California State University – Los Angeles

Department of Mathematics

Master's Degree Comprehensive Examination

Analysis   Sample Exam

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Do at least two (2) problems from Section 1 below, and at least three (3) problems from Section 2 below. All problems count equally. If you attempt more than two problems from Section 1, the best two will be used. If you attempt more than three problems from Section 2, the best three will be used. Be sure to show your work for all answers.

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only.
(3) Begin each problem on a new page.
(4) Assemble the problems you hand in in numerical order.

Exams are graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

SECTION 1 – Do two (2) problems from this section. If you attempt all three, then the best two will be used for your grade.

Sample #1. Consider the sequence defined by $x_0 = 1$ and

$$x_{n+1} = 1 + \frac{1}{x_n}$$

for all integers $n \geq 0$.

(a) Show that the sequence satisfies

$$1 \leq x_n \leq 2$$

for all non-negative integers n.

(b) Prove that (x_n) has a convergent subsequence (x_{n_k}). Hint: Use your answer from (a).

Sample #2. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers.

(a) Define what it means for $(x_n)_{n=1}^{\infty}$ to be a "Cauchy sequence."

(b) Use your answer from (a) to prove that if $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence, then $\{x_n \mid n \in \mathbb{N}\}$ is bounded. (Here $\mathbb{N}$ denotes the set of positive integers.)

Sample #3.

Let $f : D \to \mathbb{R}$ be a continuous function on an open interval D. Prove that the function $f_+ : D \to \mathbb{R}$ defined by

$$f_+(x) = \max\{f(x), 0\}$$

is continuous.

SECTION 2 – Do three (3) problems from this section. If you attempt more than three, then the best three will be used for your grade.

Sample #4. Assuming that the integrals below exist, use the definition of Riemann integrals to show that

$$\int_{-a}^{a} f(x^2)\ dx = 2\int_{0}^{a} f(x^2)\ dx.$$

Sample #5. Give an example of a measure space $(X, \mathbb{F}, \mu)$ and a function $f : X \to \mathbb{R}$ such that f is not measurable, but $|f|$ is measurable. Prove your assertion.

Sample #6. Consider the measure space $(\mathbb{N}, P(\mathbb{N}), \mu)$, where μ is counting measure on $\mathbb{N}$. For each $k \in \mathbb{N}$, let $f_k : \mathbb{N} \to \mathbb{R}$ be the function defined by

$$f_k(n) = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that $f_k \to 0$ pointwise.

(b) Show that $\displaystyle\int_{\mathbb{N}} f_k\ d\mu \to 1$.

(c) Explain why this does not contradict the Dominated Convergence Theorem.

Sample #7. Let E be an arbitrary countable subset of the interval $[0, 1]$, let E^c denote its complement, and let m be Lebesgue measure on $\mathbb{R}$. Define a function f by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap E, \\ x^2 & \text{if } x \in [0,1] \cap E^c. \end{cases}$$

Compute

$$\int_{[0,1]} f \ dm.$$

Justify all steps.