California State University – Los Angeles

Department of Mathematics

Master's Degree Comprehensive Examination Solutions

Analysis Fall 2025

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Do at least two (2) problems from Section 1 below, and at least three (3) problems from Section 2 below. All problems count equally. If you attempt more than two problems from Section 1, the best two will be used. If you attempt more than three problems from Section 2, the best three will be used. Be sure to show your work for all answers.

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only.
(3) Begin each problem on a new page.
(4) Assemble the problems you hand in in numerical order.

Exams are graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Throughout this test, let $\mathbb{N}$ denote the set of positive integers; let $\mathbb{Z}$ denote the set of integers; let $\mathbb{Q}$ denote the set of rational numbers; let $\mathbb{R}$ denote the set of real numbers; and let $\mathbb{C}$ denote the set of complex numbers.

NOTE: All solutions in this document were generated by AI and then double-checked by a committee member.

SECTION 1 – Do two (2) problems from this section. If you attempt all three, then the best two will be used for your grade.

Fall 2025 #1. Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ x, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Find all points $x \in \mathbb{R}$ such that f is continuous at x. You do *not* need to prove that your answer is correct.

We claim that f is continuous exactly at $x = 0$ and $x = 1$.

Let $x_0 \in \mathbb{R}$ and suppose f is continuous at x_0. Take a sequence (q_n) of rationals with $q_n \to x_0$ and a sequence (r_n) of irrationals with $r_n \to x_0$. Then by continuity,

$$\lim_{n\to\infty} f(q_n) = f(x_0) = \lim_{n\to\infty} f(r_n).$$

But

$$\lim_{n\to\infty} f(q_n) = \lim_{n\to\infty} q_n^2 = x_0^2, \qquad \lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} r_n = x_0.$$

Hence $x_0^2 = x_0$, so $x_0 \in \{0, 1\}$.

We now check continuity at these points.

At $x = 0$. For any sequence $x_n \to 0$ in $\mathbb{R}$, we have

$$f(x_n) = \begin{cases} x_n^2, & x_n \in \mathbb{Q}, \\ x_n, & x_n \notin \mathbb{Q}, \end{cases}$$

and in either case $f(x_n) \to 0$. Also $f(0) = 0^2 = 0$. Thus f is continuous at 0.

At $x = 1$**.** Similarly, for any sequence $x_n \to 1$,

$$f(x_n) = \begin{cases} x_n^2, & x_n \in \mathbb{Q}, \\ x_n, & x_n \notin \mathbb{Q}, \end{cases}$$

so $f(x_n) \to 1$. Since $f(1) = 1^2 = 1$, f is continuous at 1.

Therefore f is continuous exactly at $x = 0$ and $x = 1$.

Fall 2025 #2. For each subset of $\mathbb{R}$ below, answer the following questions, and in each case justify your answer: (i) Is it closed? (ii) Is it bounded? (iii) Is it compact?

(a) $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$
(b) $\{m \mid m \in \mathbb{Z}\}$
(c) $\left\{x \in \mathbb{R} \mid -\sqrt{2} \leq x \leq \sqrt{2}\right\}$

(a) $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.

- A is *bounded*: $0 < \frac{1}{n} \leq 1$ for all n.
- A is *not closed*: 0 is a limit point of A (since $\frac{1}{n} \to 0$), but $0 \notin A$.
- A is therefore *not compact*. In $\mathbb{R}$, compact sets are exactly those that are closed and bounded (Heine–Borel). A is not closed.

(b) $B = \{m : m \in \mathbb{Z}\}$.

- B is *unbounded*: clearly $|m| \to \infty$ as $m \to \pm\infty$ in $\mathbb{Z}$.
- B is *closed*: if (m_k) is a convergent sequence in B, then its limit must be an integer (since eventually the sequence is constant), so all limit points of B lie in B.
- B is *not compact*: it is not bounded, hence not compact by Heine–Borel.

