Algebra Comprehensive Exam -- Fall 2007

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Answer five (5) questions only! You must answer *at least one* from each of Groups, Rings, and Fields. Show enough work to adequately support your answers.

Groups

1. Let G be a group of order 147. Prove that G contains a nontrivial normal abelian subgroup.

2. Prove that $\mathrm{Aut}(\mathbf{Z}_n) \cong \mathbf{U}_n$. [$\mathrm{Aut}(G) = \{ \varphi: G \to G \mid \varphi \text{ is an isomorphism}\}$]
   [$\mathbf{U}_n = \{ k \in \mathbf{Z}_n \mid \mathrm{GCD}(k, n) = 1 \}$ is the group of units in the ring $\mathbf{Z}_n$. Also known as $\mathbf{Z}_n^{\times}$.]

3. Let p be a prime and assume G is a finite p-group.
   a) Show that the center of G is non-trivial (i.e. $Z(G) \neq \{e\}$).
   b) Let K be a normal subgroup of G of order p. Show that $K \subseteq Z(G)$.

Rings

1. Let R be a finite commutative ring with more that one element and with no zero divisors. Prove that R is a field

2. Let R be a commutative ring with identity. Define $\rho(I) = \{x \in R \mid x^n \in I \text{, for some } n \geq 1\}$.
   Prove the following: a) If I is an ideal of R, then $\rho(I)$ is an ideal of R.
   b) If $I \subseteq J$ are ideals, then $\rho(I) \subseteq \rho(J)$.
   c) $\rho(\rho(I)) = \rho(I)$.
   d) If I and J are ideals of R, then $\rho(I \cap J) = \rho(I) \cap \rho(J)$.

3. Let F be a field. Prove that every ideal of F[X] is principal.

Fields

1 Let $\mathbf{Q}$ be the rational numbers and let E be the splitting field of $p(X) = X^4 - 2$ over $\mathbf{Q}$.
   a) Find $[E : \mathbf{Q}]$ and explain your answer.
   b) Show that $\mathrm{Gal}(E/\mathbf{Q})$ is not abelian.

2. Let E be an extension field of F with $[E : F] = 7$.
   a) Show that $F(\alpha) = F(\alpha^3)$, for all $\alpha \in E$.
   b) Show that $F(\alpha) = F(\alpha^9)$, for all $\alpha \in E$.

3. Let G be a finite group. Prove that there exists a Galois field extension K/L whose Galois group is isomorphic to G.