California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Linear Analysis Spring 2013
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Do five of the following eight problems.
If you attempt more than 5, the best 5 will be used.
Please
(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

MISCELLANEOUS FACTS

$\sin(a+b) = \sin a \cos b + \cos a \sin b$ $\quad$ $\cos(a+b) = \cos a \cos b - \sin a \sin b$

$2 \sin a \sin b = \cos(a-b) - \cos(a+b)$ $\quad$ $2 \cos a \cos b = \cos(a-b) + \cos(a+b)$

$2 \sin a \cos b = \sin(a+b) + \sin(a-b)$ $\quad$ $2 \cos a \sin b = \sin(a+b) - \sin(a-b)$

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax) \qquad \int \cos^2(ax)\,dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} \qquad \int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

$$\int x\sin bx\,dx = \frac{1}{b^2}\sin bx - \frac{x}{b}\cos bx \qquad \int x\cos bx\,dx = \frac{1}{b^2}\cos bx + \frac{x}{b}\sin bx$$

Spring 2013 # 1. Let $C^2(\mathbb{R})$ be the set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that the second derivative, f'', exists and is continuous. For f in $C^2(\mathbb{R})$, let $(Lf)(t) = f''(t) + 3f(t)$.

a. Show that $C^2(\mathbb{R})$ is a vector subspace of $C(\mathbb{R})$.

b. Show that L is a linear transformation from $C^2(\mathbb{R})$ into $C(\mathbb{R})$, the space of all continuous real valued functions on $\mathbb{R}$.

c. Let $\mathcal{W} = \{f \in C^2(\mathbb{R}) \mid f''(t) + 3f(t) = 0 \text{ for all } t\}$. Show that $\mathcal{W}$ is a vector subspace of $C^2(\mathbb{R})$.

d. Suppose we change the criterion in part **(c)** to $f''(t) + 3f(t) = \cos t$. Now is $\mathcal{W}$ a vector subspace ? Why or why not ?

Spring 2013 # 2. Let $\mathcal{V}$ be the space of continuous functions on the interval $[-\pi, \pi]$ with the L^2 norm $\| f \|_2 = \left(\int_{-\pi}^{\pi} | f(t) |^2 \, dt \right)^{1/2}$. (Caution: There is no factor of $1/\pi$ here.) For f in $\mathcal{V}$, define

$$\phi_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \, dt$$

Show that ϕ_k is a bounded linear functional on $\mathcal{V}$ and find the norm $\| \phi_k \|$ of ϕ_k as a linear functional on $\mathcal{V}$.

Spring 2013 # 3. Let p_0 and p_1 be the monomials $p_0(x) = 1$ and $p_1(x) = x$, and let $\mathcal{W}$ be the subspace of the space $C([0,2])$ of all continuous function on the interval $[0,2]$ spanned by p_0 and p_1.

a. Find a basis for the subspace $\mathcal{W}$ which is orthonormal with respect to the inner product $\langle f, g \rangle = \displaystyle\int_0^2 f(x)\overline{g(x)} \, dx$

b. Use your results from part **a** to find the function $f(x) = ax + b$ in $\mathcal{W}$ which makes the quantity $J(f) = \displaystyle\int_0^2 \left| x^3 - f(x) \right|^2 \, dx$ as small as possible.

Spring 2013 # 4. Consider the operator defined for f in $L^2([0,1])$ by $(Kf)(x) = \int_0^1 x^2 t^2 f(t) \, dt$. and the integral equation

$$f(x) = g(x) + \lambda \int_0^1 x^2 t^2 f(t) \, dt = g(x) + \lambda (Kf)(x)$$

a. Find any nonzero eigenvalues and the corresponding eigenvectors for K

b. Find a resolvent kernel $R(x, t; \lambda)$ so that solutions to the integral equation are given for each g by

$$f(x) = g(x) + \lambda \int_0^1 R(x, t; \lambda) g(t) \, dt.$$

c. Find a function $f(x)$ for $0 \leq x \leq 1$ such that

$$f(x) = x + \lambda \int_0^1 x^2 t^2 f(t) \, dt.$$

Spring 2013 # 5. For each continuous function f on the interval $[0, 1]$, define a function Tf by

$$(Tf)(x) = x - \lambda \int_0^x (x - t) f(t)\, dt.$$

a. Find a range of values for the parameter λ for which the transformation T is a contraction with respect to the supremum norm. Justify your answer.

(Recall that the supremum norm is defined on $C([0, 1])$ by $\| g \|_\infty = \sup\{| g(x) | \mid x \in [0, 1]\}$.)

b. Describe the Picard iteration process for solving the integral equation

$$f(x) = x - \lambda \int_0^x (x - t) f(t)\, dt$$

specifying the transformation to be iterated and explaining how this leads to a solution. With $f_0(x) = x$ for all x as the starting function, compute the first two iterates, $f_1(x)$ and $f_2(x)$.

Spring 2013 # 6. Let $f : \mathbb{R} \to \mathbb{R}$ be the 2π-periodic function given for $-\pi \leq x < \pi$ by $f(x) = x$.

a. Compute the Fourier series for f. (Either exponential or trigonometric form, your choice).

b. Give a statement of the Parseval identity and compute what it tells you for the example in part **a**.

Spring 2013 # 7. Consider the following formulas for $v = (x, y)$ and $w = (a, b)$ in $\mathbb{R}^2$.

a. Decide for each of the following whether it defines norm on $\mathbb{R}^2$. If "yes", prove it. If "no", show that it is not.

(i) $\| v \|_i = | xy |$ **(ii)** $\| v \|_{ii} = | x | - 2 | y |$

b. Decide for each of the following whether it defines an inner product on $\mathbb{R}^2$. If "yes", prove it. If "no", show that it is not.

(i) $\langle v, w \rangle_i = 3xa + 5yb$ **(ii)** $\langle v, w \rangle_{ii} = xy + ab$

Spring 2013 # 8. In each of the following, T is a function from the vector space $C([0, 1], \mathbb{R})$ of all continuous real valued functions on $[0, 1]$ to $\mathbb{R}$. For each, decide whether T is linear. If it is, prove it. If it is not, show by an example or an explanation that it is not.

a. $T : C([0, 1], \mathbb{R}) \to \mathbb{R}$ given by $T(f) = 5 + \int_0^1 f(t) \cos t\, dt$.

b. $T : C([0, 1], \mathbb{R}) \to \mathbb{R}$ given by $T(f) = f(0) + 7f(1/2)$.

End of Exam