(c) $C = \{x \in \mathbb{R} : -\sqrt{2} \leq x \leq \sqrt{2}\}$.

- C is *bounded*: $|x| \leq \sqrt{2}$ for all $x \in C$.

- C is *closed*: it is a closed interval $[-\sqrt{2}, \sqrt{2}]$.
- C is *compact*: closed and bounded in $\mathbb{R}$.

Fall 2025 #3. For all positive integers n, let

$$x_n = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^n \frac{1}{n!}.$$

Show that the sequence $(x_n)_{n=1}^{\infty}$ is convergent.

Fall 2025 #3 (Cauchy–sequence solution). For all positive integers n, let

$$x_n = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^n \frac{1}{n!}.$$

Fall 2025 #3 (Cauchy–sequence solution via alternating–series tail estimate). For all positive integers n, let

$$x_n = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^n \frac{1}{n!}.$$

We show that (x_n) is a Cauchy sequence.

Write $a_k = \dfrac{1}{k!}$ for $k \geq 1$. Then (a_k) is a decreasing sequence of positive real numbers with $a_k \to 0$.

Let $\varepsilon > 0$ be given. Since $a_k \to 0$, there exists $N \in \mathbb{N}$ such that

$$a_{N+1} = \frac{1}{(N+1)!} < \varepsilon.$$

Now take any $m > n \geq N$. Consider the difference of partial sums:

$$x_m - x_n = \sum_{k=1}^{m} (-1)^{k+1} a_k - \sum_{k=1}^{n} (-1)^{k+1} a_k = \sum_{k=n+1}^{m} (-1)^{k+1} a_k.$$

This is a finite alternating sum whose first term in absolute value is a_{n+1}, and whose terms $a_{n+1}, a_{n+2}, \ldots$ are decreasing.

By the usual estimate for alternating sums of decreasing positive terms (i.e. the key step in the alternating series test), the absolute value of such a tail is at most the first omitted term:

$$|x_m - x_n| = \left| \sum_{k=n+1}^{m} (-1)^{k+1} a_k \right| \le a_{n+1} = \frac{1}{(n+1)!} \le \frac{1}{(N+1)!} < \varepsilon.$$

Since for every $\varepsilon > 0$ we can find N such that $|x_m - x_n| < \varepsilon$ whenever $m, n \ge N$, the sequence (x_n) is Cauchy. Because $\mathbb{R}$ is complete, (x_n) is therefore convergent.

SECTION 2 – Do three (3) problems from this section. If you attempt more than three, then the best three will be used for your grade.

Fall 2025 #4. Let E be a Banach space, equipped with the norm $\|\cdot\|$. Suppose that for all $x, y \in E$ we have that

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right).$$

For all $x, y \in E$, define

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\,\|x+iy\|^2 - i\,\|x-iy\|^2\right).$$

Hint: As you go along in this problem, use the previous parts of the problem. You can use a previous part even if you didn't do that part.

(a) Show that for all $x, y, z \in E$,

$$\langle x+y, z\rangle = 2\left\langle x, \frac{z}{2}\right\rangle + 2\left\langle y, \frac{z}{2}\right\rangle.$$

Let $w = z/2$, so $z = 2w$. Then

$$\langle x+y, z\rangle = \langle x+y, 2w\rangle.$$

Using the definition and the parallelogram law repeatedly, one can expand $\|x+y+2w\|^2$ and $\|x+y-2w\|^2$ and rewrite the result in terms of $\|x \pm w\|^2$ and $\|y \pm w\|^2$; the algebra gives precisely

$$\langle x + y, 2w\rangle = 2\langle x, w\rangle + 2\langle y, w\rangle.$$

Replacing w with $z/2$ yields the desired identity.

(Any full solution can include the detailed expansions; the key point is systematic use of the parallelogram identity.)

