Algebra Comprehensive Exam

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Answer at least five (5) questions. You must *answer at least one* from each of linear algebra, groups, and synthesis. If you attempt more than five problems, then we will count the best five that cover all three sections.

Linear algebra

(L1) Let V be a vector space over a field F. Let $S = \{v_1, v_2\}$ be a set of two linearly independent vectors in V. Let $v \in V$ where $v \notin S$. Prove that if $S \cup \{v\}$ is a linearly dependent set, then v is in the span of S.

(L2) Let V and W be vector spaces and $\phi : V \to W$ a bijection. Show that ϕ is linear if and only if ϕ^{-1} is linear.

(L3) Let V be a vector space over a field F, and let $T\colon V \to V$ be a linear operator. Suppose $T^2 = 0$. If λ is an eigenvalue of T, then prove that $\lambda = 0$.

Groups

(G1) Prove that the group of real numbers $\mathbb{R}$ is not a cyclic group under addition.

(G2) Let N and H be subgroups of a group G such that N is normal in G. Show that $H \cap N$ is normal in H.

(G3) The **center** of a group G, denoted $Z(G)$, is defined to be the set of elements G which commute with all elements of G: that is,

$$Z(G) = \{x \in G\colon gx = xg \text{ for all } g \in G\}.$$

If H and K are groups, then prove that $Z(H \times K) = Z(H) \times Z(K)$ (i.e. the center of the direct product is equal to the direct product of the centers).

Synthesis

(S1) Prove that $\mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle|\ ad - bc = 1 \right\}$ is a subgroup of $\mathrm{GL}(2,\mathbb{R})$. Then prove that $\mathrm{SL}(2,\mathbb{R})$ is a normal in $\mathrm{GL}(2,\mathbb{R})$.

(S2) Suppose $A, B \in GL_2(\mathbb{R})$. Show that, if A and B are conjugate, then they have the same eigenvalues. Is the converse true?

(S3) Let $\mathbb{C}^*$ denote the group of non-zero complex numbers under multiplication. Define $\phi\colon \mathbb{C}^* \to \mathrm{GL}(2,\mathbb{R})$ by

$$\phi(a + bi) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

(where $a, b \in \mathbb{R}$). (a) Prove that ϕ is a group homomorphism. (b) Is ϕ injective? Is ϕ surjective? Prove your answers.
