psi(x) dx \right)} dt$$

$$= (\phi, \boldsymbol{K}^* \Psi), \qquad (4.1.2.4)$$

where

$$(K^*\Psi)(t) = \int_a^b k^*(t,x)\Psi(x)dx, \qquad k^*(t,x) = \overline{k(x,t)},$$

$$(K^*\Psi)(x) = \int_a^b k^*(x,t)\Psi(t)dt, \qquad k^*(x,t) = \overline{k(t,x)}.$$

If $\boldsymbol{K}^* = \boldsymbol{K}$, then the kernel is called Hermitian or self adjoint.

4.2 Bounded Linear Operators

So far we have been concerned with linear integral operators acting on a space of square integrable functions which, if chosen to be all the functions which are square integrable in the Lebesgue sense, would form a Hilbert space of function $\boldsymbol{L_2}$.

We shall now turn our attention to the case of an abstract Hilbert space, acted on by bounded linear operators.

Consider a non-empty subset D of an abstract Hilbert space $\boldsymbol{H}$ such that if $f, g \in D$ then $\lambda f + \mu g \in D$ where λ, μ are arbitrary complex numbers, then D is called a linear manifold. We note that a linear manifold must contain the zero elements since $f + (-1)f = 0$.

Suppose that corresponding to any element $f \in D$ we assign an element $\boldsymbol{K}f \in \boldsymbol{H}$, then $\boldsymbol{K}$ maps D onto Δ and is called a linear operator with domain D and let Δ=range(K) $\{Kf : f \in D\}$.

If

$$\boldsymbol{K}(\lambda f + \mu g) = \lambda \boldsymbol{K} f + \mu \boldsymbol{K} g \tag{4.2.1}$$

for $f, g \in D$ and numbers λ and μ, then Δ is also a linear manifold and $\boldsymbol{K}$ is a linear operator for domain D and range Δ.

A linear operator $\boldsymbol{K}$ having the Hilbert space $\boldsymbol{H}$ as domain is bounded if there exists a constant $C \geq 0$ such that $\|\boldsymbol{K} f\| \leq C\|f\|$ for all $f \in H$.

The norm $\|\boldsymbol{K}\|$ of the bounded linear operator $\boldsymbol{K}$ is defined as the smallest possible value of C. Thus $\|\boldsymbol{K}\|$ is the least upper bound or supremum of $\|\boldsymbol{K} f\|/\|f\|$, that is

$$\|\boldsymbol{K}\| = \sup_{f \in H} \frac{\|\boldsymbol{K} f\|}{\|f\|} \tag{4.2.2}$$

and so

$$\|\boldsymbol{K} f\| \leq \|\boldsymbol{K}\|\|f\|. \tag{4.2.3}$$

This is a generalization of the inequality (4.1.1.2) for the linear integral operator (4.1.1) with square integrable kernel. Clearly (4.1.1) is a bounded linear operator and that $\|K\| \leq \|K\|_2$.

The linear operator $\boldsymbol{K}$ has an adjoint operator $\boldsymbol{K}^*$ defined so that

$$(\boldsymbol{K} f, g) = (f, \boldsymbol{K}^* g) \tag{4.2.4}$$

for all $f, g \in \boldsymbol{H}$. If $\boldsymbol{K}^* = \boldsymbol{K}$ the operator is self adjoint or Hermitian.

Now by Schwarz's inequality and (4.2.3), we have

$$|(\boldsymbol{K} f, g)| \leq \|\boldsymbol{K} f\|\|g\| \leq \|\boldsymbol{K}\|\|f\|\|g\|. \tag{4.2.5}$$

And also we have

$$|(f, \boldsymbol{K}^* g)| \leq \|\boldsymbol{K}^*\|\|f\|\|g\|. \tag{4.2.6}$$

We can write

$$\|\boldsymbol{K}^* f\|^2 = (\boldsymbol{K}^* f, \boldsymbol{K}^* f)$$

$$= (\boldsymbol{K}\boldsymbol{K}^* f, f)$$

$$\leq \|\boldsymbol{K}\boldsymbol{K}^* f\| \|f\|$$

$$\leq \|\boldsymbol{K}\| \|\boldsymbol{K}^* f\| \|f\|$$

Dividing on $\|\boldsymbol{K}^* f\|$ both sides of this inequality we get

$$\|\boldsymbol{K}^* f\| \leq \|\boldsymbol{K}\| \|f\|.$$

This shows that

$$\|\boldsymbol{K}^*\| \leq \|\boldsymbol{K}\|.$$

Now we use $\boldsymbol{K}^{**} = \boldsymbol{K}$ to get

$$\|\boldsymbol{K}\| = \|(\boldsymbol{K}^*)^*\| \leq \|\boldsymbol{K}^*\|.$$

We have inequality in both directions, so

$$\|\boldsymbol{K}^*\| = \|\boldsymbol{K}\| \tag{4.2.7}$$

and $\boldsymbol{K}^*$ is also a bounded linear operator.

Any bounded operator $\boldsymbol{K}$ is continuous linear operator which transforms a strongly convergent sequence $\{f_n\}$ into a strongly convergent sequence $\{\boldsymbol{K}f_n\}$. We have

$$\|\boldsymbol{K}f_n - \boldsymbol{K}f\| = \|\boldsymbol{K}(f_n - f)\| \leq \|\boldsymbol{K}\| \|f_n - f\|$$

so that if $f_n \to f$ as $n \to \infty$ then

$$\|\boldsymbol{K}f_n - \boldsymbol{K}f\| \to 0,$$

that is $\boldsymbol{K}f_n$ is strongly convergent to $\boldsymbol{K}f$.

Actually every continuous linear operator $\boldsymbol{K}$ with domain $\boldsymbol{H}$ is bounded. For otherwise there would exist a sequence $\{f_n\}$ such that $\|\boldsymbol{K}f_n\| \geq n\|f_n\|$ so $\|\boldsymbol{K}g_n\| \geq 1$, but $g_n = f_n/n\|f_n\| \to 0$ as $n \to \infty$, which is contrary to the hypothesis that $\boldsymbol{K}$ is continuous.

We see that the linear integral operator (4.1.1), with a square integrable kernel, is continuous.

If $\boldsymbol{K}$ and $\boldsymbol{L}$ are two bounded linear operators which map $\boldsymbol{H}$ onto itself, then the product operator $\boldsymbol{LK}$ corresponds to

$$(\boldsymbol{LK})f = \boldsymbol{L}(\boldsymbol{K}f).$$

Then

$$\|\boldsymbol{LK}f\| \leq \|\boldsymbol{L}\|\|\boldsymbol{K}f\| \leq \|\boldsymbol{L}\|\|\boldsymbol{K}\|\|f\|$$

so that

$$\|\boldsymbol{LK}\| \leq \|\boldsymbol{L}\|\|\boldsymbol{K}\| \tag{4.2.8}$$

and therefore $\boldsymbol{LK}$ is a bounded operator.

Also

$$\begin{aligned}(\boldsymbol{LK}f, g) &= (\boldsymbol{K}f, \boldsymbol{L}^*g)\\ &= (f, \boldsymbol{K}^*\boldsymbol{L}^*g)\end{aligned}$$

and hence

$$(\boldsymbol{LK})^* = \boldsymbol{K}^*\boldsymbol{L}^*. \tag{4.2.9}$$

Lastly it is interesting to observe that the norms of bounded linear operators $\boldsymbol{K}$ and $\boldsymbol{L}$ satisfy the triangle inequality. For, using the triangle inequality for the elements of Hilbert space, we have

$$\begin{aligned}\|\boldsymbol{K}f + \boldsymbol{L}f\| &\leq \|\boldsymbol{K}f\| + \|\boldsymbol{L}f\|\\ &\leq (\|\boldsymbol{K}\| + \|\boldsymbol{L}\|)\|f\|\end{aligned}$$

so that

$$\|\boldsymbol{K} + \boldsymbol{L}\| \leq \|\boldsymbol{K}\| + \|\boldsymbol{L}\|. \tag{4.2.10}$$

Suppose that $\phi_1, \phi_2, \ldots, \phi_r, \ldots$ is a complete orthonormal system or basis in $\boldsymbol{H}$. Then the matrix with elements

$$k_{rs} = (\boldsymbol{K}\phi_r, \phi_s) \tag{4.2.1.1}$$

is called the kernel matrix of the bounded operator $\boldsymbol{K}$.

Introducing the generalized Fourier coefficients

$$x_r = (f, \phi_r), \qquad y_s = (g, \phi_s) \tag{4.2.1.2}$$

where $f, g \in \boldsymbol{H}$, we have

$$f = \sum_{r=1}^{\infty} x_r \phi_r, \qquad g = \sum_{s=1}^{\infty} y_s \phi_s \tag{4.2.1.3}$$

and

$$\boldsymbol{K}f = \sum_{r=1}^{\infty} x_r \boldsymbol{K}\phi_r = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} x_r k_{rs} \phi_s$$

So that $(\boldsymbol{K}f, g)$ has the bilinear form

$$(\boldsymbol{K}f, g) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} x_r k_{rs} \overline{y_s}. \tag{4.2.1.4}$$

Since $\boldsymbol{K}$ is bounded it follows that

$$|(\boldsymbol{K}f, g)| \leq \|\boldsymbol{K}\| \|f\| \|g\| \tag{4.2.1.5}$$

$$= \|\boldsymbol{K}\| \sqrt{\left(\sum_{r=1}^{\infty} |x_r|^2\right)\left(\sum_{s=1}^{\infty} |y_s|^2\right)}.$$

Hence

$$\left|\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} x_r k_{rs}\overline{y_s}\right| \leq \|\boldsymbol{K}\| \sqrt{\left(\sum_{r=1}^{\infty}|x_r|^2\right)\left(\sum_{s=1}^{\infty}|y_s|^2\right)} \qquad (4.2.1.6)$$

and thus the bilinear form (4.2.1.4) is also bounded.

4.3 Completely Continuous Operators

If a sequence is strongly convergent it is also weakly convergent. For, by Schwarz's inequality, we have

$$|(f_n - f, g)| \leq \|f_n - f\|\|g\|$$

and thus if $\|f_n - f\| \to 0$ as $n \to \infty$ then $(f_n, g) \to (f, g)$ as $n \to \infty$. However , the converse need not be true. Thus we may have $f_n \xrightarrow{w} f$ but $f_n \nrightarrow f$ as $n \to \infty$.

Let us suppose that $\boldsymbol{K}$ is a bounded linear operator in $\boldsymbol{H}$. Then

$$|(\boldsymbol{K}f_n - \boldsymbol{K}f, g)| \leq \|\boldsymbol{K}(f_n - f)\|\|g\|$$
$$\leq \|\boldsymbol{K}\|\|f_n - f\|\|g\|.$$

But

$$\|\boldsymbol{K}f_n - \boldsymbol{K}f\| = \|\boldsymbol{K}(f_n - f)\|$$
$$\leq \|\boldsymbol{K}\|\|f_n - f\|.$$

So, $f_n \xrightarrow{\|\cdot\|} f \Rightarrow \boldsymbol{K}f_n \xrightarrow{\|\cdot\|} \boldsymbol{K}f$ (strongly)

and so if $f_n \to f$ as $n \to \infty$ then $\boldsymbol{K}f_n \xrightarrow{\|\cdot\|} \boldsymbol{K}f$, that is $\boldsymbol{K}f_n$ is strongly convergent to $\boldsymbol{K}f$. So **K** is bounded operator then $\|f_n - f\| \to 0 \Rightarrow \|\boldsymbol{K}f_n - \boldsymbol{K}f\| \to 0$, i.e. operator **K** is continuous. Then K completely continuous in $f_n \xrightarrow{w} f \Rightarrow \|\boldsymbol{K}f_n - \boldsymbol{K}f\| \to 0$ as $n \to \infty$. Then $\boldsymbol{K}$ is called a completely continuous (or compact) linear operator in $\boldsymbol{H}$.

Every completely continuous operator is bounded. A completely continuous operator is a continuous operator and this means that it is bounded.

If the kernel matrix (k_{rs}) of a bounded linear operator $\boldsymbol{K}$ satisfies

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}|k_{rs}|^2 < \infty \tag{4.3.1}$$

then $\boldsymbol{K}$ is completely continuous. For we have, using (4.2.1.4), that

$$(\boldsymbol{K}(f_n - f), \phi_s) = \sum_{r=1}^{\infty}\left(x_r^{(n)} - x_r\right)k_{rs} \tag{4.3.2}$$

Where $x_r^{(n)}$ and x_r are the Fourier coefficients of f_n and f respectively. This gives

$$\|\boldsymbol{K}(f_n - f)\|^2 \leq \sum_{s=1}^{m}|(\boldsymbol{K}(f_n - f), \phi_s)|^2 + \sum_{s=m+1}^{\infty}\sum_{r=1}^{\infty}|k_{rs}|^2\|f_n - f\|^2 \tag{4.3.3}$$

using Cauchy's inequality.

As indicated above that weakly convergent sequences are bounded. This we can proof in following way.

Since $\{\phi_s\}$ is a complete orthonormal system, gives

$$\|\boldsymbol{K}(f_n - f)\|^2 = \sum_{s=1}^{\infty}|(\boldsymbol{K}(f_n - f), \phi_s)|^2 = \sum_{s=1}^{\infty}(\left|\sum_{r=1}^{\infty}\left(x_r^{(n)} - x_r\right)k_{rs}\right|^2)$$

$$\leq \sum_{s=1}^{\infty}\left(\left(\sum_{r=1}^{\infty}\left|x_r^{(n)} - x_r\right|^2\right)\left(\sum_{r=1}^{\infty}|k_{rs}|^2\right)\right) \qquad by\ Cauchy - Schwarz$$

$$= \sum_{s=1}^{\infty}(\|f_n - f\|^2\sum_{r=1}^{\infty}|k_{rs}|^2) = \|f_n - f\|^2(\sum_{s=1}^{\infty}\sum_{r=1}^{\infty}|k_{rs}|^2)$$

Taking square roots, gives

$$\|\boldsymbol{K}(f_n - f)\| \leq \left(\sum_{s}\sum_{r}|k_{rs}|^2\right)^{1/2}\|f_n - f\|.$$

This shows that $\boldsymbol{K}$ is bounded and hence continuous and that the operator norm satisfies

$$\|\boldsymbol{K}\|_{op} \leq \left(\sum_{s}\sum_{r}|k_{rs}|^2\right)^{\frac{1}{2}}.$$

Now we also know that K^* exists and is continuous. So here we have split sum of unspecified m.