(b) Show that for all $x, z \in E$,

$$2\left\langle x, \frac{z}{2}\right\rangle = \langle x, z\rangle.$$

Again write $w = z/2$. Then $z = 2w$, and

$$\langle x, z\rangle = \langle x, 2w\rangle.$$

Applying the definition and the homogeneity of the norm $\|\alpha u\| = |\alpha|\|u\|$, a direct computation yields

$$\langle x, 2w\rangle = 2\langle x, w\rangle.$$

Thus $2\langle x, z/2\rangle = \langle x, z\rangle$.

(c) Show that for all $\lambda \in \mathbb{C}$ and $x, y \in E$,

$$\langle \lambda x, y\rangle = \lambda\langle x, y\rangle.$$

First assume $\lambda \in \mathbb{R}$, $\lambda > 0$. Using the definition and $\|\lambda u\| = |\lambda|\|u\|$, one checks that

$$\langle \lambda x, y\rangle = \lambda\langle x, y\rangle.$$

For $\lambda < 0$ this follows from $\langle -x, y\rangle = -\langle x, y\rangle$ (which itself is obtained by applying (a) and (b) with $x + y = 0$).

For general $\lambda = a + ib \in \mathbb{C}$, write

$$\langle (a+ib)x, y\rangle = \langle ax, y\rangle + \langle ibx, y\rangle = a\langle x, y\rangle + i\langle bx, y\rangle = a\langle x, y\rangle + ib\langle x, y\rangle = \lambda\langle x, y\rangle,$$

using the real homogeneity already established and the definition of $\langle ix, y\rangle$ (again via the norm and parallelogram law). Thus the identity holds for all $\lambda \in \mathbb{C}$.

(d) Show that $\langle \cdot, \cdot\rangle$ is an inner product on E.

We must verify:

(i) *Linearity in the first variable:* Parts (a) and (b) together imply

$$\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle,$$

and part (c) gives $\langle \lambda x, z\rangle = \lambda\langle x, z\rangle$.

(ii) *Conjugate symmetry:* A direct computation from the definition shows

$$\langle y, x\rangle = \overline{\langle x, y\rangle},$$

using the parallelogram law and the fact that $\|x + iy\|^2$ and $\|x - iy\|^2$ appear with conjugate coefficients.

(iii) *Positive-definiteness:* Taking $y = x$ in the definition gives

$$\langle x, x\rangle = \frac{1}{4}\Big(\|2x\|^2 - \|0\|^2 + i\|x+ix\|^2 - i\|x-ix\|^2\Big) = \frac{1}{4}(4\|x\|^2 - 0 + i\cdot 0 - i\cdot 0) = \|x\|^2 \geq 0,$$

and $\langle x, x\rangle = 0$ iff $\|x\| = 0$, i.e. iff $x = 0$.

Hence $\langle \cdot, \cdot\rangle$ is an inner product.

(e) Show that $\| \cdot \|$ is the norm associated to $\langle \cdot, \cdot\rangle$.

From (d)(iii) we have for all x,

$$\langle x, x\rangle = \|x\|^2.$$

Thus the norm induced by the inner product, $\|x\|_{\langle\cdot,\cdot\rangle} := \sqrt{\langle x, x\rangle}$, coincides with the given norm $\|x\|$.

And now, here are all the excruciating details for parts (a), (b), and (c). Note that $\Re$ here means "real part" and $\Im$ means "imaginary part."

Fall 2025 #4: Detailed solutions for (a), (b), (c). Assume E is a Banach space with norm $\|\cdot\|$ satisfying the parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2\big(\|x\|^2 + \|y\|^2\big), \qquad x, y \in E.$$

For $x, y \in E$ define

$$\langle x, y\rangle = \frac{1}{4}\Big(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\Big).$$

We now prove (a), (b), (c) with full details.

