Algebra Comprehensive Exam Spring 2020

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Answer five (5) questions only. You must answer *at least one* from each of groups, rings, and fields. Indicate CLEARLY which problems you want us to grade; otherwise, we will select the first problem from each section, and then the first two additional problems answered after that. Be sure to show enough work that your answers are adequately supported. Tip: When a question has multiple parts, the later parts often (but not always) make use of the earlier parts.

Notation: Unless otherwise stated, $\mathbb{Q}, \mathbb{Z}, \mathbb{Z}_n, \mathbb{C}$, and $\mathbb{R}$ denote the sets of rational numbers, integers, integers modulo n, complex numbers, and real numbers respectively, regarded as groups or rings in the usual way.

Groups

(1) Let G be a group and p be a prime number.

(a) Prove or disprove: If the order of G is p^2, then G is abelian.
(b) Prove or disprove: If the order of G is p^3, then G is abelian.

(2) Prove that every group of order 35 is cyclic.

(3) Let A_5 be the alternating group on 5 letters. Show that A_5 is a simple group. Hint: The group A_5 has exactly four distinct conjugacy classes, which contain exactly 1, 15, 20, and 24 elements, respectively—you may use this fact without proving it.

Rings

(1) Let $f\colon R \to S$ be a ring homomorphisim. Let I and J be ideals in rings R and S, respectively.

(a) Prove that the inverse image $f^{-1}(J)$ is an ideal in R that contains the kernel of f.
(b) Prove that if f is surjective, then $f(I)$ is an ideal in S.
(c) Provide an example that shows if f is NOT surjective, then $f(I)$ need not be an ideal in S.

(2) Let R be the set of all rational numbers of the form $a/2^k$ where a is an integer and k is a nonnegative integer.

(a) Prove that R is a commutative ring with unity, under usual addition and multiplication.
(b) Prove that R is not a field.
(c) Is $\mathbb{Z}$ an ideal of R? Prove that your answer is correct.

(3) Let R be a commutative ring with identity 1 with $1 \neq 0$. Let I be an ideal of R with $I \neq R$. Prove that R/I is an integral domain if and only if I is a prime ideal of R.

Fields

(1) Let G be the Galois group of $x^4 - 2$ over $\mathbb{Q}$.

(a) List the elements of G.

(b) Prove that G is not abelian.

(2) Let $\mathbb{Z}_5[x]$ be the ring of polynomials with coefficients in $\mathbb{Z}_5$ with indeterminate x.

(a) Prove that $x^2 - 3$ is irreducible in $\mathbb{Z}_5[x]$.

(b) Let $I = \langle x^2 - 3 \rangle$ be the principal ideal generated by $x^2 - 3$ in $\mathbb{Z}_5[x]$. Let $K = \mathbb{Z}_5[x]/I$. Prove that K is a field containing exactly 25 elements.

(c) Let $R = \mathbb{Z}[\sqrt{3}] = \{a + b\sqrt{3} \mid a, b \in \mathbb{Z}\}$. You may assume without proof that R is a commutative ring with identity under usual addition and multiplication. Let K be as in (b). Define $f\colon R \to K$ by $f(a + b\sqrt{3}) = a + bx + I$. Prove that f is a surjective ring homomorphism.

(d) Let $J = \langle 5 \rangle$ be the principal ideal generated by 5 in R. Prove that J is a maximal ideal of R. Hint: Notice that $J = \ker(f)$.

(3) Prove or disprove: Any finite integral domain must be a field.