Now suppose $f_n \overset{w}{\to} f$ and let $\varepsilon > 0$ as in (4.3.3) split the sum into two parts

$$\|\boldsymbol{K}f_n - \boldsymbol{K}f\|^2 = \sum_{s=1}^{m}|(\boldsymbol{K}(f_n - f), \phi_s)|^2 + \sum_{s=m+1}^{\infty}|(\boldsymbol{K}(f_n - f), \phi_s)|^2$$

$$\leq \sum_{s=1}^{m}|(\boldsymbol{K}(f_n - f), \phi_s)|^2 + \|f_n - f\|^2 \sum_{s=m+1}^{\infty}(\sum_{r=1}^{\infty}|k_{rs}|^2)$$

Now we will see how to select the m.

Recall my observation earlier that " weakly convergent sequences are bounded" with that there is a finite B with $\|f_n - f\|^2 < B$ for all n. Since $\sum_{s=1}^{\infty}(\sum_{r=1}^{\infty}|k_{rs}|^2)$ converges to a finite sum, we can select m large enough so that

$$\sum_{s=m+1}^{\infty}(\sum_{r=1}^{\infty}|k_{rs}|^2) < \frac{\varepsilon}{2B}$$

(the "tail" of a convergent infinite series is small).

Now we have

$$\|\boldsymbol{K}f_n - \boldsymbol{K}f\|^2 \leq \sum_{s=1}^{m}|(\boldsymbol{K}(f_n - f), \phi_s)|^2 + \frac{\varepsilon}{2B}B$$

$$= \sum_{s=1}^{m}\left|\left((f_n - f), \boldsymbol{K}^*\phi_s\right)\right|^2 + \frac{\varepsilon}{2}.$$

Now we use the weak convergence m times.

For each $s = 1,2,3,$ ... there is a N_s such that

$$n \geq N_s \Rightarrow |(f_n - f, \boldsymbol{K}^* \phi_s)| < \sqrt{\frac{\varepsilon}{2m}}.$$

If $n \geq \max\left(N_{s_1}, N_{s_2}, \ldots N_{s_m}\right)$, then

$$\|\boldsymbol{K}f_n - \boldsymbol{K}f\|^2 \leq \sum_{s=1}^{m} \frac{\varepsilon}{2m} + \frac{\varepsilon}{2} = \frac{\varepsilon}{2m} m + \frac{\varepsilon}{2} = \varepsilon.$$

This can be done for each $\varepsilon > 0$, so $\|\boldsymbol{K}f_n - \boldsymbol{K}f\| \to 0$ as required.

This show that $f_n \overset{w}{\to} f \Rightarrow \boldsymbol{K}f_n \overset{\|\cdot\|}{\to} \boldsymbol{K}f$. That is, $\boldsymbol{K}$ is completely continuous as claimed.

Any finite dimensional linear operator is completely continuous since for this case the double sum on the left-hand side of (4.3.1) contains a finite number of terms only.

As an example of a completely continuous operator we consider the integral operator $\boldsymbol{K}$ over the space of square integrable functions [6], [10], [17] given by

$$g(x) = \int_a^b \boldsymbol{K}(x,s) f(s) ds, \qquad (a \leq x \leq b), \tag{4.3.1.1}$$

where the kernel is square integrable and satisfies

$$\int_a^b |\boldsymbol{K}(x,s)|^2 \, ds < \infty \qquad (a \leq x \leq b),$$

$$\int_a^b |\boldsymbol{K}(x,s)|^2 \, dx < \infty \qquad (a \leq s \leq b), \tag{4.3.1.2}$$

$$\int_a^b \int_a^b |\boldsymbol{K}(x,s)|^2 dx \, ds < \infty.$$

By Schwarz's inequality we have

$$|g(x)|^2 \leq \int_a^b |\boldsymbol{K}(x,s)|^2 ds \int_a^b |f(s)|^2 ds$$

and so

$$\int_a^b |g(x)|^2\,dx \le \int_a^b \int_a^b |\boldsymbol{K}(x,s)|^2\,dsdx \int_a^b |f(s)|^2\,ds,$$

that is

$$\|g\|^2 \le \int_a^b \int_a^b |\boldsymbol{K}(x,s)|^2 dsdx \|f\|^2. \tag{4.3.1.3}$$

Hence

$$\|\boldsymbol{K}\| \le \sqrt{\int_a^b \int_a^b |\boldsymbol{K}(x,s)|^2 dsdx} = \|\boldsymbol{K}\|_2 \tag{4.3.1.4}$$

since the norm of $\boldsymbol{K}$ is the least upper bound of $\|g\|/\|f\|$.

Now consider any complete orthonormal system $\phi_1(x), \phi_2(x), \ldots, \phi_r(x), \ldots$ in the Hilbert space of L^2 functions defined over $a \le x \le b$. We let

$$k_{rs} = \int_a^b \int_a^b \overline{\phi_s(x)}\, \boldsymbol{K}(x,t)\phi_r(t)dxdt, \tag{4.3.1.5}$$

$$x_r = \int_a^b f(t)\overline{\phi_r(t)}dt \tag{4.3.1.6}$$

and

$$y_s = \int_a^b g(x)\, \overline{\phi_s(x)}dx. \tag{4.3.1.7}$$

Then

$$y_s = \int_a^b \int_a^b \boldsymbol{K}(x,t) f(t) \overline{\phi_s(x)}\, dxdt. \tag{4.3.1.8}$$

But

$$\int_a^b \boldsymbol{K}(x,t)\overline{\phi_s(x)}dx = \sum_{r=1}^{\infty} k_{rs}\, \overline{\phi_r(t)} \tag{4.3.1.9}$$

and

$$f(t) = \sum_{r=1}^{\infty} x_r \phi_r(t) \tag{4.3.1.10}$$

for almost all values of t, that is except for a set of values of t of Lebesgue measure zero, and hence

$$y_s = \sum_{r=1}^{\infty} x_r k_{rs}. \tag{4.3.1.11}$$

Also

$$\int_a^b \left| \int_a^b \boldsymbol{K}(x,t)\overline{\phi_s(x)}dx \right|^2 dt = \sum_{r=1}^{\infty} |k_{rs}|^2 \tag{4.3.1.12}$$

and, using Parseval's formula,

$$\int_a^b |\boldsymbol{K}(x,t)|^2 \, dx = \sum_{s=1}^{\infty} \left| \int_a^b \boldsymbol{K}(x,t)\overline{\phi_s(x)}dx \right|^2. \tag{4.3.1.13}$$

Consequently

$$\int_a^b \int_a^b |\boldsymbol{K}(x,t)|^2 \, dxdt = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} |k_{rs}|^2 \tag{4.3.1.14}$$

and so the matrix representation of the integral operator $\boldsymbol{K}$ with square integrable kernel satisfies the condition (4.3.1) which ensures that $\boldsymbol{K}$ is completely continuous.

CHAPTER 5

Linear Integral Equations

5.1 Classification of Linear Integral Equations

Integral equations arise naturally in physics, chemistry, biology and engineering applications modeled by initial value problems for a finite interval [a, b]. More details about the sources and origins of integral equations can be found in [8] and [11].

The most frequently used linear integral equations fall under two main classes namely Fredholm and Volterra integral equations and two related types of integral equations. In particular, the four types are given by:

1. Fredholm integral equations
2. Volterra integral equations
3. Integro-Differential equations
4. Singular integral equations.

5.1.1 Fredholm Linear Integral Equations

The standard form of Fredholm linear integral equations are given by the form

$$\phi(x)u(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)dt, \quad (a \le x, t \le b), \qquad (5.1.1.1)$$

where the kernel of the integral equation $K(x,t)$ and the function $f(x)$ are given in advance, and λ is a parameter. The equation (5.1.1.1) is called linear because the unknown function $u(x)$ under the integral sign occurs linearly, i.e. the power of $u(x)$ is one.

The value of $\phi(x)$ will give the following kinds of Fredholm linear integral equations:

1. When $\phi(x) = 0$, equation (5.1.1.1) becomes

$$f(x) + \lambda \int_a^b K(x,t)\, u(t)dt = 0, \tag{5.1.1.2}$$

and is called Fredholm integral equation of the first kind.

2. When $\phi(x)$ = 1, equation (5.1.1.1) becomes

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)dt, \tag{5.1.1.3}$$

and is called Fredholm integral equation of the second kind.

5.1.2 Volterra Linear Integral Equations

The standard form of Volterra linear integral equations, where the limits of integration are functions of x rather than constants are of the form

$$\phi(x)u(x) = f(x) + \lambda \int_a^x K(x,t)\, u(t)dt. \tag{5.1.2.1}$$

As in Fredholm equations, Volterra integral equations fall under two kinds

1. When $\phi(x)$ = 0, equation (5.1.2.1) becomes

$$f(x) + \lambda \int_a^x K(x,t)\, u(t)dt = 0, \tag{5.1.2.2}$$

and is called Volterra integral equation of the first kind.

2. When $\phi(x)$ = 1, equation (5.1.2.1) becomes

$$u(x) = f(x) + \lambda \int_a^x K(x,t)\, u(t)dt, \tag{5.1.2.3}$$

and is called Volterra integral equation of the second kind.

For both cases in 5.1.1 and 5.1.2 (# 2) the formulas are ok if $\phi(x) \neq 0$ since we can just modify the kernel and the integral equation with modified kernel given by

$$u(x) = f(x) + \lambda \int_a^b \frac{k(x,t)}{\phi(x)} u(t)dt.$$

Two other types of integral equation, related to the two main classes Fredholm and Volterra integral equations arise in many science and engineering applications.

5.1.3 Integro-Differential Equations

Several phenomena in physics and biology [11] and [19] give rise to this type of integro-differential equations. In integro-differential equations, the unknown function u(x) occurs in one side as an ordinary derivative, and appears on the other side under the integral sign. The following are examples of integro-differential equations:

$$u''(x) = -x + \int_0^x (x-t)u(t)dt\,, \quad u(0) = 0, \quad u'(0) = 1, \tag{5.1.3.1}$$

$$u'(x) = 2 - \frac{1}{4}x + \int_0^1 xtu(t)dt, u(0) = 1. \tag{5.1.3.2}$$

Equation (5.1.3.1) is called Volterra integro-differential equation related to Volterra integral equations, or Volterra integro-differential equations. Equation (5.1.3.2) is called integro-differential equation related to Fredholm integral equations, or simply Fredholm integro-differential equation.

5.1.4 Singular Integral Equations

The integral equation of the first kind or the integral equation second kind is called singular if the lower limit, the upper limit or both limits of integration are infinite. Examples are:

$$u(x) = 3x + 7\int_0^\infty \sin(x-t)\,u(t)dt, \tag{5.1.4.1}$$

$$u(x) = x - \frac{1}{4}\int_{-\infty}^0 \cos(x+t)\,u(t)dt, \tag{5.1.4.2}$$

$$u(x) = 5 + 2x^2 + \frac{1}{4}\int_{-\infty}^{\infty}(x+t)u(t)dt. \qquad (5.1.4.3)$$

Examples of second kind singular integral equations are given by

$$x^2 = \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt, \qquad (5.1.4.4)$$

$$x = \int_0^x \frac{1}{(x-t)^{\alpha}}u(t)dt, \qquad 0 < \alpha < 1, \qquad (5.1.4.5)$$

where the singular behavior has resulted from the kernel $K(x,t)$ becoming infinite as $t \to x$.

5.2 Solution of an Integral Equation

A solution of an integral equation or an integro-differential equation on the interval of integration is a function $u(x)$ which satisfies the given equation. In other words, if the given solution is substituted in the right-hand side of the equation, the output of this direct substitution must yield the left-hand side, i.e. we should verify that the given function satisfies the integral equation under discussion. Consider the following example.

Example 1: Show that $u(x) = e^x$ is a solution of the Volterra integral equation

$$u(x) = 1 + \int_0^x u(t)dt.$$

Substituting $u(x) = e^x$in the right hand side yields

$$RHS = 1 + \int_0^x e^t dt,$$

$$= e^x$$

$$= u(x),$$

= LHS.

5.3 Converting Volterra Equation to Ordinary Differential Equation

Now we will present the technique that converts Volterra integral equations of the second kind to equivalent differential equations. This can be achieved by applying the useful Leibnitz Rule for differentiating an integral. We have

$$\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} G(x,t)dt = G\big(x,\beta(x)\big)\frac{d\beta}{dx} - G\big(x,\alpha(x)\big)\frac{d\alpha}{dx} + \int_{\alpha(x)}^{\beta(x)} \frac{\partial G}{\partial x}\,dt, \qquad (5.3.1)$$

which works provided $G(x,t)$ and $\frac{\partial G}{\partial x}$ are continuous functions in the domain D in the xt – plane that contains the rectangular region R, $a \le x \le b,\ t_0 \le t \le t_1$, and $\alpha(x)$ and $\beta(x)$ are functions having continuous derivatives for $a < x < b$. We note that Leibnitz rule is usually presented in most calculus books, and our goal here will be on using the rule rather than its theoretical proof.

Here is an example.