(a) Show that for all $x, y, z \in E$ we have

$$\langle x+y, z\rangle = 2\Big\langle x, \frac{z}{2}\Big\rangle + 2\Big\langle y, \frac{z}{2}\Big\rangle.$$

Set $w = z/2$, so $z = 2w$. We will show

$$\langle x+y, 2w\rangle = 2\langle x, w\rangle + 2\langle y, w\rangle.$$

Real parts. Using the definition,

$$\Re\langle u, v\rangle = \frac{1}{4}\big(\|u+v\|^2 - \|u-v\|^2\big).$$

So

$$4\,\Re\langle x+y, 2w\rangle = \|x+y+2w\|^2 - \|x+y-2w\|^2,$$

and

$$4\,\Re\big(2\langle x, w\rangle + 2\langle y, w\rangle\big) = 2\big(\|x+w\|^2 - \|x-w\|^2\big) + 2\big(\|y+w\|^2 - \|y-w\|^2\big).$$

Now apply the parallelogram law twice.

First with $a = x + w$ and $b = y + w$:

$$\|a+b\|^2 + \|a-b\|^2 = 2\|a\|^2 + 2\|b\|^2.$$

That is,

$$(1) \qquad \|x+y+2w\|^2 + \|x-y\|^2 = 2\|x+w\|^2 + 2\|y+w\|^2.$$

Second with $a = x - w$ and $b = y - w$:

$$\|x + y - 2w\|^2 + \|x - y\|^2 = 2\|x - w\|^2 + 2\|y - w\|^2. \tag{2}$$

Subtracting (2) from (1) gives

$$\|x+y+2w\|^2-\|x+y-2w\|^2 = 2\Big(\|x+w\|^2-\|x-w\|^2+\|y+w\|^2-\|y-w\|^2\Big).$$

Comparing with the expressions above,

$$4\,\Re\langle x + y, 2w\rangle = 4\,\Re\big(2\langle x, w\rangle + 2\langle y, w\rangle\big),$$

hence

$$\Re\langle x + y, 2w\rangle = \Re\big(2\langle x, w\rangle + 2\langle y, w\rangle\big).$$

Imaginary parts. Similarly,

$$\Im\langle u, v\rangle = \frac{1}{4}\big(\|u + iv\|^2 - \|u - iv\|^2\big),$$

so

$$4\,\Im\langle x + y, 2w\rangle = \|x + y + 2iw\|^2 - \|x + y - 2iw\|^2,$$

and

$$4\,\Im\big(2\langle x, w\rangle+2\langle y, w\rangle\big) = 2\big(\|x+iw\|^2-\|x-iw\|^2\big)+2\big(\|y+iw\|^2-\|y-iw\|^2\big).$$

Now we repeat the parallelogram-law argument with w replaced by iw. Note that the parallelogram law holds for all vectors in E, including iw. So exactly as above we obtain

$$\|x+y+2iw\|^2-\|x+y-2iw\|^2 = 2\Big(\|x+iw\|^2-\|x-iw\|^2+\|y+iw\|^2-\|y-iw\|^2\Big),$$

which implies

$$\Im\langle x + y, 2w\rangle = \Im\big(2\langle x, w\rangle + 2\langle y, w\rangle\big).$$

Since the real and imaginary parts agree, we have

$$\langle x + y, 2w\rangle = 2\langle x, w\rangle + 2\langle y, w\rangle,$$

i.e.

$$\langle x + y, z\rangle = 2\Big\langle x, \frac{z}{2}\Big\rangle + 2\Big\langle y, \frac{z}{2}\Big\rangle.$$

This proves (a).

(b) Show that for all $x, z \in E$ we have

$$2\left\langle x, \frac{z}{2}\right\rangle = \langle x, z\rangle.$$

We use part (a) with $y = 0$. For all x, z we have

$$\langle x + 0, z\rangle = 2\left\langle x, \frac{z}{2}\right\rangle + 2\left\langle 0, \frac{z}{2}\right\rangle.$$

So

$$\langle x, z\rangle = 2\left\langle x, \frac{z}{2}\right\rangle + 2\left\langle 0, \frac{z}{2}\right\rangle.$$

It remains to show $\langle 0, w\rangle = 0$ for all $w \in E$.

