Algebra Comprehensive Exam

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Directions: Answer 5 questions only. You must *answer at least one* from each of linear algebra, groups, and synthesis. Indicate clearly which problems you want us to grade. You are graded on your logic, reasoning, and understanding.

Linear Algebra

(L1) Let V be a vector space with basis $\{v_1, \ldots, v_n\}$. Let $T\colon V \to V$ be a linear map, and assume T is one-to-one. Prove that $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for V.

(L2) Let V be a vector space and $L\colon V \to V$ be a linear transformation. Let λ be an eigenvalue of L. Prove that

$$E_\lambda = \{v \in V \mid L(v) = \lambda v\}$$

is a subspace of V.

(L3) Let $T\colon V \to W$ be a linear transformation from an n-dimensional space V to an m-dimensional space W. Show that $\dim(\ker(T)) + \dim(\operatorname{im}(T)) = n$.

Groups

(G1) Let S_3 denote the symmetric group, i.e. the group of all permutations of the set $\{1, 2, 3\}$. Define $H = \{1, (12)\}$.

(a) How many left cosets does H have?

(b) List the elements of each left coset of H.

(c) Is H a normal subgroup of G?

(G2) Let $\phi\colon G_1 \to G_2$ be a group homomorphism where G_1 and G_2 are groups. Prove that if G_1 is cyclic and ϕ is onto then G_2 is cyclic.

(G3) Let G be a finite group, and let H be a nonempty subset of G. Prove that H is a subgroup of G if and only if H is closed under the operation of G.

Synthesis

(S1) Let $H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \neq 0 \right\}$. Prove that H is a normal subgroup of $GL_2(\mathbb{R})$.

(S2) Let $\mathbb{R}^*$ denote the group of non-zero real numbers under multiplication. Define a function $\phi\colon GL_n(\mathbb{R}) \to \mathbb{R}^*$ by

$$\phi(A) = \begin{cases} 1 & \text{if } \det(A) > 0; \\ -1 & \text{if } \det(A) < 0. \end{cases}$$

Prove that ϕ is a group homomorphism, and find the order of $GL_n(\mathbb{R})/\ker(\phi)$.

(S3) Let V be a vector space over $\mathbb{R}$. We know that V is also an Abelian group under addition (you do not need to prove this fact). Prove that, for every $v \in V$, the order of v (as an element of the group under addition) is either 1 or ∞.