Algebra Comprehensive Exam Spring 2021, new style
Solutions

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Answer five (5) questions only. You must answer *at least one* from each of section: (I) Linear algebra, (II) Group theory, and (III) Synthesis: linear algebra and group theory. Indicate CLEARLY which problems you want us to grade; otherwise, we will select the first problem from each section, and then the first two additional problems answered after that. Be sure to show enough work that your answers are adequately supported. Tip: When a question has multiple parts, the later parts often (but not always) make use of the earlier parts.

Notation: Unless otherwise stated, $\mathbb{Q}, \mathbb{Z}, \mathbb{Z}_n, \mathbb{C}$, and $\mathbb{R}$ denote the sets of rational numbers, integers, integers modulo n, complex numbers, and real numbers respectively, regarded as groups or fields or vector spaces in the usual way.

Linear algebra

(1) Let V be a vector space over a field F. Let $S = \{x, y, w\}$ be a set of linearly independent vectors of V. Let $z \in V$. Prove that $S \cup \{z\}$ is linearly dependent if and only if z is in the span of the vectors from S.

Solution:
https://textbooks.math.gatech.edu/ila/linear-independence.html
See "Criteria for linear independence."

(2) Let A be a square matrix, and let A^T denote the transpose of A. Prove that λ is an eigenvalue for A if and only if λ is an eigenvalue for A^T.
https://math.stackexchange.com/questions/1384950/prove-that-lambda-is-an-eigenvalue-of-a-if-and-only-if-la
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(3) Let U be a subspace of a finite dimensional vector space V over a field F. Show that there is a linear transformation $\phi : V \to V$ such that $U = \ker \phi$.

Solution: Let $\{v_1, \ldots, v_k\}$ be a basis for U. Extend this to a basis $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ of V. Define $\phi\colon V \to V$ by

$$\phi(a_1v_1 + \cdots a_kv_k + a_{k+1}v_{k+1} + \cdots + a_nv_n) = a_{k+1}v_{k+1} + \cdots + a_nv_n.$$

Then ϕ is linear, and

$$\ker \phi = \{a_1v_1 + \cdots a_kv_k + a_{k+1}v_{k+1} + \cdots + a_nv_n \mid a_{k+1} = \cdots = a_n = 0\} = U.$$

Groups

(1) Let G be a group. Let H and K be normal subgroups of G. Prove that (i) $H \cap K$ is a subgroup of G, and (ii) $H \cap K$ is normal in G.

Solution:
https://martin-thoma.com/intersection-two-normal-subgroups-normal-subgroup/

(2) Let G be a group. Let $a, b \in G$. Let

$$D_{2n} = \{1, r, r^2, \dots, r^{n-1}, s, sr, sr^2, \dots, sr^{n-1}\}$$

be the dihedral group with $2n$ elements, where 1 denotes the identity element, and $s^2 = r^n = 1$, and $rs = sr^{n-1}$. Prove that there exists a homomorphism $\phi\colon D_{2n} \to G$ such that $\phi(r) = a$ and $\phi(s) = b$ if and only if $|a|$ divides n and $|b| \le 2$ and $ab = ba^{-1}$. Here $|x|$ denotes the order of the element x.

Solution:
First, suppose such a homomorphism exists. We will show that $|a|$ divides n and $|b| \le 2$ and $ab = ba^{-1}$.
Because $\phi(r) = a$, we have that $e = \phi(1) = \phi(r^n) = \phi(r)^n = a^n$, from which it follows that $|a|$ divides n. Here e denotes the identity element of G, and we make use of the fact that a homomorphism maps the identity element to the identity element.
Simlarly, because $\phi(s) = b$, we have that $1 = \phi(s^2) = \phi(s)^2 = b^2$, from which it follows that $|b|$ divides 2, so $|b| = 1$ or $|b| = 2$.
Finally, $ab = \phi(r)\phi(s) = \phi(rs) = \phi(sr^{-1}) = \phi(s)\phi(r)^{-1} = ba^{-1}$.