By definition,

$$\langle 0, w\rangle = \frac{1}{4}\Big(\|0+w\|^2 - \|0-w\|^2 + i\|0+iw\|^2 - i\|0-iw\|^2 \Big) = \frac{1}{4}\Big(\|w\|^2 - \|-w\|^2 + i\|iw\|^2 - i\|-iw\|^2 \Big).$$

But norms satisfy $\|w\| = \|-w\|$ and $\|iw\| = \|-iw\|$, so each difference is 0, hence $\langle 0, w\rangle = 0$. Therefore

$$\langle x, z\rangle = 2\left\langle x, \frac{z}{2}\right\rangle,$$

which is exactly the desired identity.

(c) Show that for all $\lambda \in \mathbb{C}$ and $x, y \in E$,

$$\langle \lambda x, y\rangle = \lambda\langle x, y\rangle.$$

We break this into two steps: real scalars and multiplication by i.

Step 1: Additivity in the first variable. Combining (a) and (b) we get, for all $x, y, z \in E$,

$$\langle x + y, z\rangle = 2\left\langle x, \frac{z}{2}\right\rangle + 2\left\langle y, \frac{z}{2}\right\rangle = \langle x, z\rangle + \langle y, z\rangle.$$

Thus for each fixed z, the map $x \mapsto \langle x, z\rangle$ is additive.

Step 2: Real scalar multiples. First note that

$$\langle 0, y\rangle = 0 \quad \text{and} \quad \langle x, y\rangle = \langle 1\cdot x, y\rangle.$$

Using additivity, we get for a positive integer n:

$$\langle nx, y\rangle = \langle \underbrace{x + \cdots + x}_{n \text{ times}}, y\rangle = \underbrace{\langle x, y\rangle + \cdots + \langle x, y\rangle}_{n \text{ times}} = n\langle x, y\rangle.$$

For $n \in \mathbb{N}$, applying (b) repeatedly gives

$$\left\langle \frac{x}{2^n}, y \right\rangle = \frac{1}{2^n}\langle x, y\rangle.$$

Combining these, for any dyadic rational $r = m/2^n$ (with $m \in \mathbb{Z}$) we obtain

$$\langle rx, y\rangle = r\,\langle x, y\rangle.$$

Now fix x, y and consider the function

$$\phi(\lambda) := \langle \lambda x, y\rangle \qquad (\lambda \in \mathbb{R}).$$

From the explicit formula in terms of the norm and the continuity of the norm, ϕ is continuous in λ. Since dyadic rationals are dense in $\mathbb{R}$ and $\phi(\lambda) = \lambda\langle x, y\rangle$ holds on that dense set, by continuity it holds for all real λ:

$$\langle \lambda x, y\rangle = \lambda\langle x, y\rangle \qquad \text{for all } \lambda \in \mathbb{R}.$$

Step 3: Multiplication by i. We now compute $\langle ix, y\rangle$ explicitly from the definition.

We use that $\|iu\| = \|u\|$ for all $u \in E$ (since $\|iu\| = |i|\|u\| = \|u\|$) and that multiplying a vector by i or $-i$ does not change its norm.

First,

$$ix + y = i(x - iy), \qquad ix - y = i(x + iy),$$

so

$$\|ix + y\|^2 = \|x - iy\|^2, \qquad \|ix - y\|^2 = \|x + iy\|^2.$$

Also

$$ix + iy = i(x + y), \qquad ix - iy = i(x - y),$$

so

$$\|ix + iy\|^2 = \|x + y\|^2, \qquad \|ix - iy\|^2 = \|x - y\|^2.$$

Thus

$$\begin{aligned}\langle ix, y\rangle &= \frac{1}{4}\Big(\|ix + y\|^2 - \|ix - y\|^2 + i\|ix + iy\|^2 - i\|ix - iy\|^2\Big)\\ &= \frac{1}{4}\Big(\|x - iy\|^2 - \|x + iy\|^2 + i\|x + y\|^2 - i\|x - y\|^2\Big).\end{aligned}$$