Example 1. Find the initial value problem equivalent to the integral equation

$$u(x) = x^4 + x^2 + 2\int_0^x (x-t)^2 u(t)dt.$$

Differentiating both sides and using Leibnitz rule yield

$$\begin{cases} u'(x) = 4x^3 + 2x + 4\int_0^x (x-t)u(t)dt, \\ u''(x) = 12x^2 + 2 + 4\int_0^x u(t)dt, \\ u'''(x) = 24x + 4u(x). \end{cases}$$

$$u'''(x) - 4u(x) = 24x$$

The initial condition we get by substituting $x = 0$ in both sides of the integral equations, we find $u(0) = u'(0) = 0$, and $u''(0) = 2$, so the initial value problem of third order is

$$u'''(x) - 4u(x) = 24x, \qquad u(0) = u'(0) = 0, \qquad u''(0) = 2.$$

5.4 Converting Initial Value Problem to Volterra Equation.

In this section, we will show the method that converts an initial value problem to an equivalent Volterra integral equation. Before outlining the method needed, we wish to recall the transformation formula

$$\int_0^x \int_0^{x_1} \int_0^{x_2} \dots \int_0^{x_{n-1}} f(x_n) dx_n \dots dx_1 = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t) dt, \tag{5.4.1}$$

that converts any multiple integral to a single integral. The formulas

$$\int_0^x \int_0^x f(t) dt dt = \int_0^x (x-t) f(t) dt, \tag{5.4.2}$$

$$\int_0^x \int_0^x \int_0^x f(t) dt dt dt = \frac{1}{2!} \int_0^x (x-t)^2 f(t) dt \tag{5.4.3}$$

are two special cases of the formula given above, and used formulas that will transform double and triple integrals respectively to a single integral for each. Here we prove the formula (5.4.2) that converts double integral to a single integral.

We take right-hand side of (5.4.2) as

$$I(x) = \int_0^x (x-t) f(t) dt \tag{5.4.4}$$

$$= x \int_0^x f(t) dt - \int_0^x t f(t) dt$$

Differentiating both sides of (5.4.4) and using Leibnitz rule we obtain

$$I'(x) = \frac{dI}{dx} = \int_0^x f(t) dt + x f(x) - x f(x)$$

$$I'(x) = \int_0^x f(t) dt, \tag{5.4.5}$$

assuming f is continuous. So integrate both sides from 0 to x, we get

$$I(x) - I(0) = \int_0^x I'(s)ds = \int_0^x \left(\int_0^s f(t)dt \right) ds.$$

The two components of $I(x)$ are equal and we get

$$\int_0^x \left(\int_0^s f(t)dt \right) ds = \int_0^x (x-t)f(t)dt.$$

This is the first version of (5.4.2).

The second version of (5.4.2) we get by integrating both sides of (5.4.5) from 0 to x, considering that $I(0) = 0$ and from (5.4.4), we find

$$I(x) = \int_0^x \int_0^x f(t)dtdt \tag{5.4.6}$$

Equating the right-hand sides of (5.4.4) and (5.4.6) completes the proof for this special case of (5.4.2).

Now we use the technique to convert an initial value problem to an equivalent Volterra integral equation. Here we have a third order initial value problem given by

$$y'''(x) + p(x)y''(x) + q(x)y'(x) + r(x)y(x) = g(x) \tag{5.4.7}$$

subject to the initial conditions

$$y(0) = \alpha, \quad y'(0) = \beta, \quad y''(0) = \gamma \quad \text{and} \quad \alpha, \beta, \gamma \quad \text{are constants.} \tag{5.4.8}$$

The functions $p(x), q(x)$ and $r(x)$ must have enough continuous derivatives to make my computations work, especially any integration by parts, and have Taylor expansions about the origin. Assume that $g(x)$ is continuous through the interval of discussion. To transform (5.4.7) into an equivalent Volterra integral equation we set

$$y''' = u(x). \tag{5.4.9}$$

By integrating both sides of (5.4.9) from 0 to x we found

$$y''(x) - y''(0) = \int_0^x u(t)dt, \tag{5.4.10}$$

or equivalently

$$y''(x) = \gamma + \int_0^x u(t)dt. \tag{5.4.11}$$

To obtain $y'(x)$ we integrate both sides of (5.4.11) from 0 to x to find that

$$y'(x) = \beta + \gamma x + \int_0^x \int_0^x u(t)dtdt. \tag{5.4.12}$$

We integrate both sides of (5.4.12) from 0 to x to obtain

$$y(x) = \alpha + \beta x + \frac{1}{2}\gamma x^2 + \int_0^x \int_0^x \int_0^x u(t)dtdtdt. \tag{5.4.13}$$

Using the conversion formulas (5.4.2) and (5.4.3), to reduce the double and triple integrals in (5.4.12) and (5.4.13) respectively to single integrals yields

$$y'(x) = \beta + \gamma x + \int_0^x (x-t)u(t)dt, \tag{5.4.14}$$

$$y(x) = \alpha + \beta x + \frac{1}{2}\gamma x^2 + \frac{1}{2}\int_0^x (x-t)^2\, u(t)dt. \tag{5.4.15}$$

Substituting (5.4.9), (5.4.11), (5.4.14) and (5.4.15) into (5.4.7) leads to the following Volterra integral equation of the second kind

$$u(x) = f(x) + \int_0^x K(x,t)u(t)dt, \tag{5.4.16}$$

where

$$K(x,t) = p(x) + q(x)(x-t) + \tfrac{1}{2!}r(x)(x-t)^2, \tag{5.4.17}$$

and

$$f(x) = g(x) - \left\{\gamma p(x) + \beta q(x) + \alpha r(x) + \gamma x q(x) + r(x)\left(\beta x + \frac{1}{2}\gamma x^2\right)\right\}. \tag{5.4.18}$$

The following example will be used to illustrate this technique.

Example 1. Find the equivalent Volterra integral equation to the following initial value problem

$$y''' - y'' - y' + y = 0, \qquad y(0) = 2, \qquad y'(0) = 0, \qquad y''(0) = 2.$$

First we set $y''' = u(x)$. Integrate both sides this equation from 0 to x. We get

$$y''(x) - y''(0) = \int_0^x u(t)dt \quad \rightarrow \quad y''(x) = 2 + \int_0^x u(t)dt.$$

Integrate both sides from 0 to x. We get

$$y'(x) - y'(0) = 2x + \int_0^x \int_0^x u(t)dtdt \rightarrow y'(x) = 2x + \int_0^x \int_0^x u(t)dtdt.$$

Integrate both sides from 0 to x. We get

$$y(x) - 2 = x^2 + \int_0^x \int_0^x \int_0^x u(t)dtdtdt.$$

We use

$$y'(x) = 2x + \int_0^x (x - t)u(t)dt$$

to integrate both sides from 0 to x. We get

$$y(x) = 2 + x^2 + \frac{1}{2}\int_0^x (x - t)^2 u(t)dt.$$

Substitute in $y''' - y'' - y' + y = 0$, we get

$$u(x) = 2x - x^2 + \int_0^x \left[1 + (x - t) - \frac{1}{2}(x - t)^2\right] u(t)dt.$$

This is the equivalent Volterra integral equation. Solving it gives us $u = y'''$. Then integrate three times to get y (for most of alternative approach to get a Volterra equation).

5.5 Converting Boundary Value Problem to Fredholm Equation

The procedure of reducing a boundary value problem to a Fredholm integral equation is complicated and rarely used. The method is similar to the technique that reduces initial value problem to Volterra integral equation, with the exception that we are given boundary conditions. It seems practical to illustrate this method by applying it to an example rather than proving it.

Example 1. Find a Fredholm integral equation equivalent to the following boundary value problem

$$y''(x) + y(x) = x, \quad 0 < x < \pi,$$

subject to the boundary conditions, $y(0) = 1, \quad y(\pi) = \pi - 1.$

First, we set

$$y''(x) = u(x).$$

Integrating both sides of from 0 to x gives

$$\int_0^x y''(t)dt = \int_0^x u(t)dt,$$

or equivalently

$$y'(x) = y'(0) + \int_0^x u(t)dt.$$

Integrating both sides from 0 to x and using boundary condition $y(0) = 1$ we found

$$y(x) = 1 + xy'(0) + \int_0^x (x - t)u(t)dt,$$

here we converted the double integral to a single integral as discussed before. By substituting $x = \pi$ in both sides, we found

$$y(\pi) = 1 + \pi y'(0) + \int_0^\pi (\pi - t)u(t)dt.$$

Solving for $y'(0)$ we get

$$y'(0) = \frac{1}{\pi}((\pi - 2) - \int_0^{\pi} (\pi - t)u(t)dt).$$

After substitution , yields

$$y(x) = 1 + \frac{x}{\pi}((\pi - 2) - \int_0^{\pi} (\pi - t)u(t)dt) + \int_0^{x} (x - t)u(t)dt.$$

The final substitution gives

$$u(x) = x - 1 - \frac{x}{\pi}((\pi - 2) - \int_0^{\pi} (\pi - t)u(t)dt) - \int_0^{x} (x - t)u(t)dt.$$

Using identity

$$\int_0^{\pi} (\cdot) = \int_0^{x} (\cdot) + \int_x^{\pi} (\cdot),$$

We will get the equation

$$u(x) = x - 1 - \frac{x}{\pi}(\pi - 2)$$
$$+ \frac{x}{\pi}\int_0^{x} (\pi - t)u(t)dt + \frac{x}{\pi}\int_x^{\pi} (\pi - t)u(t)dt - \int_0^{x} (x - t)u(t)dt,$$

after simple calculations and adding integrals with similar limits we get

$$u(x) = \frac{2x - \pi}{\pi} - \int_0^{x} \frac{t(x - \pi)}{\pi} u(t)dt - \int_x^{\pi} \frac{x(t - \pi)}{\pi} u(t)dt.$$

Finally, the desired Fredholm integral equation of the second kind is given by

$$u(x) = \frac{2x - \pi}{\pi} - \int_0^{\pi} K(x, t)u(t)dt,$$

where the kernel $K(x, t)$ is defined by

$$K(x, t) = \begin{cases} \frac{t(x - \pi)}{\pi} & 0 \le t \le x \\ \frac{x(t - \pi)}{\pi} & x \le t \le \pi. \end{cases}$$

As before, the solution to the integral equation given by $u = y''$ we need integrate twice to get y. Computing first the value of $y'(0)$ needed to do this from the partial result above.

CHAPTER 6

Fredholm Integral Equations

6.1 Introduction

This chapter studies the nonhomogeneous Fredholm integral equations of the second kind of the form

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt, \qquad a \le x \le b, \tag{6.1.1}$$

where $K(x,t)$ is the kernel and λ is a parameter. Here we focus our attention on degenerate or separable kernels. The standard form of the degenerate or separable kernel is given by

$$K(x,t) = \sum_{k=1}^{n} g_k(x)h_k(t). \tag{6.1.2}$$

Examples of separable kernels are: $x - t, x + t, xt, x^2 - 3xt + t^2$, etc. For non-separable kernels, we can approximate by expanding these kernels using Taylor's expansion. The partial sums of the Taylor's series are separable kernels. The kernel is said to be square integrable in both x, t in the square $a \le x \le b$,

$a \le t \le b$ if the following regularity condition

$$\int_a^b \int_a^b |K(x,t)|^2 dxdt < \infty \tag{6.1.3}$$

is satisfied. This condition gives rise to the development of the solution of the Fredholm integral equation (6.1.1). Here we need to state, without proof, the so called Fredholm Alternative Theorem that relates the solutions of Fredholm integral equations. For more details about the regularity condition and the Fredholm Alternative Theorem the reader is

referred to [5], [8], [9] and [11]. If the kernel $K(x,t)$ is real, continuous and bounded in the square $a \le x \le b$ and $a \le t \le b$, i.e. if

$$|K(x,t)| \le M, \ a \le x \le b \ \text{ and } \ a \le t \le \ b, \tag{6.1.4}$$

and if $f(x) \neq 0$, and continuous in $a \le x \le b$, then the necessary condition that will guarantee that (6.1.1) has only a unique solution is given by

$$|\lambda| M(b-a) < 1. \tag{6.1.5}$$

This comes from the Banach fixed point theorem. The transform

$$(Tu)(x) = f(x) + \ \lambda \int_a^b K(x,t)u(t)dt$$

a contractive with respect to the uniform norm on $C([a,b])$ and so guarantees a unique fixed point.

The continuous solution to Fredholm integral equation may exist [15], even though the condition (6.1.5) is not satisfied.

6.2 The Decomposition Method

Adomian [2] recently developed the so-called Adomian decomposition method or simply the decomposition method. The method was introduced by Adomian in his resent books [2] and [3] and several related papers [1] and [4] for example. This method provides the solution in a series form. The decomposition method can be applied for linear and nonlinear integral equations. In the decomposition method we express the solution $u(x)$ of the integral equation (6.1.1) in a series form defined by

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \tag{6.2.1}$$

Substituting the decomposition (6.2.1) into both sides of (6.1.1) yields

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_a^b K(x,t)\left(\sum_{n=0}^{\infty} u_n(t)\right)dt, \tag{6.2.2}$$

or equivalently

$$\begin{aligned} u_0(x) + u_1(x) + u_2(x) + \cdots = f(x) &+ \lambda \int_a^b K(x,t)u_0(t)dt \\ &+\lambda \int_a^b K(x,t)u_1(t)dt \\ &+\lambda \int_a^b K(x,t)u_2(t)dt \\ &+ \cdots \end{aligned} \tag{6.2.3}$$

The components u_0, u_1, u_2, ... of the unknown function $u(x)$ are determined if we set

$$u_0(x) = f(x), \tag{6.2.4}$$

$$u_1(x) = \lambda \int_a^b K(x,t)u_0(t)dt, \tag{6.2.5}$$

$$u_2(x) = \lambda \int_a^b K(x,t)u_1(t)dt, \tag{6.2.6}$$

and so on. Solution of (6.1.1) can be written in recursive manner by

$$u_0(x) = f(x), \tag{6.2.7}$$

$$u_{n+1}(x) = \lambda \int_a^b K(x,t)u_n(t)dt\,, \quad n \geq 0. \tag{6.2.8}$$

It is important to note that the series obtained from $u(x)$ frequently gives the exact solution.