Now we prove the converse. That is, suppose that $|a|$ divides n and $|b| \le 2$ and $ab = ba^{-1}$. We will show that there exists a homomorphism $\phi\colon D_{2n} \to G$ such that $\phi(r) = a$ and $\phi(s) = b$.
Define $\phi\colon D_{2n} \to G$ by $\phi(s^i r^j) = b^i a^j$ for all $i, j \in \mathbb{Z}$.
Because the same input can be represented in more than one way, we must show that ϕ is well-defined.
Suppose $s^i r^j = s^k r^m$ for some integers i, j, k, m. We will show that $b^i a^j = b^k a^m$.
From $s^i r^j = s^k r^m$ we get that $i \equiv k \pmod 2$ and $j \equiv m \pmod n$.
Because $|b| \le 2$, we know that $|b| = 1$ or $|b| = 2$, so in either case, $b^2 = e$. Hence $b^{i-k} = e$.
Because $|a|$ divides n, we have that $a^n = e$, so $a^{j-m} = e$.
Multiplying these two equations, we get $b^{i-k}a^{j-m} = e$. Multiply by b^k on the left and a^m on the right to get $b^i a^j = b^k a^m$.
Therefore ϕ is well-defined.
Next, we will show that ϕ is a homomorphism.
Let $s^i r^j, s^k r^m \in D_{2n}$. Then

$$\phi(s^i r^j \cdot s^k r^m) = \phi(s^{i+k} \cdot r^{m-j}) = b^{i+k} \cdot a^{m-j} = b^i a^j \cdot b^k a^m = \phi(s^i r^j)\phi(s^k r^m).$$

Here we get $r^j \cdot s^k = s^k r^{-j}$ by repeatedly applying the relation $rs = sr^{n-1}$. Same goes for a and b in lieu of r and s.
Finally, $\phi(r) = \phi(s^0 r^1) = b^0 a^1 = a$ and $\phi(s) = \phi(s^1 r^0) = b^1 a^0 = b$.

(3) Let A_4 be the alternating group on 4 letters. Let $K = \{\iota, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$. Here ι denotes the identity element of A_4.

(i) Prove that K is a subset of A_4.
(ii) Prove that K is a subgroup of A_4.

(iii) You may assume without proof that K is a normal subgroup of A_4. (It is.) Find a familiar group isomorphic to the quotient group A_4/K, and prove that your answer is correct.

Solutions

(i) All four elements of K are even permutations.

(ii) We are given that $\iota \in K$.
Let $a = (1\ 2)(3\ 4), b = (1\ 3)(2\ 4), c = (1\ 4)(2\ 3)$.
Then $a^2 = b^2 = c^2 = \iota$. Also, $ab = ba = c, ac = ca = b, bc = cb = a$. So K is closed under the group operation.
Finally, every element in K is its own inverse, so K is closed under inverses.

(iii) We know that $|A_4| = 4!/2 = 24/2 = 12$. So $|A_4/K| = |A_4|/|K| = 12/4 = 3$. Because 3 is prime, by a corollary to Lagrange's theorem, we have that A_4/K is cyclic of order 3, hence isomorphic to the group of integers mod 3 under addition.

Synthesis: Linear algebra and group theory

(1) Let G be the group of all 2×2 invertible matrices with entries from the real numbers. Here the group operation is multiplication of matrices. Let

$$H = \{A \in G \mid \det(A) = 1\}.$$

Prove that H is a subgroup of G.

We have that $H \subset G$ by def. of H.
We have $I \in H$ because $\det(I) = 1$. Here I denote the 2×2 identity matrix.
Suppose $A, B \in H$. Then $\det(AB) = \det(A)\det(B) = 1^2 = 1$, so $AB \in H$.
Suppose $A \in H$. Then $\det(A^{-1}) = \det(A)^{-1} = 1$, so $A^{-1} \in H$.

