**California State University – Los Angeles**

**Department of Mathematics**

**Master's Degree Comprehensive Examination**

**Analysis Spring 2022**

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Do at least five of the following seven problems. All problems count equally. If you attempt more than five, the best five will be used.

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only.
(3) Begin each problem on a new page.
(4) Assemble the problems you hand in in numerical order.

**Exams are graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.**

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**SECTION 1 – Do two (2) problems from this section. If you attempt all three, then the best two will be used for your grade.**

**Spring 2022 #1.** Are the following sets open, closed or neither. Justify your answer.

(a) $\mathbb{Q}$.

(b) $\{x \in \mathbb{R} \setminus \mathbb{Q} : x^3 < 27\}$.

(c) $\{1 + (-1)^n/n : n \in \mathbb{Z}^+\} \bigcup \{1\}$

(d) $\bigcup_{n=1}^{\infty} (1 - 1/n, 1 + 1/n)$.

**Spring 2022 #2.** Let $D$ be a nonempty set. Suppose that $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are functions such that $f(x) > 0$ and $g(x) > 0$ for all $x \in D$. Compare

$$\sup_{x\in D}[f(x)\cdot g(x)] \quad \text{and} \quad \left(\sup_{x\in D} f(x)\right)\cdot\left(\sup_{x\in D} g(x)\right),$$

and justify your answer.

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**Spring 2022 #3.** Let $D$ be an interval of $\mathbb{R}$, $f$ a continuous function from $D$ to $\mathbb{R}$, and $a \in D$ an interior point of $D$. Prove the following two statements are equivalent:

(a) $f$ is continuous at $a$.

(b) For any sequence $\{x_n\}$ in $D$ such that $x_n \to a$, we have $f(x_n) \to f(a)$.

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**SECTION 2 – Do three (3) problems from this section. If you attempt more than three, then the best three will be used for your grade.**

**Spring 2022 #4.**

(a) Suppose $\langle\cdot,\cdot\rangle$ is an inner product on a vector space $\mathcal{V}$ over $\mathbb{C}$, and $\|\cdot\|$ is the associated norm. Show that

$$\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2,$$

for all $v$ and $w$ in $\mathcal{V}$.

(b) Show that there is no possible inner product on $\mathbb{R}^2$ for which the norm $\|(x,y)\| = |x| + |y|$ is the associated norm.

**Spring 2022 #5.** For each continuous function $f$ on the interval $[0,2]$ define a function $Tf$ by

$$(Tf)(x) = x + \lambda \int_0^x xtf(t)\,dt. \tag{1}$$

(a) Suppose $\mathcal{X}$ is a metric space with metric $d$. A mapping $T : \mathcal{X} \to \mathcal{X}$ is said to be a *proper contraction* on $\mathcal{X}$ if there is a real constant $c$ with $0 \leq c < 1$ and $d(T(x), T(y)) \leq c \cdot d(x, y)$ for all $x, y \in \mathcal{X}$.

Find a range of values for the parameter $\lambda$ for which the transformation $T$ defined in (1) is a proper contraction on $C([0, 2])$ with respect to the supremum norm. Justify your answer.

(b) Describe the iterative process for solving the integral equation

$$f(x) = x + \lambda \int_0^x xtf(t)\,dt$$

specifying the transformation to be iterated and explaining how this leads to a solution. With $f_0(x) = 0$ for all $x$ as the starting function, compute the first two iterates, $f_1(x)$ and $f_2(x)$.

(c) Show that if $f$ is a solution to the integral equation of Part (b), then it is also a solution to the differential equation

$$f''(x) - \lambda x^2 f'(x) - 3\lambda x f(x) = 0$$

with $f(0) = 0, f'(0) = 1$.

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**Spring 2022 #6.** Let $\mathcal{X}$ be the space $C([0, 2\pi]; \mathbb{R})$ of all continuous real valued functions on the interval $[0, 2\pi]$ equipped with the supremum norm, $\|f\|_\infty = \sup\{|f(t)| : t \in [0, 2\pi]\}$. Let $\mathcal{Y}$ be the subspace $C^1([0, 2\pi]; \mathbb{R})$ consisting of those $f$ in $\mathcal{X}$ which are differentiable with $f'$ also continuous. Let $T : \mathcal{Y} \to \mathcal{X}$ be the linear transformation given by $Tf = f'$.

(a) Show that $T : (\mathcal{Y}, \|\cdot\|_\infty) \to (\mathcal{X}, \|\cdot\|_\infty)$ is not continuous. (Hint: the functions $f_k(x) = \sin(kx), k = 1, 2, \ldots$ might be useful.)

(b) Show that the formula $[[f]] = \|f\|_\infty + \|f'\|_\infty$ gives a norm on $\mathcal{Y}$.

(c) Show that $T : (\mathcal{Y}, [[\cdot]]) \to (\mathcal{X}, \|\cdot\|_\infty)$ is continuous.

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**Spring 2022 #7.** Let $f(t) = |t|$ for $t \in [-\pi, \pi]$, and extend it to be $2\pi$-periodic on $\mathbb{R}$.

(a) Compute the Fourier series for $f$. (Either exponential or trigonometric form, your choice).

(b) Use the result of Part (a) to show that

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{\pi^4}{96}.$$