Algebra Comprehensive Exam

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Directions: Answer 5 questions only. You must *answer at least one* from each of linear algebra, groups, and synthesis. Indicate clearly which problems you want us to grade. You are graded on your logic, reasoning, and understanding.

Linear Algebra

(L1) Let V be a vector space with basis $\{v_1, \ldots, v_n\}$. Let $T\colon V \to V$ be a linear map, and assume T is one-to-one. Prove that $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for V.

`Solution:` Since V has dimension n, we just need to show that $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a linearly independent set. Suppose that $c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n) = 0$. Since T is linear we have that $T(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = 0$. Since T is one-to-one and $T(0) = 0$ we have that $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$. Since $\{v_1, \ldots, v_n\}$ is a basis for V this implies that $c_1 = c_2 = \cdots = c_n = 0$. Thus $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a linearly independent set.

(L2) Let V be a vector space and $L\colon V \to V$ be a linear transformation. Let λ be an eigenvalue of L. Prove that

$$E_\lambda = \{v \in V \mid L(v) = \lambda v\}$$

is a subspace of V.

`Solution:` (a) Since L is a linear transformation we have that $L(0) = 0 = \lambda \cdot 0$. Thus $0 \in E_\lambda(L)$. (b) Let $v_1, v_2 \in E_\lambda(L)$. Then $L(v_1 + v_2) = L(v_1) + L(v_2) = \lambda v_1 + \lambda v_2 = \lambda(v_1 + v_2)$. Thus $v_1 + v_2 \in E_\lambda(L)$. (c) Let $w \in E_\lambda(L)$ and let c be a scalar. Then $L(w) = \lambda w$. Thus $L(cw) = cL(w) = c(\lambda w) = \lambda(cw)$. Thus $cw \in E_\lambda(L)$. By parts (a,b,c) we have that $E_\lambda(L)$ is a subspace of V.

(L3) Let $T\colon V \to W$ be a linear transformation from an n-dimensional space V to an m-dimensional space W. Show that $\dim(\ker(T)) + \dim(\operatorname{im}(T)) = n$.

`Solution:` There are many different kinds of proof, depending on our perspective and our starting assumptions. The proof we present here is based on the proof of Theorem 2.3 (Dimension Theorem) in Friedberg, Insel, and Spence, *Linear Algebra.*

Let $k = \dim(\ker(T))$, and let $\{v_1, \ldots, v_k\}$ be a basis of $\ker(T)$. Since $\ker(T)$ is a subspace of V, we can extend the basis $\{v_1, \ldots, v_k\}$ of $\ker(T)$ to a basis $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ of V. Now let $S = \{T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\}$. We will prove that S is a basis of $\operatorname{im}(T)$.

First we show that $\operatorname{span} S = \operatorname{im}(T)$. Since $\operatorname{span}\{v_1, \ldots, v_n\} = V$, we have

$$\operatorname{im}(T) = \operatorname{span}\{T(v_1), \ldots, T(v_n)\}.$$

But $T(v_1) = \cdots = T(v_k) = 0$ (since $v_1, \ldots, v_k \in \ker(T)$), so

$$\begin{aligned} \operatorname{im}(T) &= \operatorname{span}\{0, \ldots, 0, T(v_{k+1}), \ldots, T(v_n)\} \\ &= \operatorname{span}\{T(v_{k+1}), \ldots, T(v_n)\} = \operatorname{span} S. \end{aligned}$$

Now we show that S is independent. Let $b_{k+1}, \dots, b_n$ be scalars such that

$$b_{k+1}T(v_{k+1}) + \cdots + b_nT(v_n) = 0.$$

Then, since T is linear, we get

$$T(b_{k+1}v_{k+1} + \cdots + b_nv_n) = 0,$$

and so $b_{k+1}v_{k+1} + \cdots + b_nv_n \in \ker(T)$. Since $\{v_1, \dots, v_k\}$ is a basis of $\ker(T)$, there exist scalars $c_1, \dots, c_k$ such that

$$b_{k+1}v_{k+1} + \cdots + b_nv_n = c_1v_1 + \cdots + c_kv_k.$$

We can rearrange this equation to obtain

$$-c_1v_1 - \cdots - c_kv_k + b_{k+1}v_{k+1} + \cdots + b_nv_n = 0.$$

But $\{v_1, \dots, v_n\}$ is a basis of V, so we conclude that $b_{k+1} = \cdots = b_n = 0$.

Therefore, S is a basis of $\mathrm{im}(T)$. Since S is a set of $n - k$ vectors, we have $\dim(\mathrm{im}(T)) = n - k$. Therefore,

$$\dim(\ker(T)) + \dim(\mathrm{im}(T)) = k + (n - k) = n.$$

Groups

(G1) Let S_3 denote the symmetric group, i.e. the group of all permutations of the set $\{1, 2, 3\}$. Define $H = \{1, (12)\}$.
- (a) How many left cosets does H have?
- (b) List the elements of each left coset of H.
- (c) Is H a normal subgroup of G?

`Solution:`
- (a) 3, since $|S_3|/|H| = 6/2 = 3$.
- (b) $\{1, (12)\}, \{(13), (123)\}, \{(23), (132)\}$. If anyone accidentally does the right cosets instead of the left cosets, then those are: $\{1, (12)\}, \{(13), (132)\}, \{(23), (123)\}$.
- (c) No, because, for instance, $(13)H = \{(13), (123)\}$ but $H(13) = \{(13), (132)\}$, so $(13)H \neq H(13)$.