(2) Let V be a vector space of dimension 2 over $\mathbb{R}$. Let $GL(V)$ be the set of all bijective linear transformations from V to V. Then $GL(V)$ is a group; you do not need to prove that. We say that a function $f\colon V \to V$ is a *dilation* if there is a nonzero real number a such that $f(v) = av$ for all $v \in V$. Prove that the center of $GL(V)$ equals the set of dilations.
Hint: Let ϕ be an element of Z, where Z is the center of $GL(V)$. Write $\phi(e_1) = ae_1 + be_2$ and $\phi(e_2) = ce_1 + de_2$, where $\{e_1, e_2\}$ is a basis for V. Now consider the following two elements of $GL(V)$, namely, $f\colon e_1 \mapsto e_1 + e_2, e_2 \mapsto e_2$ and $g\colon e_1 \mapsto -e_1, e_2 \mapsto e_2$.

Solution:
Let D be the set of dilations. We will show that $D = Z$.
First we will show that $D \subset Z$.
Let f be a dilation. Let $g \in GL(V)$. We will show that $f \circ g = g \circ f$.
By def. of dilation, there exists $\alpha \in \mathbb{R}$ such that $f(v) = \alpha v$ for all $v \in V$.
Then $(f \circ g)(v) = \alpha g(v) = g(\alpha v) = (g \circ f)(v)$ for all $v \in V$, so $f \circ g = g \circ f$.
Hence $D \subset Z$. Now we will show that $Z \subset D$. We pick up where the hint left off.
By def. of center, we know that $f \circ \phi = \phi \circ f$.
So $f \circ \phi(e_1) = \phi \circ f(e_1)$.
So $f(ae_1 + be_2) = \phi(e_1 + e_2) = \phi(e_1) + \phi(e_2)$.

So $ae_1 + (a+b)e_2 = (a+c)e_1 + (b+d)e_2$.
From this we get that $a = a + c$ and $a + b = b + d$, so $c = 0$ and $a = d$.
Now we use that $g \circ \phi = \phi \circ g$.
So $g \circ \phi(e_1) = \phi \circ g(e_1)$.
So $g(ae_1 + be_2) = \phi(-e_1) = -\phi(e_1)$.
So $-ae_1 + be_2 = -ae_1 - be_2$, which implies that $b = -b$, whence $b = 0$.
To recap, we have $a = d$ and $b = c = 0$.
So $\phi(e_1) = ae_1$ and $\phi(e_2) = ae_2$. It follows that $\phi(v) = av$ for all $v \in V$, so $\phi \in D$.

(3) Let

$$V = \{a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{12}x_1x_2 + a_{23}x_2x_3 + a_{13}x_1x_3 \mid a_{11}, a_{12}, \ldots, a_{33} \in \mathbb{R}\}$$

be the 6-dimensional vector space of all quadratic homogeneous polynomials in variables x_1, x_2 and x_3 with coefficients in $\mathbb{R}$. (The operations of addition and scalar multiplication in this vector space are the usual addition and scalar multiplication for polynomials.) Let S_3 act on V by permuting the variables. For example, with $(1\,2\,3) \in S_3$ and $3x_1^2 - 5x_2x_3 \in V$ we get $(1\,2\,3) \cdot (3x_1^2 - 5x_2x_3) = 3x_2^2 - 5x_3x_1$. Find a basis for the subspace of all polynomials that are fixed by all elements of S_3.

Solution: Suppose $f = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{12}x_1x_2 + a_{23}x_2x_3 + a_{13}x_1x_3$. Then f is fixed by $(1\,2)$ if and only if $a_{11} = a_{22}$ and $a_{13} = a_{23}$. It is fixed by $(1\,3)$ if and only if $a_{11} = a_{33}$ and $a_{12} = a_{23}$, and it is fixed by $(2\,3)$ if and only if $a_{22} = a_{33}$ and $a_{12} = a_{13}$. Then f is fixed by all three transpositions if and only if it has the form

$$f = a_{11}(x_1^2 + x_2^2 + x_3^2) + a_{12}(x_1x_2 + x_2x_3 + x_3x_1)$$

But, if f has this form, then it is also fixed by $(1\,2\,3)$ and $(3\,2\,1)$. So $\{x_1^2 + x_2^2 + x_3^2, x_1x_2 + x_2x_3 + x_3x_1\}$ is a basis for the subspace of all polynomials that are fixed by all elements of S_3.