On the other hand,

$$\begin{aligned}i\langle x, y\rangle &= i \cdot \frac{1}{4}\Big(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2\Big)\\ &= \frac{1}{4}\Big(i\|x + y\|^2 - i\|x - y\|^2 + i^2\|x + iy\|^2 - i^2\|x - iy\|^2\Big)\\ &= \frac{1}{4}\Big(i\|x + y\|^2 - i\|x - y\|^2 - \|x + iy\|^2 + \|x - iy\|^2\Big).\end{aligned}$$

Comparing these two expressions, we see they are identical:

$$\langle ix, y\rangle = i\langle x, y\rangle.$$

Step 4: General complex scalars. Let $\lambda = a + ib \in \mathbb{C}$ with $a, b \in \mathbb{R}$. Then

$$\langle \lambda x, y\rangle = \langle (a + ib)x, y\rangle = \langle ax, y\rangle + \langle ibx, y\rangle$$

by additivity in the first variable. Using real homogeneity from Step 2 and the i–homogeneity from Step 3, we get

$$\langle ax, y\rangle = a\langle x, y\rangle, \qquad \langle ibx, y\rangle = i\,b\langle x, y\rangle.$$

Hence

$$\langle \lambda x, y\rangle = a\langle x, y\rangle + ib\langle x, y\rangle = (a + ib)\langle x, y\rangle = \lambda\langle x, y\rangle.$$

This completes the proof of (c).

Fall 2025 #5. Let $C[0, 1]$ the vector space of continuous real-valued functions on $[0, 1]$. For all $f \in C[0, 1]$, define

$$\|f\| = |f(0)| + \int_0^1 |f(x)|\,dx.$$

You may assume without proof that $\|\cdot\|$ defines a norm on $C[0,1]$. Let d be the metric associated to $\|\cdot\|$. Define $g\colon[0,1]\to\mathbb{R}$ by $g(x)=1$; define $h\colon[0,1]\to\mathbb{R}$ by $h(x)=0$; and define $j\colon[0,1]\to\mathbb{R}$ by $j(x)=x^2$. Let $A=\{f\in C[0,1]\;:\;f(0)=0\}$.

(a) Find $d(g,h)$.
(b) Find $d(g,j)$.
(c) Find $d(g,A)$.
(d) Is there a function $f\in A$ such that $d(g,f)=d(g,A)$? Prove that your answer is correct.

(a) Find $d(g,h)$.

We have $g-h=1-0=1$, so

$$d(g,h)=\|g-h\|=|(g-h)(0)|+\int_0^1|g(x)-h(x)|\,dx=|1|+\int_0^1 1\,dx=1+1=2.$$

(b) Find $d(g,j)$.

Here $g-j=1-x^2$. Since $1-x^2\geq 0$ on $[0,1]$,

$$d(g,j)=\|g-j\|=|(1-x^2)(0)|+\int_0^1(1-x^2)\,dx=1+\Big[x-\frac{x^3}{3}\Big]_0^1=1+\Big(1-\frac{1}{3}\Big)=\frac{5}{3}.$$

(c) Find $d(g,A)$.

For $f\in A$ we have $f(0)=0$, so

$$d(g,f)=\|g-f\|=|g(0)-f(0)|+\int_0^1|g(x)-f(x)|\,dx=1+\int_0^1|1-f(x)|\,dx.$$

Thus

$$d(g,A)=\inf_{f\in A}d(g,f)=1+\inf_{f\in A}\int_0^1|1-f(x)|\,dx.$$

We claim that $\inf_{f\in A}\int_0^1 |1-f(x)|\,dx = 0$. Indeed, for each $\varepsilon > 0$, define $f_\varepsilon \in C[0,1]$ by

$$f_\varepsilon(x) = \begin{cases} \frac{x}{\varepsilon}, & 0 \le x \le \varepsilon, \\ 1, & \varepsilon \le x \le 1. \end{cases}$$

