California State University – Los Angeles

Department of Mathematics

Master's Degree Comprehensive Examination

Analysis Sample Exam

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Do at least two (2) problems from Section 1 below, and at least three (3) problems from Section 2 below. All problems count equally. If you attempt more than two problems from Section 1, the best two will be used. If you attempt more than three problems from Section 2, the best three will be used. Be sure to show your work for all answers.

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only.
(3) Begin each problem on a new page.
(4) Assemble the problems you hand in in numerical order.

Exams are graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

SECTION 1 – Do two (2) problems from this section. If you attempt all three, then the best two will be used for your grade.

Sample #1. Let $\mathbb{R}$ denote the set of real numbers, and let $\mathbb{Q}$ denote the set of rational numbers.

Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Prove that for all $a \in \mathbb{R}$, we have that f is not continuous at $x = a$.

Proof. By contradiction. Suppose f is continuous at $x = a$. Then for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon$$

whenever

$$|x - a| < \delta.$$

Choose $\epsilon_0 = 1$. If $a \in \mathbb{Q}$, then there exists $x \in \mathbb{R} \setminus \mathbb{Q}$ such that

$$|x - a| < \delta$$

by the density of the irrational numbers. Thus we have

$$|f(x) - f(a)| = 1 \geq \epsilon_0.$$

It follows that f is not continuous at $x = a$. An analogous argument works in the case that a is irrational. □

Sample #2. Use the definition of limits to show that

$$\lim_{n\to\infty} \frac{1}{(n+1)^2} = 0.$$

Proof. Let $\epsilon > 0$. We must find $N \in \mathbb{N}$ such that

$$\left| \frac{1}{(n+1)^2} - 0 \right| < \epsilon$$

whenever $n \geq N$. Observe that this is equivalent to

$$\frac{1}{(n+1)^2} < \epsilon.$$

To find N, observe that the Archimedean property of the reals tells us that for $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\frac{1}{\epsilon} < N.$$

Next, we observe that for $n \in \mathbb{N}$, we have

$$n < n + 1 < (n+1)^2.$$

Thus, whenever $n \geq N$, we will have

$$\frac{1}{\epsilon} < (n+1)^2,$$

which we may rewrite as

$$\frac{1}{(n+1)^2} < \epsilon.$$

It follows that

$$\frac{1}{(n+1)^2} \to 0.$$

□

Sample #3. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in $\mathbb{R}$. Show that

$$\liminf(x_n + y_n) \geq \liminf x_n + \liminf y_n.$$

Proof. Because $\{x_n\}$ and $\{y_n\}$ are bounded sequences, so is $\{x_n + y_n\}$, so all the lim infs here exist.

Let $L = \liminf(x_n + y_n)$. Let $L_x = \liminf x_n$ and $L_y = \liminf y_n$. We will show that $L \geq L_x + L_y$.

Let $\epsilon > 0$. Then there exists $M_x \in \mathbb{N}$ such that if $n \geq M_x$, then $x_n \geq L_x - \frac{1}{2}\epsilon$. Similarly, there exists $M_y \in \mathbb{N}$ such that if $n \geq M_y$, then $y_n \geq L_y - \frac{1}{2}\epsilon$.

Let $M = \max\{M_x, M_y\}$. It follows that if $n \geq M$, then

$$x_n + y_n \geq (L_x - \frac{1}{2}\epsilon) + (L_y - \frac{1}{2}\epsilon) = L_x + L_y - \epsilon.$$

Because ϵ was arbitrary, it follows that $L \geq L_x + L_y$. □

SECTION 2 – Do three (3) problems from this section. If you attempt more than three, then the best three will be used for your grade.

Sample #4. Show that the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

is *not* Riemann integrable on $[0, 1]$.

Proof. It suffices to show that f is not Darboux integrable on $[0, 1]$. For this, it suffices to find ϵ_0 such that for any partition $\{x_i\}_{i=0}^n$ of $[0, 1]$, the upper and lower Darboux sums satisfy

$$S - s \geq \epsilon_0.$$

Let $\{x_i\}_{i=0}^n$ be an arbitrary partition of $[0, 1]$, so that

$$0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1.$$

Recall that the rational numbers are *dense* in $\mathbb{R}$. Thus, for any pair x_{i-1}, x_i, there exists a rational number r_i with

$$x_{i-1} < r_i < x_i.$$

Since r_i is rational, we have $f(r_i) = 1$.

In the same way, the irrational numbers are also dense in $\mathbb{R}$. Thus, for any pair x_{i-1}, x_i, there exists an irrational number q_i with

$$x_{i-1} < q_i < x_i.$$

Since q_i is irrational, we have $f(q_i) = 0$.