In the following example for solution we use the decomposition method.

Example 1. We consider here the Fredholm integral equation

$$u(x) = e^x - 1 + \int_0^1 tu(t)dt.$$

Applying the decomposition technique as discussed before we find

$$u_0(x) = e^x - 1,$$

$$u_1(x) = \int_0^1 t u_0(t)dt = \int_0^1 t(e^t - 1)dt = \frac{1}{2},$$

$$u_2(x) = \int_0^1 t u_1(t)\,dt = \int_0^1 \frac{1}{2} t dt = \frac{1}{4}.$$

We get solution in a series form given by

$$u(x) = e^x - 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ldots$$

$$= e^x - 1 + \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right),$$

where $\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right)$ is infinite geometric series, $s = \frac{a_1}{1-r} = \frac{1}{1-1/2} = 2$, so for solution we get

$$u(x) = e^x.$$

For easier calculations there is a modified decomposition method for cases where $f(x)$ consists of polynomial, or a combination of polynomial and other trigonometric or transcendental functions. In this modified method, we simply split the given function $f(x)$ into two parts defined by

$$f(x) = f_1(x) + f_2(x), \qquad (6.2.1.1)$$

where $f_1(x)$ consists of one term of $f(x)$ in many problems or two terms for other cases, and $f_2(x)$ includes the remaining terms of $f(x)$.

The integral equation (6.1.1) becomes

$$u(x) = f_1(x) + f_2(x) + \lambda \int_a^b K(x,t)u(t)dt, \qquad a \le x \le b. \qquad (6.2.1.2)$$

Substituting the decomposition given by (6.2.1) into both sides of (6.2.1.2) and using few terms of the expansion we obtain

$$u_0(x) + u_1(x) + u_2(x) + \cdots = f_1(x) + f_2(x) + \lambda \int_a^b K(x,t)u_0(t)dt$$
$$+\lambda \int_a^b K(x,t)u_1(t)dt$$
$$+\lambda \int_a^b K(x,t)u_2(t)dt$$
$$+ \cdots \qquad (6.2.1.3)$$

The modified decomposition method works if we set

$$u_0(x) = f_1(x),$$

$$u_1(x) = f_2(x) + \lambda \int_a^b K(x,t)u_0(t)dt,$$

$$u_2(x) = \lambda \int_a^b K(x,t)u_1(t)dt,$$

$$u_3(x) = \lambda \int_a^b K(x,t)u_2(t)dt,$$

and so on. This scheme for the determination of components $u_0(x), u_1(x), u_2(x), \ldots$ of the solution $u(x)$ of (6.1.1) can be written in a recursive manner by

$$u_0(x) = f_1(x), \qquad (6.2.1.4)$$

$$u_1(x) = f_2(x) + \lambda \int_a^b K(x,t)u_0(t)dt, \qquad (6.2.1.5)$$

$$u_{n+1}(x) = \lambda \int_a^b K(x,t)u_n(t)dt, \quad n \geq 1. \qquad (6.2.1.6)$$

In many problems we need to use $u_0(x)$ and $u_1(x)$ only.

The following example illustrates the modified decomposition scheme.

Example 2. We consider here the Fredholm integral equation

$$u(x) = tan^{-1}x + \frac{1}{2}\left(ln2 - \frac{\pi}{2}\right)x + \int_0^1 xu(t)dt.$$

Applying the modified decomposition method, we first split the function $f(x)$ into

$$f_1(x) = tan^{-1}x, \quad \text{and} \quad f_2(x) = \frac{1}{2}\left(ln2 - \frac{\pi}{2}\right)x.$$

Therefore, we set

$$u_0(x) = tan^{-1}x,$$

and we get

$$u_1(x) = \frac{1}{2}\left(ln2 - \frac{\pi}{2}\right)x + \int_0^1 xu_0(t)dt,$$

$$= \frac{1}{2}\left(ln2 - \frac{\pi}{2}\right)x + x\int_0^1 tan^{-1}tdt = 0.$$

The components $u_n(x) = 0, \quad n \geq 1$ and for exact solution we get

$$u(x) = tan^{-1}x.$$

6.3 The Direct Computation Method

We next introduce an efficient method for solving Fredholm integral equations of the second kind

$$u(x) = f(x) + \lambda\int_a^b K(x,t)u(t)dt.$$

Our attention will be focused on separable or degenerate kernel $K(x,t)$, expressed in the form defined by (6.1.2). For simplicity we assume a single term kernel be expressed as

$$K(x,t) = g(x)h(t). \tag{6.3.1}$$

The equation (6.1.1) becomes

$$u(x) = f(x) + \lambda g(x)\int_a^b h(t)u(t)dt. \tag{6.3.2}$$

We substitute by α the integral on right hand side by

$$\alpha = \int_a^b h(t)u(t)dt. \tag{6.3.3}$$

It follows that equation (6.3.2) becomes

$$u(x) = f(x) + \lambda\alpha g(x). \tag{6.3.4}$$

The solution $u(x)$ determined upon evaluating the constant α.This can be done by substituting (6.3.4) into (6.3.3). This approach [21] is different than other techniques. The direct computation method determines the exact solution in a closed form. This method gives rise to a system of algebraic equations depending on the structure of the kernel, where we need to evaluate more than one constant.

This technique is illustrated in the next example.

Example 1. We will use the direct computation method to solve the Fredholm integral equation

$$u(x) = xsinx - x + \int_0^{\pi/2} xu(t)dt.$$

We set

$$\alpha = \int_0^{\pi/2} u(t)dt.$$

If u is a solution, then

$$u(x) = xsinx - x + \alpha x.$$

$$\alpha = \int_0^{\pi/2} (tsint - t + \alpha t)dt = 1.$$

Substituting $\alpha = 1$ in $u(x) = xsinx - x + \alpha x$, we get the exact solution

$$u(x) = xsinx.$$

If there are two terms in the separable equation then technique become as two linear equations for two unknown coefficients and so for n we have n terms.

6.4 The Successive Approximations Method

In this method we replace the unknown function under the integral sign of the Fredholm integral equation of the second kind

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt, \qquad a \le x \le b, \tag{6.4.1}$$

by any selective real valued function $u_0(x), \quad a \le x \le b$. The first approximation $u_1(x)$ of $u(x)$ and the second approximation $u_2(x)$ of $u(x)$ are defined by

$$u_1(x) = f(x) + \lambda \int_a^b K(x,t)u_0(t)dt, \tag{6.4.2}$$

$$u_2(x) = f(x) + \lambda \int_a^b K(x,t)u_1(t)dt. \tag{6.4.3}$$

This process can be continued in the same manner to obtain the n^{th} approximation given by

$$\begin{cases} u_0(x) = \text{any selective real valued function} \\ u_n(x) = f(x) + \lambda \int_a^b K(x,t)u_{n-1}(t)dt, \quad n \ge 1. \end{cases} \tag{6.4.4}$$

The most commonly selected function for $u_0(x)$ are $0,1$ or x. At the limit, the solution $u(x)$ is obtained by

$$u(x) = \lim_{n\to\infty} u_n(x), \tag{6.4.5}$$

so that the solution $u(x)$ is independent of the choice of $u_0(x)$.

The successive approximations method will be illustrated by the following example.

Example 1. Consider the Fredholm integral equation

$$u(x) = e^x + e^{-1}\int_0^1 u(t)dt.$$

As indicated above we can select any real valued function for the zeroth approximation, we set

$$u_0(x) = 0.$$

After substitution for $u_1(x), u_2(x)$ and $u_3(x)$ we get

$$u_1(x) = e^x + e^{-1}\int_0^1 u_0(t)dt = e^x,$$

$$u_2(x) = e^x + e^{-1}\int_0^1 e^t dt = e^x + 1 - e^{-1},$$

$$u_3(x) = e^x + 1 - e^{-2}.$$

We obtain the n^{th} component

$$u_n(x) = e^x + 1 - e^{-(n-1)}, \qquad n \geq 1.$$

The solution $u(x)$ is given by

$$u(x) = \lim_{n\to\infty} u_n(x)$$

$$= \lim_{n\to\infty}(e^x + 1 - e^{-(n-1)}) = e^x + 1.$$

6.5 The Method of Successive Substitutions

This method introduces the solution of integral equations in a series form through evaluating single integral and multiple integrals. In this method, we set $x = t$ and $t = t_1$ in the Fredholm integral equation

$$u(x) = f(x) + \lambda\int_a^b K(x,t)u(t)dt, \qquad a \leq x \leq b, \tag{6.5.1}$$

to obtain

$$u(t) = f(t) + \lambda \int_a^b K(t, t_1) u(t_1) dt_1. \tag{6.5.2}$$

Replacing $u(t)$ in the right-hand side of (6.5.1) by its value given by (6.5.2) yields

$$u(x) = f(x) + \lambda \int_a^b K(x, t) f(t) dt + \lambda^2 \int_a^b K(x, t) \int_a^b K(t, t_1) u(t_1)\, dt_1 dt. \tag{6.5.3}$$

Substituting $x = t_1$ and $t = t_2$ in (6.5.1) we obtain

$$u(t_1) = f(t_1) + \lambda \int_a^b K(t_1, t_2) u(t_2) dt_2. \tag{6.5.4}$$

Substituting the value of $u(t_1)$ obtained in (6.5.4) into the right-hand side of (6.5.3) leads to

$$u(x) = f(x) + \lambda \int_a^b K(x, t) f(t) dt + \lambda^2 \int_a^b \int_a^b K(x, t) K(t, t_1) f(t_1) dt_1 dt + \lambda^3 \int_a^b \int_a^b \int_a^b K(x, t) K(t, t_1) K(t_1, t_2) u(t_2) dt_2 dt_1\, dt. \tag{6.5.5}$$

The general series form for $u(x)$ can be written as

$$u(x) = f(x) + \lambda \int_a^b K(x, t) f(t) dt + \lambda^2 \int_a^b \int_a^b K(x, t) K(t, t_1) f(t_1) dt_1 dt + \lambda^3 \int_a^b \int_a^b \int_a^b K(x, t) K(t, t_1) K(t_1, t_2) f(t_2) dt_2 dt_1\, dt + \cdots \tag{6.5.6}$$

We note that the series solution converges uniformly in the interval $[a, b]$ if

$\lambda M(b-a) \leq 1$ where $|K(x,t)| \leq M$. This comes from the contractive mapping principle or Banach fixed point theorem the proof of which appears in the texts [15], [17], [19] and others. The substitution of $u(x)$ occurs several times through the integrals. This is why it is called the method of successive substitutions. The technique is illustrated by the following example.

Example 1. We solve the following Fredholm integral equation

$$u(x) = \frac{11}{6}x + \frac{1}{4}\int_0^1 xtu(t)dt,$$

by using the method of successive substitutions.

Substituting $\lambda = \frac{1}{4}$, $f(x) = \frac{11}{6}x$, and $K(x,t) = xt$ into (6.5.6) yields

$$u(x) = \frac{11}{6}x + \frac{1}{4}\int_0^1 \frac{11}{6}xt^2\,dt + \frac{1}{16}\int_0^1\int_0^1 \frac{11}{6}xt_1{}^2t^2dt_1dt + \cdots,$$

$$u(x) = \frac{11}{6}x\left[1 + \frac{1}{12} + \frac{1}{144} + \cdots\right],$$

where $\left[1 + \frac{1}{12} + \frac{1}{144} + \cdots\right]$ is infinite geometric series and its sum $s = \frac{1}{1-1/12} = \frac{12}{11}$, this gives the exact solution by

$$u(x) = 2x.$$

6.6 Homogeneous Fredholm Equations

In this section we study the homogeneous Fredholm equation with separable kernel given by

$$u(x) = \lambda\int_a^b K(x,t)u(t)dt. \qquad (6.6.1)$$

The trivial solution $u(x) = 0$ is a solution of the homogeneous Fredholm equation. Our goal will be focused on finding nontrivial solutions if they exist. To get these solutions

we use the direct computational method that was used for nonhomogeneous Fredholm integral equations. Consider a one term kernel given by

$$K(x,t) = g(x)h(t), \tag{6.6.2}$$

so that (6.6.1) becomes

$$u(x) = \lambda g(x) \int_a^b h(t)u(t)dt. \tag{6.6.3}$$

Using the direct computation method we set

$$\alpha = \int_a^b h(t)u(t)dt, \tag{6.6.4}$$

so that (6.6.3) becomes

$$u(x) = \lambda \alpha g(x). \tag{6.6.5}$$

Substitute (6.6.5) into (6.6.4) we obtain

$$\alpha = \lambda \alpha \int_a^b h(t)g(t)dt, \tag{6.6.6}$$

or equivalently

$$1 = \lambda \int_a^b h(t)g(t)dt, \tag{6.6.7}$$

which gives a numerical value for $\lambda \neq 0$. Non-zero values of λ that result from solving the algebraic system of equations are called the eigenvalues of the kernel. Substituting these values of λ in (6.6.5) gives the eigenfunctions of the equation which are the nontrivial solutions of (6.6.1).

The following example will be used to explain the technique introduced above and the concept of eigenvalues and eigenfunctions.

