California State University – Los Angeles

Department of Mathematics

Master's Degree Comprehensive Examination

Analysis   Sample Exam

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Do at least two (2) problems from Section 1 below, and at least three (3) problems from Section 2 below. All problems count equally. If you attempt more than two problems from Section 1, the best two will be used. If you attempt more than three problems from Section 2, the best three will be used. Be sure to show your work for all answers.

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only.
(3) Begin each problem on a new page.
(4) Assemble the problems you hand in in numerical order.

Exams are graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

SECTION 1 – Do two (2) problems from this section. If you attempt all three, then the best two will be used for your grade.

Sample #1. Consider the sequence defined by $x_0 = 1$ and

$$x_{n+1} = 1 + \frac{1}{x_n}$$

for all integers $n \geq 0$.

(a) Show that the sequence satisfies

$$1 \leq x_n \leq 2$$

for all non-negative integers n.

Proof. By induction on n. The result clearly holds for $n = 0$. So assume the result holds for $n = k$. Then for $n = k + 1$, we have

$$x_{k+1} = 1 + \frac{1}{x_k}.$$

By assumption, we have that

$$1 \leq x_k \leq 2,$$

so that

$$\frac{1}{2} \leq \frac{1}{x_k} \leq 1.$$

This implies that

$$\frac{3}{2} \leq 1 + \frac{1}{x_k} \leq 2.$$

Since the middle term in this inequality is equal to x_{k+1}, we obtain

$$\frac{3}{2} \leq x_{k+1} \leq 2.$$

The desired result follows. □

(b) Prove that (x_n) has a convergent subsequence (x_{n_k}). Hint: Use your answer from (a).

Proof. By part (a), the sequence is bounded. The desired result then follows from the Bolzano-Weierstrass Theorem. □

Sample #2. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers.

(a) Define what it means for $(x_n)_{n=1}^{\infty}$ to be a "Cauchy sequence."

Solution. A sequence $(x_n)_{n=1}^{\infty}$ is Cauchy if, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \epsilon$$

whenever $n, m \geq N$. □

(b) Use your answer from (a) to prove that if $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence, then $\{x_n \mid n \in \mathbb{N}\}$ is bounded. (Here $\mathbb{N}$ denotes the set of positive integers.)

Proof. Assume $(x_n)_{n=1}^{\infty}$ is Cauchy, fix $\epsilon > 0$, and let N be the natural number corresponding to ϵ. Then for any $n \geq N$, we will have

$$|x_n - x_N| < \epsilon$$

for $n \geq N$. This implies that

$$-\epsilon < x_n - x_N < \epsilon,$$

which we rewrite as

$$x_N - \epsilon < x_n < \epsilon + x_N.$$

Thus x_n is bounded, at least for $n \geq N$.

For $n = 1, \ldots, N-1$, we have

$$\min\{x_1, \ldots, x_{N-1}\} \leq x_n \leq \max\{x_1, \ldots, x_{N-1}\}.$$

Setting $m = \min\{x_1, \ldots, x_{N-1}\}$ and $M = \max\{x_1, \ldots, x_{N-1}\}$, we have

$$\min\{m, x_N - \epsilon\} \leq x_n \leq \max\{M, x_N + \epsilon\}$$

for every $n \in \mathbb{N}$. It follows that the sequence is bounded. $\square$

Sample #3.

Let $f : D \to \mathbb{R}$ be a continuous function on an open interval D. Prove that the function $f_+ : D \to \mathbb{R}$ defined by

$$f_+(x) = \max\{f(x), 0\}$$

is continuous.

SOLUTIONS:

Solution #1:

`https://www.reddit.com/r/learnmath/comments/s1eshq/show_that_if_fg_are_continuous_then_maxfg_is/`

Solution #2:

This is easy to see if we re-write the function f_+ as follows:

$$f_+(x) = \max\{f(x), 0\} = \frac{f(x) + |f(x)|}{2}.$$

SECTION 2 – Do three (3) problems from this section. If you attempt more than three, then the best three will be used for your grade.

Sample #4. Assuming that the integrals below exist, use the definition of Riemann integrals to show that

$$\int_{-a}^{a} f(x^2)\ dx = 2\int_{0}^{a} f(x^2)\ dx.$$

Proof. It suffices to show that

$$\left| \int_{-a}^{a} f(x^2)\ dx - 2\int_{0}^{a} f(x^2)\ dx \right| < \epsilon$$

for every $\epsilon > 0$. Assume both integrals exist. Then there are partitions π_1 of $[-a, a]$ and π_2 of $[0, a]$ such that for any refinement π_1' of π_1 and π_2' of π_2 with associated selections σ_1 and σ_2, respectively, we will have

$$\left| \int_{-a}^{a} f(x^2)\ dx - S(f, \pi_1', \sigma_1) \right| < \frac{\epsilon}{2}$$

and

$$\left| \int_{0}^{a} f(x^2)\ dx - S(f, \pi_2', \sigma_2) \right| < \frac{\epsilon}{4}.$$

