California State University – Los Angeles

Department of Mathematics

Master's Degree Comprehensive Examination

Analysis   Fall 2025

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Do at least two (2) problems from Section 1 below, and at least three (3) problems from Section 2 below. All problems count equally. If you attempt more than two problems from Section 1, the best two will be used. If you attempt more than three problems from Section 2, the best three will be used. Be sure to show your work for all answers.

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only.
(3) Begin each problem on a new page.
(4) Assemble the problems you hand in in numerical order.

Exams are graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Throughout this test, let $\mathbb{N}$ denote the set of positive integers; let $\mathbb{Z}$ denote the set of integers; let $\mathbb{Q}$ denote the set of rational numbers; let $\mathbb{R}$ denote the set of real numbers; and let $\mathbb{C}$ denote the set of complex numbers.

SECTION 1 – Do two (2) problems from this section. If you attempt all three, then the best two will be used for your grade.

Fall 2025 #1. Define $f \colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ x, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Find all points $x \in \mathbb{R}$ such that f is continuous at x. Prove that your answer is correct.

Fall 2025 #2. For each subset of $\mathbb{R}$ below, answer the following questions, and in each case justify your answer: (i) Is it closed? (ii) Is it bounded? (iii) Is it compact?

(a) $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$
(b) $\{m \mid m \in \mathbb{Z}\}$
(c) $\mathbb{Q}$

Fall 2025 #3. For all positive integers n, let

$$x_n = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^n \frac{1}{n!}.$$

Show that the sequence $(x_n)_{n=1}^{\infty}$ is convergent.

SECTION 2 – Do three (3) problems from this section. If you attempt more than three, then the best three will be used for your grade.

Fall 2025 #4. Let E be a Banach space, equipped with the norm $\|\cdot\|$. Suppose that for all $x, y \in E$ we have that

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right).$$

For all $x, y \in E$, define

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\,\|x+iy\|^2 - i\,\|x-iy\|^2\right).$$

Hint: As you go along in this problem, use the previous parts of the problem. You can use a previous part even if you didn't do that part.

(a) Prove that for all $x, y, z \in E$, we have:

$$\langle x+y, z\rangle = 2\left\langle x, \frac{z}{2}\right\rangle + 2\left\langle y, \frac{z}{2}\right\rangle.$$

(b) Prove that for all $x, z \in E$, we have:

$$2\left\langle x, \frac{z}{2}\right\rangle = \langle x, z\rangle.$$

(c) Prove that for all $\lambda \in \mathbb{C}$ for all $x, y \in E$, we have:

$$\langle \lambda x, y\rangle = \lambda\,\langle x, y\rangle.$$

(d) Prove that $\langle \cdot, \cdot\rangle$ is an inner product on E.

(e) Prove that $\|\cdot\|$ is the norm associated to the inner product $\langle \cdot, \cdot\rangle$.

Fall 2025 #5. Let $C[0,1]$ the vector space of continuous real-valued functions on $[0,1]$. For all $f \in C[0,1]$, define

$$\|f\| = |f(0)| + \int_0^1 |f(x)|\,dx.$$

You may assume without proof that $\|\cdot\|$ defines a norm on $C[0,1]$. Let d be the metric associated to $\|\cdot\|$. Define $g\colon [0,1] \to \mathbb{R}$ by $g(x) = 1$; define $h\colon [0,1] \to \mathbb{R}$ by $h(x) = 0$; and define $j\colon [0,1] \to \mathbb{R}$ by $j(x) = x^2$. Let $A = \{f \in C[0,1] \;:\; f(0) = 0\}$.

(a) Find $d(g, h)$.

(b) Find $d(g, j)$.

(c) Find $d(g, A)$.

(d) Is there a function $f \in A$ such that $d(g, f) = d(g, A)$? Prove that your answer is correct.

Fall 2025 #6. Let

$$l^2(\mathbb{C}) = \left\{ x = (x_n)_{n\in\mathbb{N}} : \quad \sum |x_n|^2 < \infty \right\}.$$

As usual, for a sequence $(x_n) \in l^2(\mathbb{C})$, define

$$\|(x_n)\|_2 = \sqrt{\sum |x_n|^2}.$$

You may assume without proof that $\|\cdot\|_2$ defines a norm on $l^2(\mathbb{C})$.

Let (α_n) be a bounded sequence of complex numbers.

Define $U\colon l^2(\mathbb{C}) \to l^2(\mathbb{C})$, $x = (x_1, x_2, \cdots, x_n, \cdots) \mapsto (0, \alpha_1 x_1, \alpha_2 x_2, \cdots, \alpha_n x_n, \cdots)$.

(a) Verify that U is well-defined. In other words, prove that if $(x_1, x_2, \cdots, x_n, \cdots) \in l^2(\mathbb{C})$, then

$$(0, \alpha_1 x_1, \alpha_2 x_2, \cdots, \alpha_n x_n, \cdots) \in l^2(\mathbb{C}).$$

(b) Prove that U is linear.

(c) Prove that U is continuous.

(d) Find $\|U\|$, the operator norm of U.

Fall 2025 #7. Let $f(t) = t^2$ for $t \in [-\pi, \pi]$, and extend it to be 2π-periodic on $\mathbb{R}$.

(a) Find the Fourier series for f in trigonometric form.

(b) Use the result of Part (a) together with Parseval's identity to prove that

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}.$$

Be sure to carefully justify both how you know that Parseval's identity applies in this situation as well as why this equation follows from it.