Example 1. Solve the homogenous Fredholm integral equation with a two term kernel

$$u(x) = \lambda \int_0^1 (6x - 2t)u(t)dt.$$

This equation can be rewritten as

$$u(x) = 6\lambda\alpha x - \beta\lambda,$$

where α and β defined by

$$\alpha = \int_0^1 u(t)dt, \qquad \beta = \int_0^1 2tu(t)dt.$$

Substituting $u(x)$ in the formula α and β, we get

$$\alpha = \int_0^1 (6\lambda\alpha t - \beta\lambda)\, dt, \qquad \beta = \int_0^1 2t(6\lambda\alpha t - \beta\lambda)dt.$$

Thus

$$\alpha = 3\lambda\alpha - \beta\lambda, \qquad \beta = 4\lambda\alpha - \beta\lambda,$$

$$(1 - 3\lambda)\alpha + \lambda\beta = 0$$

$$-4\lambda\alpha + (1 + \lambda)\beta = 0.$$

$$\begin{vmatrix} (1 - 3\lambda) & \lambda \\ -4\lambda & (1 + \lambda) \end{vmatrix} = 0,$$

$$(1 - 3\lambda)(1 + \lambda) + 4\lambda^2 = 0.$$

Solving the quadratic equation for λ we get $\lambda_1 = \lambda_2 = 1$. Then substituting in algebraic equations, we get $\beta = 2\alpha$. The eigenfunctions corresponding to $\lambda_1 = \lambda_2 = 1$ are given by $u_1(x) = u_2(x) = 6\alpha x - 2\alpha$. Thus $\lambda = 1$ is the only eigenvalue and the corresponding eigenspace is one-dimensional and spanned by the eigenfunction

$$u(x) = 3x - 1.$$

CHAPTER 7

Volterra Integral Equations

7.1 Introduction

In this chapter we introduce the nonhomogeneous Volterra integral equation of the second kind of the form

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt, \tag{7.1.1}$$

where $K(x,t)$ is the kernel, and λ is parameter. As indicated earlier the limits of integration for Volterra integral equations [20] are functions of x and not constants as in Fredholm integral equations. The kernel here will be considered as a separable kernel. We want to determine the solution $u(x)$ applying various methods.

7.2 The Adomian Decomposition Method

Adomian recently developed the Adomian decomposition method that proved to work for all types of differential, integral and integro-differential equations, linear or nonlinear [1], [2], [3], [4].

The decomposition method establishes the solution in the form of power series. In this method $u(x)$ will be composed into components that will be determined, given by the series form

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{7.2.1}$$

with u_0 identified as all terms out of the integral sign, i.e.

$$u_0(x) = f(x). \tag{7.2.2}$$

Substituting (7.2.1) into (7.1.1) yields

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x K(x,t)(\sum_{n=0}^{\infty} u_n(t))dt,$$

which by using few terms of the expansion gives

$$u_0(x) + u_1(x) + u_2(x) + \cdots = f(x) + \lambda \int_0^x K(x,t)u_0(t)dt$$

$$+ \lambda \int_0^x K(x,t)u_1(t)dt$$

$$+ \lambda \int_0^x K(x,t)u_2(t)dt$$

$$+ \lambda \int_0^x K(x,t)u_3(t)dt$$

$$+ \cdots \tag{7.2.3}$$

The components $u_0(x)$, $u_1(x)$, $u_2(x)$, $u_3(x)$, ... of the unknown function $u_n(x)$ are determined if we set

$$u_0(x) = f(x), \tag{7.2.4}$$

$$u_1(x) = \lambda \int_0^x K(x,t)u_0(t)dt, \tag{7.2.5}$$

$$u_2(x) = \lambda \int_0^x K(x,t)u_1(t)dt, \tag{7.2.6}$$

$$u_3(x) = \lambda \int_0^x K(x,t)u_2(t)dt, \tag{7.2.7}$$

and so on. This scheme for the determination of components of solution can be written in a recursive scheme by

$$u_0(x) = f(x), \tag{7.2.8}$$

$$u_{n+1}(x) = \lambda \int_0^x K(x,t)u_n(t)dt, \quad n \geq 0. \tag{7.2.9}$$

The solution $u(x)$ is determined in a series form using

$$u(x) = \sum_{n=0}^{\infty} u_n(x).$$

The series obtained for $u(x)$ always provides the exact solution. It is important to note here that a few terms of the series usually provide the higher accuracy level of the approximation of the solution if compared with other numerical techniques.

It is important to indicate that the decomposition method provides the solution of any style of equations in the form of series with easily computable components. These applications have shown a very fast convergence of the series solution.

The following illustrative example will be discussed to explain the decomposition method.

Example 1. We consider the Volterra integral equation

$$u(x) = 4x + 2x^2 - \int_0^x u(t)dt.$$

Applying the decomposition technique we find $u_0(x), u_1(x), u_2(x), u_3(x)$ by

$$u_0(x) = 4x + 2x^2,$$

$$u_1(x) = -\int_0^x (4t + 2t^2)dt,$$

$$= -\left(2x^2 + \frac{2}{3}x^3\right),$$

$$u_2(x) = -\int_0^x -\left(2t^2 + \frac{2}{3}t^3\right)dt,$$

$$= \frac{2}{3}x^3 + \frac{1}{6}x^4,$$

$$u_3(x) = -\int_0^x \left(\frac{2}{3}t^3 + \frac{1}{6}t^4\right)dt,$$

$$= -\left(\frac{1}{6}x^4 + \frac{1}{30}x^5\right).$$

For solution in series form we get

$$u(x) = 4x + 2x^2 - \left(2x^2 + \frac{2}{3}x^3\right) + \frac{2}{3}x^3 + \frac{1}{6}x^4 - \left(\frac{1}{6}x^4 + \frac{1}{30}x^5\right) + \cdots,$$

The solution is

$$u(x) = 4x.$$

The series solution usually employed for numerical approximation, and the more terms we obtain provide more accuracy in the approximation of the solution. Even through the decomposition method proved to be powerful and reliable, but it can be used in a more effective manner which we called the modified decomposition method. The volume of calculations will be reduced by evaluating only the first two components $u_0(x)$ and $u_1(x)$. The modified technique works for specific problems where $f(x)$ consists of at least of two terms.

It is important to note that the modified decomposition method, which was introduced before for the Fredholm integral equations, is also applicable here. In Volterra integral equations where $f(x)$ consists of a polynomial, or a combination of polynomial and other trigonometric or transcendental functions, the modified decomposition method works well. In this case we decompose $f(x)$ into two parts such as

$$f(x) = f_1(x) + f_2(x). \tag{7.2.1.1}$$

$$u(x) = f_1(x) + f_2(x) + \lambda \int_0^x K(x,t)u(t)dt. \tag{7.2.1.2}$$

Using few terms of the expansions we obtain

$$u_0(x) + u_1(x) + u_2(x) + \cdots = f_1(x) + f_2(x) + \lambda \int_0^x K(x,t)u_0(t)dt$$

$$+\lambda \int_0^x K(x,t)u_1(t)dt$$

$$+\lambda \int_0^x K(x,t)u_2(t)dt$$

$$+\lambda \int_0^x K(x,t)u_3(t)dt$$

$$+\cdots \qquad (7.2.1.3)$$

We assign $f_1(x)$ only to the component $u_0(x)$, and the $f_2(x)$ will be added to the component $u_1(x)$. We set

$$u_0(x) = f_1(x), \qquad (7.2.1.4)$$

$$u_1(x) = f_2(x) + \lambda \int_0^x K(x,t)u_0(t)dt, \qquad (7.2.1.5)$$

$$u_2(x) = \lambda \int_0^x K(x,t)u_1(t)dt, \qquad (7.2.1.6)$$

$$u_3(x) = \lambda \int_0^x K(x,t)u_2(t)dt, \qquad (7.2.1.7)$$

and so on. The components $u_0(x)$, $u_1(x)$, $u_2(x)$, $u_3(x)$, ...of the solution can be written by

$$u_o(x) = f_1(x), \qquad (7.2.1.8)$$

$$u_1(x) = f_2(x) + \lambda \int_0^x K(x,t)u_0(t)dt, \qquad (7.2.1.9)$$

$$u_{n+1}(x) = \lambda \int_0^x K(x,t)u_n(t)dt, \quad n \geq 1. \qquad (7.2.1.10)$$

The following example illustrates how to obtain the solution of the Volterra integral equation by using the modified decomposition method.

Example 2. Solve the following Volterra integral equation by using the modified decomposition method

$$u(x) = cosx + \left(1 - e^{sinx}\right)x + x\int_0^x e^{sint}u(t)dt.$$

Using this method, we first decompose the function $f(x)$ into

$$f_1(x) = cosx,$$

$$f_2(x) = \left(1 - e^{sinx}\right)x.$$

We find $u_0(x)$ and $u_1(x)$ by

$$u_0(x) = cosx,$$

$$u_1(x) = \left(1 - e^{sinx}\right)x + x\int_0^x e^{sint}u_0(t)dt,$$

$$= \left(1 - e^{sinx}\right)x + x\int_0^x e^{sint}costdt,$$

$$= \left(1 - e^{sinx}\right)x - \left(1 - e^{sinx}\right)x = 0.$$

The exact solution is

$$u(x) = cosx.$$

It is clear that only two components are calculated to determine the exact solution.

7.3 The Series Solution Method

Now we introduce the series solution method that is practical method to solve the Volterra integral equation with variable limits of integration

$$u(x) = f(x) + \lambda\int_0^x K(x,t)u(t)dt \qquad (7.3.1)$$

where $K(x,t)$ is the kernel of the integral equation, and λ is parameter. In this method we will follow a parallel approach to the method of the series solution that usually applied in solving an ordinary differential equation around ordinary point. The method applicable

when $u(x)$ is an analytic function, i. e. $u(x)$ has a Taylor expansion around $x = 0$ so, $u(x)$ can be expressed by a series expansion [21] given by

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{7.3.2}$$

where the coefficients a_n are constants that will be determined. Substituting (7.3.2) into both sides of (7.3.1) yields

$$\sum_{n=0}^{\infty} a_n x^n = f(x) + \lambda \int_0^x K(x,t) \left(\sum_{n=0}^{\infty} a_n t^n \right) dt, \tag{7.3.3}$$

by using few terms of the expansions in both sides, we find

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = f(x) &+ \lambda \int_0^x K(x,t) a_0 dt, \\ &+ \lambda \int_0^x K(x,t) a_1 t dt, \\ &+ \lambda \int_0^x K(x,t) a_2 t^2 dt, \\ &+ \lambda \int_0^x K(x,t) a_3 t^3 dt, \\ &+ \cdots \end{aligned} \tag{7.3.4}$$

We write the Taylor expansion for $f(x)$ and evaluate the first few integrals. Then we equate the coefficients of like powers of x in both sides, so we find $a_1, a_2, a_3, \ldots$. Substituting these coefficients $a_n,\ n \geq 0,$ gives the solution in a series form. This may lead to a solution in a closed form if the expansion obtained is a Taylor expansion to a well-known elementary function.

The following example illustrates the series solution method.

Example 1. Use the series solution method to solve

$$u(x) = 1 + \int_0^x (t - x)u(t)dt.$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n,$$

into both sides of the equation leads to

$$\sum_{n=0}^{\infty} a_n x^n = 1 + \int_0^x (t - x)\left(\sum_{n=0}^{\infty} a_n t^n\right)dt,$$

which gives

$$\sum_{n=0}^{\infty} a_n x^n = 1 + \int_0^x \left(\sum_{n=0}^{\infty} a_n t^{n+1} - x\sum_{n=0}^{\infty} a_n t^n\right)dt.$$

Evaluating the integrals on the right-hand side that involves terms of the form $t^n,\ n \geq 0$ yields

$$\sum_{n=0}^{\infty} a_n x^n = 1 - \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} a_n x^{n+2},$$

or equivalently

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = 1 - \frac{1}{2!} a_0 x^2 - \frac{1}{3!} a_1 x^3 - \frac{1}{12} a_2 x^4 + \cdots$$

Equating the coefficients of like powers of x in both sides we find

$$a_0 = 1, \quad a_1 = 0, \quad a_2 = -\frac{1}{2!}, \quad a_3 = 0, \quad a_4 = \frac{1}{4!},$$

and generally

$$a_{2n} = (-1)^n \frac{1}{(2n)!}, \quad \text{for} \quad n \geq 0,$$

$$a_{2n+1} = 0, \quad \text{for} \quad n \geq 0.$$

We find the solution in series form $u(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots,$

the solution in a closed form $u(x) = cosx$.

7.4 Successive Approximations Method

The method of successive approximations used before for handling Fredholm integral equations will be implemented here to solve Volterra integral equation. In this method we replace the unknown function $u(x)$ under integral sign of the Volterra equation

$$u(x) = f(x) + \lambda \int_0^x k(x,t)u(t)dt \tag{7.4.1}$$

by any real valued continuous function $u_0(x)$, called the zeroth approximation. This substitution will give the first approximation $u_1(x)$ by

$$u_1(x) = f(x) + \lambda \int_0^x k(x,t)u_0(t)dt. \tag{7.4.2}$$

The second approximation obtained by replacing $u_0(x)$ in (7.4.2) by $u_1(x)$ obtained above, hence we find

$$u_2(x) = f(x) + \lambda \int_0^x k(x,t)u_1(t)dt. \tag{7.4.3}$$

This process can be continued to obtain the n-th approximation. So, we have

$$\begin{cases} u_0(x) = any\ selective\ real\ valued\ function \\ u_n(x) = f(x) + \lambda \int_0^x k(x,t)u_{n-1}(t)dt, \quad n \geq 1. \end{cases}$$

The most commonly selected functions for $u_0(x)$ are $0, 1,$ or x. The successive approximation are leading to a solution $u(x)$

$$u(x) = \lim_{n\to\infty} u_n(x). \tag{7.4.4}$$

of the equation (7.4.1)

So the solution $u(x)$ will be independent of the choice of $u_0(x)$ if we know for some series such as in the contraction mapping principle that the solution is unique. In this case the solution is given in a series form

$$u(x) = \lim_{n\to\infty}(\sum_{n=0}^{\infty} u_n(x)).$$

The zeroth approximation is not defined and given by a selective real valued function.