In particular, we may choose refinements π_1' and π_2' such that

$$\pi_1' = \{x_0, \ldots, x_{2k+1}, \ldots, x_n\},$$

and

$$\pi_2' = \{x_{2k+1}, \ldots, x_n\},$$

with $x_{2k+1} = 0$ and π_1' and σ_1 are symmetric about 0. With this in mind, we then must have

$$\begin{aligned} \left| \int_{-a}^{a} f(x^2)\ dx - 2\int_{0}^{a} f(x^2)\ dx \right| &= \left| \int_{-a}^{a} f(x^2)\ dx - 2\int_{0}^{a} f(x^2)\ dx \right| \\ &\leq \left| \int_{-a}^{a} f(x^2)\ dx - S(f, \pi_1', \sigma_1) \right| \\ &\quad + \left| S(f, \pi_1', \sigma_1) - 2\int_{0}^{a} f(x^2)\ dx \right| \end{aligned}$$

By construction, it is easy to see that

$$S(f, \pi_1', \sigma_1) = 2S(f, \pi_2', \sigma_2),$$

so that

$$\begin{aligned}\left|\int_{-a}^{a} f(x^2)\,dx - 2\int_0^a f(x^2)\,dx\right| &\le \left|\int_{-a}^{a} f(x^2)\,dx - S(f,\pi_1',\sigma_1)\right| \\ &\quad + \left|2S(f,\pi_2',\sigma_2) - 2\int_0^a f(x^2)\,dx\right| \\ &\le \frac{\epsilon}{2} + 2\cdot\frac{\epsilon}{4} \\ &= \epsilon.\end{aligned}$$

The desired result follows. □

Sample #5. Give an example of a measure space $(X, \mathbb{F}, \mu)$ and a function $f : X \to \mathbb{R}$ such that f is not measurable, but $|f|$ is measurable. Prove your assertion.

Solution. Let $X = \mathbb{R}$, let $\mathbb{F}$ be the collection of Lebesgue measurable subsets of $\mathbb{R}$, and let μ be Lebesgue measure. Let E be a non-measurable subset of $\mathbb{R}$, and define

$$f(x) = \chi_E - \chi_{\mathbb{R}\setminus E},$$

where χ_A denotes the characteristic function of A. If we choose

$$0 < \alpha < 1,$$

then

$$f^{-1}((\alpha,\infty)) = E,$$

so that $f(x)$ is not measurable.

On the other hand, it is easy to see that

$$|f(x)| = 1$$

for all $x \in X$. Since this is a constant function, it is measurable. □

Sample #6. Consider the measure space $(\mathbb{N}, P(\mathbb{N}), \mu)$, where μ is counting measure on $\mathbb{N}$. For each $k \in \mathbb{N}$, let $f_k : \mathbb{N} \to \mathbb{R}$ be the function defined by

$$f_k(n) = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that $f_k \to 0$ pointwise.

Proof. Fix $n \in \mathbb{N}$. Then for any $k > n$, we will have

$$f_k(n) = 0.$$

It follows that $f_k(n) \to 0$. Doing the same thing for every $n \in \mathbb{N}$ implies that

$$f_k \to 0.$$

$\square$

(b) Show that $\displaystyle\int_{\mathbb{N}} f_k \, d\mu \to 1$.

Proof. It is easy to see that we may rewrite f_k as

$$f_k = \chi_{\{k\}}.$$

Using the definition of counting measure and Lebesgue integrals for simple functions, we have

$$\int f_k \, dm = m(\{k\}) = 1.$$

It follows that

$$\lim_{k\to\infty} \int f_k \, dm = 1,$$

as desired. $\square$

(c) Explain why this does not contradict the Dominated Convergence Theorem.

Solution. The Dominated Convergence Theorem requires that the sequence f_k be dominated by some integrable function g. This would require that

$$|f_k(n)| \leq g(n)$$

for all $n, k \in \mathbb{N}$. Thus it must be the case that

$$1 = |f_k(k)| \leq g(k)$$

for all $k \in \mathbb{N}$. But the function $g(n) = 1$ is not integrable on $\mathbb{N}$, and so there is no integrable dominating function. Thus the hypotheses of the DCT are not satisfied for this sequence. $\square$

Sample #7. Let E be an arbitrary countable subset of the interval $[0,1]$, let E^c denote its complement, and let m be Lebesgue measure on $\mathbb{R}$. Define a function f by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap E, \\ x^2 & \text{if } x \in [0,1] \cap E^c. \end{cases}$$

Compute

$$\int_{[0,1]} f \; dm.$$

Justify all steps.

Solution. Since E is a countable subset of $[0,1]$, we must have that $m(E) = 0$. Thus, $f(x) = x^2$ almost everywhere. Thus

$$\begin{aligned} &\text{(because } f = g \text{ a. e.)} && \int_{[0,1]} f \; dm = \int_{[0,1]} g \; dm \\ &\text{(because } g(x) \text{ is continuous)} && \qquad\qquad = \int_0^1 g(x) \; dx \\ &\text{(by basic calculus)} && \qquad\qquad = \frac{1}{3}. \end{aligned}$$

$\square$