Then $f_\varepsilon \in A$ (since $f_\varepsilon(0) = 0$) and

$$\int_0^1 |1-f_\varepsilon(x)|\,dx = \int_0^\varepsilon \left|1-\frac{x}{\varepsilon}\right| dx = \int_0^\varepsilon \left(1-\frac{x}{\varepsilon}\right) dx = \left[x-\frac{x^2}{2\varepsilon}\right]_0^\varepsilon = \varepsilon-\frac{\varepsilon}{2} = \frac{\varepsilon}{2}.$$

Hence this integral can be made arbitrarily small, so the infimum is 0. Therefore

$$d(g, A) = 1.$$

(d) Is there $f \in A$ such that $d(g, f) = d(g, A)$?

We have $d(g, A) = 1$. If $d(g, f) = 1$, then from the formula above we must have

$$1 + \int_0^1 |1 - f(x)|\,dx = 1,$$

hence $\int_0^1 |1 - f(x)|\,dx = 0$, which implies $1 - f(x) = 0$ for all x, i.e. $f(x) \equiv 1$ on $[0, 1]$. But then $f(0) = 1 \neq 0$, so $f \notin A$. Thus no $f \in A$ can satisfy $d(g, f) = 1 = d(g, A)$.

So there is *no* function $f \in A$ such that $d(g, f) = d(g, A)$.

Fall 2025 #6. Let

$$l^2(\mathbb{C}) = \left\{x = (x_n)_{n\in\mathbb{N}} : \quad \sum |x_n|^2 < \infty\right\}.$$

As usual, for a sequence $(x_n) \in l^2(\mathbb{C})$, define

$$\|(x_n)\|_2 = \sqrt{\sum |x_n|^2}.$$

You may assume without proof that $\|\cdot\|_2$ defines a norm on $l^2(\mathbb{C})$.

Let (α_n) be a bounded sequence of complex numbers.

Define $U\colon l^2(\mathbb{C}) \to l^2(\mathbb{C})$, $x = (x_1, x_2, \cdots, x_n, \cdots) \mapsto (0, \alpha_1 x_1, \alpha_2 x_2, \cdots, \alpha_n x_n, \cdots)$.

(a) Verify that U is well-defined. In other words, prove that if $(x_1, x_2, \cdots, x_n, \cdots) \in l^2(\mathbb{C})$, then

$$(0, \alpha_1 x_1, \alpha_2 x_2, \cdots, \alpha_n x_n, \cdots) \in l^2(\mathbb{C}).$$

(b) Prove that U is linear.

(c) Prove that U is continuous.

(d) Find $\|U\|$, the operator norm of U.

Let

$$\ell^2(\mathbb{C}) = \Big\{x = (x_n)_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^2 < \infty\Big\},$$

and define

$$\|(x_n)\|_2 = \Big(\sum_{n=1}^{\infty} |x_n|^2\Big)^{1/2}.$$

Let (α_n) be a bounded sequence of complex numbers: there exists $M \geq 0$ such that $|\alpha_n| \leq M$ for all n.

Define $U : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$ by

$$U(x_1, x_2, \dots) = (0, \alpha_1 x_1, \alpha_2 x_2, \dots).$$

(a) U is well-defined.

Let $x = (x_n) \in \ell^2(\mathbb{C})$. Then

$$\sum_{n=1}^{\infty} |\alpha_n x_n|^2 \leq \sum_{n=1}^{\infty} M^2 |x_n|^2 = M^2 \sum_{n=1}^{\infty} |x_n|^2 < \infty.$$

Thus $Ux \in \ell^2(\mathbb{C})$, so U is well-defined.

(b) U is linear.

For $x = (x_n)$ and $y = (y_n)$ and $\lambda, \mu \in \mathbb{C}$,

$$U(\lambda x+\mu y) = U(\lambda x_1+\mu y_1, \lambda x_2+\mu y_2, \dots) = (0, \alpha_1(\lambda x_1+\mu y_1), \dots) = \lambda Ux+\mu Uy.$$

Hence U is linear.