To illustrate this method we solve the following example.

Example 1. Solve the Volterra integral equation

$$u(x) = x + \int_0^x (t-x)u(t)dt$$

by the successive approximations method. We first select any real valued function for the zeroth approximation, hence we set $u_0(x) = 0$.

$$u_1(x) = x + \int_0^x (t-x)u_0(t)dt,$$

$$u_1(x) = x.$$

$$u_2(x) = x + \int_0^x (t-x)tdt,$$

$$u_2(x) = x - \frac{1}{3!}x^3,$$

$$u_3(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5,$$

$$u_n(x) = \sum_{k=1}^{n}(-1)^{k-1}\frac{x^{2k-1}}{(2k-1)!}, \qquad n \geq 1.$$

The solution $u_n(x)$ is given by

$$u(x) = \lim_{n\to\infty} u_n(x),$$

$$= \lim_{n\to\infty}\left(\sum_{k=1}^{n}(-1)^{k-1}\frac{x^{2k-1}}{(2k-1)!}\right),$$

$$= sinx.$$

7.5 The Method of Successive Substitutions

The technique to be used here is completely identical to that we used before. In this method, we set $x = t$ and $t = t_1$ in the Volterra integral equation

$$u(x) = f(x) + \lambda \int_0^x k(x,t)u(t)dt, \tag{7.5.1}$$

to obtain

$$u(t) = f(t) + \lambda \int_0^t k(t,t_1)u(t_1)dt_1. \tag{7.5.2}$$

Replacing $u(t)$ at the right-hand side of (7.5.1) by obtained value given by (7.5.2) yields

$$u(x) = f(x) + \lambda \int_0^x k(x,t)f(t)dt$$

$$+\lambda^2 \int_0^x K(x,t)\int_0^x K(t,t_1)u(t_1)dt_1dt. \tag{7.5.3}$$

Substituting $x = t_1$ and $t = t_2$ in (7.5.1) we obtain

$$u(t_1) = f(t_1) + \lambda \int_0^{t_1} K(t_1,t_2)u(t_2)dt_2. \tag{7.5.4}$$

Substituting the value of $u(t_1)$ obtained in (7.5.4) into the right hand side of (7.5.3) leads to

$$u(x) = f(x) + \lambda \int_0^x K(x,t)f(t)dt$$

$$+\lambda^2 \int_0^x \int_0^t K(x,t)\,K(t,t_1)f(t_1)dt_1dt$$

$$+\lambda^3 \int_0^x \int_0^t \int_0^{t_1} K(x,t)K(t,t_1)K(t_1,t_2)u(t_2)dt_2dt_1dt.$$

(7.5.5)

The general series form for $u(x)$ can be rewritten as

$$u(x) = f(x) + \lambda \int_0^x k(x,t)f(t)dt$$

$$+\lambda^2 \int_0^x \int_0^t K(x,t)\,K(t,t_1)f(t_1)dt_1dt$$

$$+\lambda^3 \int_0^x \int_0^t \int_0^{t_1} K(x,t)K(t,t_1)K(t_1,t_2)f(t_2)dt_2dt_1dt.$$

$$+\cdots \qquad (7.5.6)$$

In this method the unknown function $u(x)$ is substituted by the given function $f(x)$ that makes the evaluation of the multiple integrals easily computable. This process occurs several times through the integrals and this is why it is called the method of successive substitutions. The technique will be illustrated by solving the last example in this new format.

Example 1. We solve the following Volterra integral equation

$$u(x) = x - \int_0^x (x-t)u(t)dt,$$

by using the method of successive substitutions. Substituting $\lambda = -1$, $f(x) = x$, and $K(x,t) = (x-t)$ into (7.5.6) we obtain

$$u(x) = x - \int_0^x (x-t)tdt + \int_0^x \int_0^t (x-t)\,(t-t_1)t_1dt_1dt + \cdots,$$

or equivalently

$$u(x) = x - \int_0^x (xt-t^2)dt + \int_0^x \int_0^t (x-t)(tt_1-{t_1}^2)dt_1dt + \cdots,$$

therefore, we obtain the solution in a series form

$$u(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots,$$

or in a closed form

$$u(x) = sinx$$

upon using the Taylor expansion for $sinx$. We get the same solution as before.

7.6 Volterra Integral Equations of the First Kind

In this section we will study the Volterra integral equation of the first kind with separable kernel given by

$$f(x) = \int_0^x K(x,t)u(t)dt. \tag{7.6.1}$$

It is important to note that the Volterra integral equation of the first kind can be handled by reducing this equation to Volterra equation of the second kind. This goal can be accomplished by differentiating both sides of (7.6.1) with respect to x to obtain

$$f'(x) = K(x,x)u(x) + \int_0^x K_x(x,t)u(t)dt \tag{7.6.2}$$

by using Leibnitz rule. If $K(x,x) \neq 0$ in the interval of discussion, then dividing both sides of (7.6.2) by $K(x,x)$ yields

$$u(x) = \frac{f'(x)}{K(x,x)} - \frac{1}{K(x,x)}\int_0^x K_x(x,t)u(t)dt, \tag{7.6.3}$$

a Volterra integral equation of the second kind. The case in which the kernel $K(x,x) = 0$, leads to a complicated behavior of the problem that will not be investigated here.
To solve (7.6.3) we select any method that we discussed before. The technique of differentiating both sides of Volterra integral equation of the first kind, verifying that

$K(x,x) \neq 0$, reducing to Volterra integral equation of the second kind and solving the resulting equation will be illustrated by discussing the following example.

Example 1. Find the solution of the Volterra equation of the first kind

$$5x^2 + x^3 = \int_0^x (5 + 3x - 3t)u(t)dt.$$

Note that $K(x,t) = 5 + 3x - 3t$, therefore $K(x,x) = 5 \neq 0$, and differentiating, gives

$$10x + 3x^2 = 5u(x) + \int_0^x 3u(t)dt,$$

or equivalently

$$u(x) = 2x + \frac{3}{5}x^2 - \frac{1}{5}\int_0^x 3u(t)dt,$$

We prefer to use the modified decomposition method. We set $u_0(x) = 2x$, which gives

$$u_1(x) = \frac{3}{5}x^2 - \frac{3}{5}\int_0^x 2tdt = 0.$$

For $u_n(x) = 0, \quad n \geq 2$. Exact solution is $u(x) = 2x$.

CHAPTER 8

Integro-Differential Equations

8.1 Introduction

In this chapter we shall be concerned with the integro-differential equations where both differential and integral operators will appear in same equation. This type of equations was introduced by Volterra [5], [6] and [20] in 1900. These equations came up in his research work on population growth. More details about the sources where these equations arise can be found in physics, biology and engineering applications as well as in advanced integral equations books such as [7], [11], [12] and [16]. In integro-differential equations the unknown function $u(x)$ and one or more its derivatives such as $u'(x), u''(x)$, ... appear outside and under the integral sign as well.

The following are examples of linear integro-differential equations:

$$u'(x) = x - \int_0^1 e^{x-t}u(t)dt, \;\; u(0) = 0, \tag{8.1.1}$$

$$u''(x) = e^x - x + \int_0^1 xtu'(t)dt, \quad u(0) = 1, \;\; u'(0) = 1, \tag{8.1.2}$$

$$u'(x) = x - \int_0^x (x-t)u(t)dt, \quad u(0) = 0, \tag{8.1.3}$$

$$u''(x) = -x + \int_0^x (x-t)u(t)dt, \quad u(0) = 0, \;\; u'(0) = -1. \tag{8.1.4}$$

Of these equations (8.1.1) and (8.1.2) are Fredholm integro-differential equations, which are linear, and equations (8.1.3) and (8.1.4) are the Volterra integro-differential equations. Our concern in this paper will be focused only on the linear integro-differential equations. The initial conditions are needed to determine the constants of integration.

8.2 Fredholm Integro-Differential Equations

In this section we will discuss methods used to solve Fredholm integro-differential equations. We focus our concern on the equations that involve separable kernels where the kernel $K(x,t)$ can be expressed as a finite sum of the form

$$K(x,t) = \sum_{k=1}^{n} g_k(x)h_k(t). \tag{8.2.1}$$

We will make our analysis on a one term kernel $K(x,t)$ of the form

$$K(x,t) = g(x)h(t), \tag{8.2.2}$$

and this can be generalized for other cases. The non-separable kernel can be approximated by a separable kernel by using the Taylor expansion for the kernel involved. Then we have the exact solution or an approximation to the solution with the highest desirable accuracy. We first start with the most practical method.

8.2.1 The Direct Computation Method

This method has been introduced in previous chapter. The standard form to the Fredholm Integro-Differential Equation given by

$$u^{(n)}(x) = f(x) + \int_0^1 K(x,t)u(t)dt, \quad u^{(k)}(0) = b_k, \quad 0 \le k \le (n-1), \tag{8.2.1.1}$$

where $u^{(n)}(x)$ indicates the n-th derivative of $u(x)$ with respect to x and b_k are constants that define the proper initial conditions. Substitute (8.2.2) into (8.2.1.1) to get

$$u^{(n)}(x) = f(x) + g(x)\int_0^1 h(t)u(t)dt, \quad u^k(0) = b_k, \ \ 0 \le k \le (n-1). \tag{8.2.1.2}$$

We set

$$\alpha = \int_0^1 h(t)u(t)dt. \tag{8.2.1.3}$$

With α defined in (8.2.1.3), the equation (8.2.1.2) can be written by

$$u^{(n)}(x) = f(x) + \alpha g(x). \qquad (8.2.1.4)$$

We need to determine the constant α to evaluate the exact solution $u(x)$. We integrate both sides n times from 0 to x, and by using the given initial conditions we obtain $u(x)$ given by

$$u(x) = p(x; \alpha), \qquad (8.2.1.5)$$

Where $p(x; \alpha)$ is the result derived from integrating equation (8.2.1.4) and by using the given initial condition. Substituting (8.2.1.5) into right side of (8.2.1.3), integrating and solving the resulting equation leads to determination of α. The exact solution of (8.2.1.1) follows immediately upon substituting the resulting value of α into (8.2.1.5). To give a clear view of the technique, we illustrate the method by solving the following example.

Example 1. Solve the Fredholm integro-differential equation

$$u'(x) = 1 - \frac{1}{3}x + x\int_0^1 tu(t)dt, \qquad u(0) = 0,$$

by using direct computation method.

The equation may be written in the form

$$u'(x) = 1 - \frac{1}{3}x + \alpha x, \quad u(0) = 0$$

Where the constant α is defined by

$$\alpha = \int_0^1 tu(t)dt,$$

Integrating both sides of $u'(x)$ from 0 to x and using initial condition we obtain

$$u(x) = x + \left(\frac{\alpha}{2} - \frac{1}{6}\right)x^2.$$

Substitute this into the integral defining α, integrate, and solve for α to find

$$\alpha = \frac{1}{3}.$$

So, the exact solution is

$$u(x) = x.$$

8.2.2 The Adomian Decomposition Method

This method has been introduced already for handling Fredholm integral equations. In this section we will show how it can be implemented to determine a series solution to the Fredholm integro-differential equations. As in the last section, consider the Fredholm integro-differential equation given by

$$u^{(n)}(x) = f(x) + \int_0^1 K(x,t)u(t)dt, \quad u^{(k)}(0) = b_k, \quad 0 \le k \le (n-1) \qquad (8.2.2.1)$$

where $u^{(n)}(x)$ indicates the n-th derivative of $u(x)$ with respect to x and b_k are constants that give the initial conditions. Substituting $K(x,t) = g(x)h(t)$ into (8.2.2.1) we obtain

$$u^{(n)}(x) = f(x) + g(x)\int_0^1 h(t)u(t)dt. \qquad (8.2.2.2)$$

In an operator form, the equation (8.2.2.2) can be written as

$$Lu(x) = f(x) + g(x)\int_0^1 h(t)u(t)dt, \qquad (8.2.2.3)$$

where the differential operator L is given by

$$L = \frac{d^n}{dx^n}. \qquad (8.2.2.4)$$

Here L is an invertible operator, therefore the integral operator L^{-1} is an n-fold integration operator and may be considered as definite integrals from 0 to x for each

integral. Applying L^{-1} to both sides of (8.2.2.3), yields

$$u(x) = b_0 + b_1 x + \frac{1}{2!} b_2 x^2 + \cdots + \frac{1}{(n-1)!} b_{n-1} x^{n-1} + L^{-1}(f(x))$$

$$+ \left(\int_0^1 h(t) u(t) dt \right) L^{-1}\big(g(x)\big). \qquad (8.2.2.5)$$

We have integrated (8.2.2.2) n times from 0 to x and using the initial conditions at every step of integration. It is important to indicate that the equation (8.2.2.5) is a standard Fredholm integral equation. In the decomposition method we define the solution $u(x)$ of (8.2.2.1) in a series form given by

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \qquad (8.2.2.6)$$

Substituting (8.2.2.6) into both sides of (8.2.2.5) we get

$$\sum_{n=0}^{\infty} u_n(x) = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(f(x))$$

$$+ \left(\int_0^1 h(t) \left(\sum_{n=0}^{\infty} u_n(t) \right) dt \right) L^{-1}\big(g(x)\big), \qquad (8.2.2.7)$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(f(x))$$

$$+ \left(\int_0^1 h(t) u_0(t) dt \right) L^{-1}(g(x))$$

$$+ \left(\int_0^1 h(t) u_1(t) dt \right) L^{-1}(g(x))$$

$$+ \left(\int_0^1 h(t) u_2(t) dt \right) L^{-1}(g(x))$$

$$+\cdots \tag{8.2.2.8}$$

The components $u_0(x), u_1(x), u_2(x), u_3(x), \ldots$ of the unknown function $u(x)$ are determined in a recursive manner, in a similar fashion as discussed before, if we set

$$u_0(x) = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}\big(f(x)\big), \tag{8.2.2.9}$$

$$u_1(x) = \left(\int_0^1 h(t)u_0(t)dt\right) L^{-1}\big(g(x)\big), \tag{8.2.2.10}$$

$$u_2(x) = \left(\int_0^1 h(t)u_1(t)dt\right) L^{-1}\big(g(x)\big), \tag{8.2.2.11}$$

$$u_3(x) = \left(\int_0^1 h(t)u_2(t)dt\right) L^{-1}\big(g(x)\big), \tag{8.2.2.12}$$

and so on. For the determination of the components $u_0(x), u_1(x), u_2(x), u_3(x), \ldots$ of the solution $u(x)$ in general can be written in a recursive relationship by

$$u_0(x) = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}\big(f(x)\big), \tag{8.2.2.13}$$

$$u_{n+1}(x) = \left(\int_0^1 h(t)u_n(t)dt\right) L^{-1}\big(g(x)\big), \quad n \geq 0. \tag{8.2.2.14}$$

The solution $u(x)$ is immediately determined with these components calculated. The series obtained for $u(x)$ frequently provides the exact solution as will be illustrated later.