Next, we observe that

$$m_i = \inf_{x_{i-1} \le x \le x_i} f(x) \le 0,$$
$$M_i = \sup_{x_{i-1} \le x \le x_i} f(x) \ge 1.$$

From this, you should be able to check that

$$S = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) \ge 1,$$
$$s = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) \le 0.$$

Thus $S - s \ge 1$. If we now choose $\epsilon_0 = \frac{1}{2}$, it follows that

$$S - s \ge \epsilon_0$$

for any partition. The desired result follows. □

Sample #5. Let μ^* be Lebesgue outer measure on $\mathbb{R}$. Show that

$$\mu^*(\mathbb{Q}) = 0.$$

Proof. Since $\mathbb{Q}$ is countable, we may write

$$\mathbb{Q} = \{x_1, x_2, x_3, x_4, \ldots\}.$$

Now let $\epsilon > 0$, and let I_n be the interval

$$I_n = \left(x_n - \frac{\epsilon}{2^{n+1}}, x_n + \frac{\epsilon}{2^{n+1}}\right)$$

Then for each $n \in \mathbb{N}$, we will have

$$x_n \in I_n,$$

so that

$$\mathbb{Q} \subset \bigcup_{n=1}^{\infty} I_n.$$

Note that the length of this interval is

$$\ell\left(I_n\right) = \frac{\epsilon}{2^{n+1}} - \left(-\frac{\epsilon}{2^{n+1}}\right) = \frac{\epsilon}{2^n},$$

which implies that

$$\begin{aligned} m^*(\mathbb{Q}) &< \sum_{n=1}^{\infty} \ell(I_n) \\ &= \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} \\ &= \epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, it follows that $m^*(\mathbb{Q}) = 0$. □

Sample #6. Consider the measure space $(\mathbb{R}, \mathcal{B}, \mu)$, where $\mathcal{B}$ is the Borel σ-algebra, and μ is Lebesgue measure. For $b, m \in \mathbb{R}$, let $f : \mathbb{R} \to \mathbb{R}$ be the linear function $f(x) = mx + b$. Show that f is measurable.

Proof. First, if $m = 0$, then f is a constant function, which is measurable. So we assume from here on out that $m \neq 0$.

By definition, f is measurable if, for any $\alpha > 0$, the set

$$\{x \in \mathbb{R} : f(x) > \alpha\}$$

is measurable. But this holds for any x which satisfies

$$mx + b > \alpha.$$

From here, we must consider 2 different cases.

Case 1: $m > 0$. This case, the inequality above will be satisfied for all x satisfying

$$x > \frac{\alpha - b}{m},$$

which is the interval $\left(\frac{\alpha-b}{m}, \infty\right)$, which is a Borel set.

Case 2: $m < 0$. In this case, the inequality above will be satisfied for all x satisfying

$$x < \frac{\alpha - b}{m},$$

which is the interval $\left(-\infty, \frac{\alpha-b}{m}\right)$, which is a Borel set.

Thus, in either case, the set $\{x \in \mathbb{R} : f(x) > \alpha\}$ is measurable, so that f is measurable. □

Sample #7. Let μ be the measure defined on the power set of $\mathbb{R}$ given by

$$\mu(A) = \begin{cases} 1 & \text{if } 0 \in A, \\ 0 & \text{otherwise.} \end{cases}$$

If $\phi : \mathbb{R} \to \mathbb{R}$ is a simple function, show that

$$\int_{\mathbb{R}} \phi \, d\mu = \phi(0).$$

Proof. By definition, if ϕ is a simple function, there exist constants $a_1, \ldots, a_n$ and disjoint sets $A_1, \ldots, A_n \in \mathcal{P}(\mathbb{R})$ such that

$$\mathbb{R} = \bigcup_{i=1}^{n} A_i$$

and

$$\phi(x) = \sum_{i=1}^{n} a_i \chi_{A_i}.$$

Moreover, by definition, we have

$$\int \phi \, d\mu = \sum_{i=1}^{n} a_i \mu(A_i).$$

Since the A_i are disjoint, there is a unique $j \in \{1, \ldots, n\}$ such that $0 \in A_j$. Thus, the sum above reduces to

$$\int \phi \, d\mu = a_j \mu(A_j) = a_j.$$

To complete the proof, we observe that

$$\begin{aligned} \phi(0) &= \sum_{i=1}^{n} a_i \chi_{A_i}(0) \\ &= a_j \chi_{A_j}(0) \\ &= a_j. \end{aligned}$$

This completes the proof. $\square$