(c) U is continuous.

For $x \in \ell^2$ we have

$$\|Ux\|_2^2 = \sum_{n=1}^{\infty} |\alpha_n x_n|^2 \leq M^2 \sum_{n=1}^{\infty} |x_n|^2 = M^2 \|x\|_2^2.$$

Thus $\|Ux\|_2 \le M\|x\|_2$ for all x, so U is bounded, hence continuous.

(d) Find $\|U\|$, the operator norm.

From (c),

$$\|Ux\|_2 \le M\|x\|_2 \qquad \text{for all } x,$$

so $\|U\| \le M := \sup_n |\alpha_n|$.

To see that $\|U\| = M$, fix $\varepsilon > 0$ and choose k such that $|\alpha_k| > M - \varepsilon$ (possible since M is the supremum). Let e_k be the sequence with 1 in the kth position and 0 elsewhere. Then $\|e_k\|_2 = 1$ and

$$Ue_k = (0, \dots, 0, \alpha_k, 0, \dots), \qquad \text{so} \qquad \|Ue_k\|_2 = |\alpha_k| > M - \varepsilon.$$

Thus

$$\|U\| \ge \|Ue_k\|_2 > M - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\|U\| \ge M$. Combined with $\|U\| \le M$, this yields

$$\|U\| = \sup_n |\alpha_n|.$$

Fall 2025 #7. Let $f(t) = t^2$ for $t \in [-\pi, \pi]$, and extend it to be 2π-periodic on $\mathbb{R}$.

(a) Find the Fourier series for f in trigonometric form.

(b) Use the result of Part (a) together with Parseval's identity to prove that

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}.$$

Be sure to carefully justify both how you know that Parseval's identity applies in this situation as well as why this equation follows from it.

Let $f(t) = t^2$ on $[-\pi, \pi]$ and extend it to be 2π-periodic.

(a) Find the Fourier series of f in trigonometric form.

Since f is even, its Fourier series has only cosine terms:

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt), \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt.$$

First,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} t^2\, dt = \frac{2}{\pi} \int_0^{\pi} t^2\, dt = \frac{2}{\pi} \cdot \frac{\pi^3}{3} = \frac{2\pi^2}{3}.$$

For $n \geq 1$,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 \cos(nt)\, dt = \frac{2}{\pi} \int_0^{\pi} t^2 \cos(nt)\, dt.$$

Integrating by parts twice (or using a known formula) gives

$$a_n = \frac{4(-1)^n}{n^2}.$$

Thus

$$t^2 \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nt).$$

(b) Use Parseval's identity to prove

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}.$$

Since $f \in L^2([-\pi, \pi])$, Parseval's identity applies:

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(t)|^2\, dt = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2,$$

where $b_n = 0$ here.

Compute the left-hand side:

$$\frac{1}{\pi} \int_{-\pi}^{\pi} t^4\, dt = \frac{2}{\pi} \int_0^{\pi} t^4\, dt = \frac{2}{\pi} \cdot \frac{\pi^5}{5} = \frac{2\pi^4}{5}.$$

For the right-hand side we use $a_0 = \frac{2\pi^2}{3}$ and $a_n = \frac{4(-1)^n}{n^2}$:

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 = \frac{1}{2}\Big(\frac{2\pi^2}{3}\Big)^2 + \sum_{n=1}^{\infty} \frac{16}{n^4} = \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\frac{1}{n^4}.$$

Parseval's identity gives

$$\frac{2\pi^4}{5} = \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\frac{1}{n^4}.$$

Hence

$$16\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{2\pi^4}{5} - \frac{2\pi^4}{9} = 2\pi^4\Big(\frac{1}{5} - \frac{1}{9}\Big) = 2\pi^4 \cdot \frac{4}{45} = \frac{8\pi^4}{45}.$$

Dividing by 16,

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

Thus

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}.$$