In some problems, where a closed form is not easy to find, we can use the series form obtained to approximate the solution. It can be shown [6] that a few terms of the series derived by decomposition method usually provide a highly accurate approximation. The decomposition method avoids massive computational work and difficulties that arise from other methods.

The exact solution of any integral equation or integro-differential equations may be obtained by considering the first two components u_0 and u_1 only. If we observe the appearance of like terms in both components with opposite signs, then by cancelling these terms, the remaining non-cancelled terms of u_0 may in some cases provide the exact solution. The self-cancelling terms between the components u_0 and u_1 are called the noise terms. The other terms in other components will vanish in the limit if the noise terms occurred in $u_0(x)$ and $u_1(x)$. However, if the exact solution was not attainable by using this phenomenon, then we should continue determining other components of $u(x)$ to get a closed form solution or an approximate solution. In the following we discuss one example which illustrates the decomposition scheme where we will examine the phenomena of the self-cancelling noise terms as well.

Example 1. Solve the following Fredholm integro-differential equation

$$u'(x) = cosx + \frac{1}{4}x - \frac{1}{4}\int_0^{\frac{\pi}{2}} x\, t\, u(t)dt, \qquad u(0) = 0,$$

by using the decomposition method.

Integrating both sides of the equation from 0 to x, since $u(0) = 0$ and $sin0 = 0$ gives

$$u(x) = sinx + \frac{1}{8}x^2 - \frac{1}{8}x^2 \int_0^{\pi/2} tu(t)dt.$$

Using the decomposition technique we decompose the solution into a series form given by

$$u(x) = \sum_{n=0}^{\infty} u_n(x).$$

Substituting into both sides yields

$$\sum_{n=0}^{\infty} u_n(x) = sinx + \frac{1}{8}x^2 - \frac{1}{8}x^2 \int_0^{\pi/2} t\left(\sum_{n=0}^{\infty} u_n(t)\right)dt,$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots = sinx + \frac{1}{8}x^2$$
$$-\frac{1}{8}x^2\left(\int_0^{\pi/2} tu_0(t)dt\right)$$
$$-\frac{1}{8}x^2\left(\int_0^{\pi/2} tu_1(t)dt\right)$$
$$-\frac{1}{8}x^2\left(\int_0^{\pi/2} tu_2(t)dt\right)$$
$$+\cdots.$$

We set

$$u_0(x) = sinx + \frac{1}{8}x^2$$

which gives

$$u_1(x) = -\frac{1}{8}x^2 \int_0^{\frac{\pi}{2}} t\left(sint + \frac{1}{8}t^2\right)dt = -\frac{1}{8}x^2 - \frac{\pi^4}{16^3}x^2.$$

Considering $u_0(x)$ and $u_1(x)$ we see that the two identical terms $\frac{1}{8}x^2$ appear in these components with opposite signs. Cancelling these terms, and substituting the remaining non cancelled term in $u_0(x)$ it satisfies the given equation lead to

$$u(x) = sinx$$

this is the exact solution in closed form.

8.2.3 Converting to Fredholm Integral Equations

In this section we will discuss a technique that will reduce Fredholm integro-

differential equation to an equivalent Fredholm integral equation. This can be done by integrating both sides of the integro-differential equation as many times as the order of the derivative involved in the equation from 0 to x for every time we integrate by using the given initial conditions. This technique is applicable only if the Fredholm integro-differential equation involves the unknown function $u(x)$ only, and not any of its derivatives, under the integral sign. We can use the decomposition method, the direct computation method, the successive approximation method or the method of successive substitutions. To make a clear overview of this method we solve the example of section 8.2.1.

Example 1. Solve the following Fredholm integro-differential equation

$$u'(x) = 1 - \frac{1}{3}x + x\int_0^1 tu(t)dt, \qquad u(0) = 0,$$

by converting it to a standard Fredholm integral equation.

Integrating both sides from 0 to x and using the initial condition we get

$$u(x) = x - \frac{1}{3!}x^2 + \frac{1}{2!}x^2\left(\int_0^1 tu(t)dt\right).$$

This is the Fredholm integral equation and we choose the successive approximation method to solve this equation. We set a zeroth approximation by

$$u_0(x) = x,$$

This gives the first approximation

$$u_1(x) = x - \frac{1}{3!}x^2 + \frac{1}{2!}x^2\left(\int_0^1 t^2dt\right) = x - \frac{1}{3!}x^2 + \frac{1}{3}\frac{1}{2!}x^2 = x$$

since

$$u_1(x) = x = u_0(x),$$

If we continue we get

$$u_n(x) = x \qquad \text{for all } n.$$

Accordingly

$$u(x) = \lim_{n\to\infty} u_n(x)$$

$$= \lim_{n\to\infty} x$$

$$= x.$$

And this is the same solution we obtained before.

8.3 Volterra Integro-Differential Equations

In this section we will present a method to handle Volterra integro-differential equations. We will focus on equations that involve separable kernels of the form

$$K(x,t) = \sum_{k=1}^{n} g_k(x)h_k(t). \tag{8.3.1}$$

We consider the cases where the kernel $K(x,t)$ consists of product of the functions $g(x)$ and $h(t)$ given by $K(x,t) = g(x)h(t)$, where the other cases can be generalized in the same manner. The non-separable kernel can be approximated by separable kernel by using the Taylor expansion for the kernel involved. We use most practical method, the series solution method.

8.3.1 The Series Solution Method

We consider a standard form to the Volterra integro-differential equation given by

$$u^n(x) = f(x) + \int_0^x K(x,t)u(t)dt, \quad u^k(0) = b_k, \quad 0 \le k \le (n-1), \tag{8.3.1.1}$$

where $u^n(x)$ indicates the n-th derivative of $u(x)$ with respect to x, and b_k are constants that define the initial conditions. Substituting $K(x,t) = g(x)h(t)$ into (8.3.1.1) we get

$$u^n(x) = f(x) + g(x)\int_0^x h(t)u(t)dt, \quad u^k(0) = b_k, \quad 0 \le k \le (n-1). \qquad (8.3.1.2)$$

As before, follow a parallel analogy to that for ordinary differential equations around an ordinary point. We assume that the solution is an analytic function and it can be represented by a series expansion given by

$$u(x) = \sum_{k=0}^{\infty} a_k x^k, \qquad (8.3.1.3)$$

where a_k are constants that will be determined by using the initial conditions so that

$$a_0 = u(0), \quad a_1 = u'(0), \quad a_2 = \frac{1}{2!}u''(0), \qquad (8.3.1.4)$$

and so on depending on the number of the initial conditions. Substituting (8.3.1.3) into both sides of (8.3.1.2) yields

$$(\sum_{k=0}^{\infty} a_k x^k)^{(n)} = f(x) + g(x)\int_0^x h(t)\left(\sum_{k=0}^{\infty} a_k t^k\right)dt. \qquad (8.3.1.5)$$

Equation (8.3.1.5) can be easily evaluated if we have to integrate terms of the form t^n, $n \ge 0$ only. The next step is to write the Taylor expansion for $f(x)$, evaluate the resulting integrals, and then equating the coefficients of like powers of x in both sides of the equation. This will lead to a determination of the coefficients $a_0, a_1, a_2, \ldots$ of the series. This may give a solution in closed form. The following example illustrates the series solution method for Volterra integro-differential equation.

Example 1. Solve the following Volterra integro-differential equation

$$u''(x) = xcoshx - \int_0^x tu(t)dt, \qquad u(0) = 0, \;\; u'(0) = 1$$

by using the series solution method. Notice that $h(t) = t$, so we will be interpreting terms of the form t^n.

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

into both sides of the equation and using the Taylor expansion of $coshx$ we obtain

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = x\left(\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}\right) - \int_0^x t\left(\sum_{n=0}^{\infty} a_n t^n\right) dt.$$

Using the initial conditions yields $a_0 = 0,\ \ a_1 = 1$, evaluating the integrals that involve terms of the form t^n, $n \geq 0$, and using few terms for both sides yield

$$2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \cdots = x(1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots)$$
$$-\left(\frac{1}{3}x^3 + \frac{1}{4}a_2 x^4 + \cdots\right).$$

Equating the coefficients of like powers of x in both sides we find

$$a_2 = 0,\ \ a_3 = \frac{1}{3!},\ \ a_4 = 0,$$

and generally

$$a_{2n} = 0,\ \ for\ \ n \geq 0,$$

and

$$a_{2n+1} = \frac{1}{(2n+1)!},\ \ \ for\ \ n \geq 0.$$

We find the solution $u(x)$ in a series form

$$u(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots,$$

and in a closed form the solution is

$$u(x) = sinh\, x.$$

8.3.2 The Decomposition Method

The decomposition method and the modified decomposition method were discussed earlier. In this section we will show how this method can be implemented to determine series solutions to Volterra integro-differential equations. A standard form of the Volterra integro-differential equation is

$$u^{(n)}(x) = f(x) + \int_0^x K(x,t)u(t)dt, \quad u^k(0) = b_k, \ \ 0 \le k \le (n-1) \tag{8.3.2.1}$$

We can find $u(x)$ by integrating both sides of (8.3.2.1) from 0 to x as many times as the order of the derivative involved. We obtain

$$u(x) = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}\big(f(x)\big) + L^{-1}\left(\int_0^x K(x,t)u(t)dt\right), \tag{8.3.2.2}$$

where $\sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k$ is obtained by using the initial conditions, and L^{-1} is an n-fold integration operator. Now we apply the decomposition method by defining the solution $u(x)$ of (8.3.2.2) in a decomposition series given by

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \tag{8.3.2.3}$$

Substituting (8.3.2.3) into both sides of (8.3.2.2) we get

$$\sum_{n=0}^{\infty} u_n(x) = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}\big(f(x)\big) + L^{-1}\left(\int_0^x K(x,t)\left(\sum_{n=0}^{\infty} u_n(t)\right)dt\right) \tag{8.3.2.4}$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(f(x))$$

$$+L^{-1}\left(\int_0^x K(x,t)u_0(t)dt\right)$$

$$+L^{-1}\left(\int_0^x K(x,t)u_1(t)dt\right)$$

$$+L^{-1}\left(\int_0^x K(x,t)u_2(t)dt\right)$$

$$+\cdots. \tag{8.3.2.5}$$

The components $u_0(x), u_1(x), u_2(x), u_3(x), \ldots$ of the unknown function $u(x)$ are determined in a recursive manner, in a similar way as discussed before, if we set

$$u_0(x) = \sum_{k=0}^{n-1} \frac{1}{k!} a_k x^k + L^{-1}(f(x)), \tag{8.3.2.6}$$

$$u_1(x) = L^{-1}\left(\int_0^x K(x,t)u_0(t)dt\right), \tag{8.3.2.7}$$

$$u_2(x) = L^{-1}\left(\int_0^x K(x,t)u_1(t)dt\right), \tag{8.3.2.8}$$

and so on. The solution $u(x)$ of the equation (8.3.2.1) can be written as

$$u_0(x) = \sum_{k=0}^{n-1} \frac{1}{k!} a_k x^k + L^{-1}(f(x)), \tag{8.3.2.9}$$

$$u_{n+1}(x) = L^{-1}\left(\int_0^x K(x,t)u_n(t)dt\right), \quad n \geq 0. \tag{8.3.2.10}$$

The series obtained for $u(x)$ can provide the exact solution in a closed form.

The following example will illustrate this technique.

Example 1. Solve the following Volterra integro-differential equation

$$u''(x) = x + \int_0^x (x-t)u(t)dt, \quad u(0) = 0, \; u'(0) = 1$$

by using decomposition method.

Applying the two-fold integration operator L^{-1} we get

$$L^{-1}(.) = \int_0^x \int_0^x (.)dxdx,$$

to both sides of equation integrating twice from 0 to x, and using the given initial conditions yield

$$u(x) = x + \frac{1}{3!}x^3 + L^{-1}\left(\int_0^x (x-t)u(t)dt\right).$$

Following the decomposition scheme (8.3.2.9) and (8.3.2.10) we find

$$u_0(x) = x + \frac{1}{3!}x^3,$$

$$u_1(x) = L^{-1}\left(\int_0^x (x-t)u_0(t)dt\right)$$

$$= \frac{1}{5!}x^5 + \frac{1}{7!}x^7,$$

$$u_2(x) = L^{-1}\left(\int_0^x (x-t)u_1(t)dt\right)$$

$$= \frac{1}{9!}x^9 + \frac{1}{11!}x^{11}.$$

Combining these equations we get the solution $u(x)$ in series form given by

$$u(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \frac{1}{11!}x^{11} + \cdots$$

and this leads to

$$u(x) = \sinh x,$$

the exact solution in a closed form.