(G2) Let $\phi\colon G_1 \to G_2$ be a group homomorphism where G_1 and G_2 are groups. Prove that if G_1 is cyclic and ϕ is onto then G_2 is cyclic.

`Solution:` Let $G_1 = \langle g \rangle$. The claim is that $\phi(g)$ will generate G_2. Let $y \in G_2$. Then since ϕ is onto there exists $x \in G_1$ with $\phi(x) = y$. Since $G_1 = \langle g \rangle$ we have that $x = g^k$ for some integer k. Thus $y = \phi(x) = \phi(g^k) = \phi(g)^k$. Thus all of $G_2 = \langle \phi(g) \rangle$.

(G3) Let G be a finite group, and let H be a nonempty subset of G. Prove that H is a subgroup of G if and only if H is closed under the operation of G.

`Solution:` If H is a subgroup of G, then H is closed under the operation of G, by definition of subgroup.

Now assume H is closed under the operation of G. In addition to that, we must prove that the identity of G is in H and that every element of H has its inverse in H, in order to conclude that H is a subgroup of G.

Since H is nonempty, there exists some $x \in H$. Since H is closed under the operation of G, it follows that $x^i \in H$ for every positive integer i. But G is a finite group, so x must have finite order in G: there exists a positive integer n such that $x^n = 1$ (the identity of G). Therefore, $1 = x^n \in H$ — the identity of G is in H.

Now let h be an arbitrary element of H. Just as above, we have $h^i \in H$ for every positive integer i, and there exists $m \geq 2$ such that $h^m = 1$. Then $h^{-1} = h^{m-1}$, which is in H since $m - 1$ is a positive integer.

Synthesis

(S1) Let $H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \neq 0 \right\}$. Prove that H is a normal subgroup of $GL_2(\mathbb{R})$.

`Solution:` We first prove that H is a subgroup. (a) The identity $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is in H. (b) If $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right), \left(\begin{smallmatrix} b & 0 \\ 0 & b \end{smallmatrix}\right) \in H$, then $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)\left(\begin{smallmatrix} b & 0 \\ 0 & b \end{smallmatrix}\right) = \left(\begin{smallmatrix} ab & 0 \\ 0 & ab \end{smallmatrix}\right) \in H$. (c) If $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \in H$, then, since $a \neq 0$, we have $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 1/a & 0 \\ 0 & 1/a \end{smallmatrix}\right) \in H$.

We now prove that H is normal. If $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \in H$ and $A \in GL_2(\mathbb{R})$, then

$$A \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) A^{-1} = A(aA^{-1}) = a(AA^{-1}) = aI = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \in H.$$

Therefore H is a normal subgroup of $GL_2(\mathbb{R})$.

(S2) Let $\mathbb{R}^*$ denote the group of non-zero real numbers under multiplication. Define a function $\phi\colon GL_n(\mathbb{R}) \to \mathbb{R}^*$ by

$$\phi(A) = \begin{cases} 1 & \text{if } \det(A) > 0; \\ -1 & \text{if } \det(A) < 0. \end{cases}$$

Prove that ϕ is a group homomorphism, and find the order of $GL_n(\mathbb{R})/\ker(\phi)$.

`Solution:` Let $A, B \in GL_n(\mathbb{R})$.

- Case I. $\det(A) > 0$ and $\det(B) > 0$. Then $\det(AB) = \det(A)\det(B) > 0$, so $\phi(AB) = 1$, and $\phi(A)\phi(B) = (1)(1) = 1$.
- Case II. $\det(A) > 0$ and $\det(B) < 0$. Then $\det(AB) = \det(A)\det(B) < 0$, so $\phi(AB) = -1$, and $\phi(A)\phi(B) = (1)(-1) = -1$.
- Case II. $\det(A) < 0$ and $\det(B) > 0$. Then $\det(AB) = \det(A)\det(B) < 0$, so $\phi(AB) = -1$, and $\phi(A)\phi(B) = (-1)(1) = -1$.
- Case I. $\det(A) < 0$ and $\det(B) < 0$. Then $\det(AB) = \det(A)\det(B) > 0$, so $\phi(AB) = 1$, and $\phi(A)\phi(B) = (-1)(-1) = 1$.

Thus we have verified that $\phi(AB) = \phi(A)\phi(B)$ in all four cases.

By the First Isomorphism Theorem, $|GL_n(\mathbb{R})/\ker(\phi)| = |\operatorname{im}(\phi)| = |\{1, -1\}| = 2$.

(S3) Let V be a vector space over $\mathbb{R}$. We know that V is also an Abelian group under addition (you do not need to prove this fact). Prove that, for every $v \in V$, the order of v (as an element of the group under addition) is either 1 or ∞.

`Solution:` Let $v \in V$. Suppose the order of v is not ∞; we will show that its order is 1. Let k be the order of v, so k is a (finite) positive integer. Then $v + \cdots + v = 0$, where v is added to itself k times. Then

$$0 = \underbrace{v + \cdots + v}_{k} = \underbrace{1v + \cdots + 1v}_{k} = (\underbrace{1 + \cdots + 1}_{k})v = kv.$$

So $kv = 0$. Since k is a non-zero real number, $1/k$ is defined. So

$$\begin{aligned} kv &= 0 \\ (1/k)kv &= (1/k)0 \\ 1v &= 0 \\ v &= 0. \end{aligned}$$

Therefore, since v is the additive identity, its order is 1.