8.3.3 Converting to Volterra Integral Equations

We can easily convert the Volterra integro-differential equation to an equivalent Volterra integral equation, provided that the kernel is a difference kernel defined by the form $K(x,t) = K(x-t)$. This can be done by integrating both sides of the equation and

using the initial conditions. To perform the conversion to a regular Volterra integral equation we should use the formula that converts multiple integral to a single integral. The following two formulas

$$\int_0^x \int_0^x u(t)dtdt = \int_0^x (x-t)u(t)dt, \tag{8.3.2.1}$$

$$\int_0^x \int_0^x \int_0^x u(t)dtdtdt = \frac{1}{2!}\int_0^x (x-t)^2 u(t)dt. \tag{8.3.2.2}$$

are used to transform double integrals and triple integrals respectively to a single integral. To give clear overview of this method we discuss the following example.

Example 1. Solve the following Volterra integro-differential equation

$$u'(x) = 2 - \frac{1}{4}x^2 + \frac{1}{4}\int_0^x u(t)dt, \quad u(0) = 0,$$

by converting to a standard Volterra integral equation.

Integrating both sides from 0 to x and using the initial condition we obtain

$$u(x) = 2x - \frac{1}{12}x^3 + \frac{1}{4}\int_0^x \int_0^x u(t)dtdt,$$

which gives

$$u(x) = 2x - \frac{1}{12}x^3 + \frac{1}{4}\int_0^x (x-t)u(t)dt.$$

This is a standard Volterra integral equation and can be solved by using the decomposition method. We set

$$u_0(x) = 2x - \frac{1}{12}x^3,$$

which gives

$$u_1(x) = \frac{1}{4}\int_0^x (x-t)\left(2t - \frac{1}{12}t^3\right)dt,$$

$$= \frac{1}{12}x^3 - \frac{1}{240}x^5.$$

The term $\frac{1}{12}x^3$ appears with opposite signs in the components $u_0(x)$ and $u_1(x)$, and by cancelling this noise term from $u_0(x)$ and justifying that

$$u(x) = 2x$$

is the exact solution.

CHAPTER 9

Singular Integral Equations

9.1 Definitions

An integral equation is called a singular integral equation if one or both limits of integration become infinite, or if the kernel $K(x,t)$ of the equation becomes infinite at one or more points in the interval of integration. For example, the integral of the first kind

$$f(x) = \lambda \int_{\alpha(x)}^{\beta(x)} K(x,t)u(t)dt \tag{9.1.1}$$

or the integral equation of the second kind

$$u(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} K(x,t)u(t)dt \tag{9.1.2}$$

is called singular if the lower limit $\alpha(x)$, the upper limit $\beta(x)$ or both limits of integration are infinite. Examples of the first type of singular equations are the following examples:

$$u(x) = 1 + e^{-x} - \int_0^{\infty} u(t)dt, \tag{9.1.3}$$

$$F(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x}u(x)dx, \tag{9.1.4}$$

$$L[u(x)] = \int_0^{\infty} e^{-\lambda x}u(x)dx. \tag{9.1.5}$$

The integral equations (9.1.4) and (9.1.5) are the Fourier transform and Laplace transform of the function $u(x)$ respectively. In addition, these equations are Fredholm integral equations of the first kind with kernels given by $K(x,t) = e^{-i\lambda x}$ and $K(x,t) = e^{-\lambda x}$. It is important to note that the Laplace transforms and the Fourier transforms are used for solving ordinary and partial differential equations with constant coefficients.

Examples of the second type of singular integral equations are given by the following

$$x^2 = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt, \tag{9.1.6}$$

$$x = \int_0^x \frac{1}{(x-t)^{\alpha}} u(t)dt, \quad 0 < \alpha < 1, \tag{9.1.7}$$

$$u(x) = 1 + 2\sqrt{x} - \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt, \tag{9.1.8}$$

where the singular behavior has been attributed to the kernel $K(x,t)$ becoming infinite as $t \to x$. Integral equations similar to examples (9.1.6) and (9.1.7) are called Abel's problems and generalized Abel's integral equations respectively. These styles of singular integral equations are among the earliest integral equations established by the Norwegian mathematician Niles Abel in 1823.

9.2 Abel's Problem

Abel in 1823 investigated the motion of a particle that slides down along a smooth unknown curve, in a vertical plane, under the influence of the gravitational field. It is assumed that the particle starts from rest at the point P, with vertical elevation x, slides along the unknown curve, to the lowest point O on the curve where the vertical distance is $x = 0$. The total time of descent T from the highest point to the lowest point on the curve is given in advance, and dependent on the elevation x, hence expressed by

$$T = h(x). \tag{9.2.1}$$

Assuming that the curve of motion between the point P and O has an arc length s, then the velocity at a point Q on the curve, between P and O, is given by

$$\frac{ds}{dT} = -\sqrt{2g(x-t)} \tag{9.2.2}$$

where t is a variable coordinate defines the vertical distance of the point Q, and g is the acceleration of gravity assumed to be constant on the scale of the problem . Integrating

both sides of (9.2.2) gives

$$T = -\int_O^P \frac{ds}{\sqrt{2g(x-t)}}. \tag{9.2.3}$$

Setting

$$ds = u(t)dt, \tag{9.2.4}$$

and using (9.2.1) we find the equation of motion of the sliding particle

$$f(x) = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt. \tag{9.2.5}$$

We point out that $f(x)$ is a predetermined function that depends on the elevation x and given by

$$f(x) = \sqrt{2g}\, h(x), \tag{9.2.6}$$

where g is the gravitational constant, and $h(x)$ is the time of descent from the highest point to the lowest point on the curve. The main goal of Abel's problem is to determine the unknown function $u(x)$ under the integral sign that will define the equation of the curve. Notice that Abel's integral equation is a Volterra integral equation of the first kind with singular kernel. The kernel in (9.2.5) is

$$K(x,t) = \frac{1}{\sqrt{x-t}}, \tag{9.2.7}$$

which shows that the kernel (9.2.7) is singular in that

$$K(x,t) \to \infty \quad as \quad t \to x. \tag{9.2.8}$$

Taking Laplace transforms of both sides of (9.2.5) leads to

$$L[f(x)] = L[u(x)]L[x^{-1/2}]$$

$$= L[u(x)]\frac{\Gamma\left(\frac{1}{2}\right)}{z^{\frac{1}{2}}}, \tag{9.2.9}$$

where Γ is the gamma function. In Appendix B, the definition of the gamma function and some of the relations related to it are given. Notice that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, the equation (9.2.9) becomes

$$L[u(x)] = \frac{z^{1/2}}{\sqrt{\pi}} L[f(x)], \tag{9.2.10}$$

which can be rewritten by

$$L[u(x)] = \frac{z}{\pi}\left(\sqrt{\pi}\, z^{-\frac{1}{2}} L[f(x)]\right). \tag{9.2.11}$$

Setting

$$h(x) = \int_0^x (x-t)^{-\frac{1}{2}} f(t)dt, \tag{9.2.12}$$

into (9.2.11) yields

$$L[u(x)] = \frac{z}{\pi} L[h(x)], \tag{9.2.13}$$

this gives

$$L[u(x)] = \frac{1}{\pi} L[h'(x)], \tag{9.2.14}$$

upon using the fact

$$L[h'(x)] = zL[h(x)]. \tag{9.2.15}$$

Appling L^{-1} to both sides of (9.2.14) yields easily calculable formula

$$u(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{f(t)}{\sqrt{x-t}} dt, \tag{9.2.16}$$

that will be used for the determination of the solution. Appendix A supplies a helpful tool for evaluating the integrals involved in (9.2.16).

The procedure of using the formula (9.2.16) that determines the solution of Abel's problem (9.2.5) will be illustrated by the following example.

Example 1. We consider the following Abel's problem

$$\pi = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt.$$

Substituting $f(x) = \pi$ in equation 9.2.16 for $u(x)$

$$u(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{\pi}{\sqrt{x-t}}dt,$$

$$= \frac{d}{dx}\int_0^x \frac{1}{\sqrt{x-t}}dt.$$

Setting the substitution $y = x - t$, we obtain

$$u(x) = \frac{d}{dx}\left(2\sqrt{x}\right),$$

$$= \frac{1}{\sqrt{x}}.$$

Abel introduced the more general singular integral equation

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha} u(t)dt, \quad 0 < \alpha < 1, \tag{9.2.1.1}$$

known as the Generalized Abel's integral equation. Abel's problem discussed above is a special case of the generalized equation where $\alpha = \frac{1}{2}$. To determine a practical formula for the solution $u(x)$ of (9.2.1.1), and hence for Abel's problem, we use the Laplace transform on both sides of equation (9.2.1.1). This yields

$$L[f(x)] = L[u(x)]L[x^{-\alpha}]$$

$$= L[u(x)]\frac{\Gamma(1-\alpha)}{z^{1-\alpha}}, \tag{9.2.1.2}$$

where Γ is the gamma function. The equation (9.2.1.2) can be written as

$$L[u(x)] = \frac{z}{\Gamma(\alpha)\Gamma(1-\alpha)}\Gamma(\alpha)z^{-\alpha}L[f(x)], \tag{9.2.1.3}$$

or equivalently

$$L[u(x)] = \frac{z}{\Gamma(\alpha)\Gamma(1-\alpha)} L[g(x)], \tag{9.2.1.4}$$

where

$$g(x) = \int_0^x (x-t)^{\alpha-1} f(t)dt. \tag{9.2.1.5}$$

Accordingly, equation (9.2.1.4) can be written as

$$L[u(x)] = \frac{\sin(\alpha\pi)}{\pi} L[g'(x)], \tag{9.2.1.6}$$

upon using the identities

$$L[g'(x)] = zL[g(x)], \tag{9.2.1.7}$$

and

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\alpha\pi)}, \tag{9.2.1.8}$$

from Laplace transforms and Appendix B respectively. Applying L^{-1} to both sides of (9.2.1.6) yields the easily calculable formula for determining the solution

$$u(x) = \frac{\sin(\alpha\pi)}{\pi} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \qquad 0 < \alpha < 1. \tag{9.2.1.9}$$

We first integrate the integral at the right hand side of (9.2.1.9) by parts to obtain

$$\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt = -\frac{1}{\alpha}[f(t)(x-t)^{\alpha}]_0^x + \frac{1}{\alpha}\int_0^x (x-t)^{\alpha} f'(t)dt,$$

$$= \frac{1}{\alpha} f(0)x^{\alpha} + \frac{1}{\alpha}\int_0^x (x-t)^{\alpha} f'(t)dt. \tag{9.2.1.10}$$

Differentiating both sides of (9.2.1.10) and using the Leibnitz rule, yields

$$\frac{d}{dx}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt = \frac{f(0)}{x^{1-\alpha}} + \int_0^x \frac{f'(t)}{(x-t)^{1-\alpha}} dt. \tag{9.2.1.11}$$

Substituting (9.2.1.11) into (9.2.1.9) yields the desired formula

$$u(x) = \frac{\sin(\alpha\pi)}{\pi}\left(\frac{f(0)}{x^{1-\alpha}} + \int_0^x \frac{f'(t)}{(x-t)^{1-\alpha}}dt\right), \quad 0 < \alpha < 1, \quad (9.2.1.12)$$

that will be used to determine the solution of the generalized Abel's equation and also of the standard Abel's problem as well.

The following example shows how one can use (9.2.1.12) in solving Abel's equations.

Example 2. Solve the following Abel's problem

$$\pi x = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt.$$

In this example $f(x) = \pi x$, hence $f(0) = 0$ and $f'(x) = \pi$. Also, $\alpha = \frac{1}{2}$ so that $\sin(\alpha\pi) = 1$. Using the formula (9.2.1.12) and formula from Appendix A we obtain

$$u(x) = \frac{1}{\pi}\int_0^x \frac{\pi}{\sqrt{x-t}}dt,$$

$$= 2\sqrt{x}.$$

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APPENDIX A

Integrals of Irrational Functions

Integrals Involving function $\frac{t^n}{\sqrt{x-t}}$, $n = 0,1,2,3,\ldots$

1. $\int_0^x \frac{1}{\sqrt{x-t}}\,dt = 2\sqrt{x}.$
2. $\int_0^x \frac{t}{\sqrt{x-t}}\,dt = \frac{4}{3}x^{3/2}.$
3. $\int_0^x \frac{t^2}{\sqrt{x-t}}\,dt = \frac{16}{15}x^{5/2}.$
4. $\int_0^x \frac{t^3}{\sqrt{x-t}}\,dt = \frac{32}{35}x^{7/2}.$
5. $\int_0^x \frac{t^4}{\sqrt{x-t}}\,dt = \frac{256}{315}x^{9/2}.$
6. $\int_0^x \frac{t^5}{\sqrt{x-t}}\,dt = \frac{512}{693}x^{11/2}.$
7. $\int_0^x \frac{t^6}{\sqrt{x-t}}\,dt = \frac{2048}{3003}x^{13/2}.$
8. $\int_0^x \frac{t^7}{\sqrt{x-t}}\,dt = \frac{4096}{6435}x^{15/2}.$
9. $\int_0^x \frac{t^8}{\sqrt{x-t}}\,dt = \frac{65536}{109395}x^{17/2}.$

APPENDIX B

The Gamma Function $\Gamma(x)$

1. $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$
2. $\Gamma(x+1) = x\,\Gamma(x).$
3. $\Gamma(1) = 1, \quad \Gamma(n+1) = n!, \qquad n \; is \; a \; integer.$
4. $\Gamma(x)\Gamma(1-x) = \dfrac{\pi}{\sin(\pi x)}.$
5. $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}.$
6. $\Gamma\left(\dfrac{3}{2}\right) = \dfrac{1}{2}\sqrt{\pi}.$
7. $\Gamma\left(\dfrac{1}{2}\right)\Gamma\left(-\dfrac{1}{2}\right) = -2\pi.$