Algebra Comprehensive Exam

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Answer at least five (5) questions. You must *answer at least one* from each of linear algebra, groups, and synthesis. If you attempt more than five problems, then we will count the best five that cover all three sections.

Linear algebra

(L1) Let V be a vector space over a field F. Let $S = \{v_1, v_2\}$ be a set of two linearly independent vectors in V. Let $v \in V$ where $v \notin S$. Prove that if $S \cup \{v\}$ is a linearly dependent set, then v is in the span of S.

Answer: Suppose that $S \cup \{v\}$ is a linearly dependent set. Then $c_1v_1 + c_2v_2 + c_3v = 0$ for some $c_1, c_2, c_3 \in F$ where c_1, c_2, c_3 are not all zero. If $c_3 = 0$, then $c_1v_1 + c_2v_2 = 0$ where c_1, c_2 are not both zero. But this would contradict that S is a linearly independent set. Thus $c_3 \neq 0$. So $v = -c_3^{-1}c_1v_1 - c_3^{-1}c_2v_2$. Thus v is in the span of S.

(L2) Let V and W be vector spaces and $\phi : V \to W$ a bijection. Show that ϕ is linear if and only if ϕ^{-1} is linear.

Answer: Since the inverse of a bijection is a bijection it suffices to prove only one direction. Suppose that ϕ is linear. Then

$$\phi(c_1\phi^{-1}(w_1) + c_2\phi^{-1}(w_2)) = c_1\phi(\phi^{-1}(w_1)) + c_2\phi(\phi^{-1}(w_2)) = c_1w_1 + c_2w_2$$

for all $w_1, w_2 \in W$ and scalars c_1 and c_2, and so

$$\phi^{-1}(c_1w_1 + c_2w_2) = \phi^{-1}(\phi(c_1\phi^{-1}(w_1) + c_2\phi^{-1}(w_2))) = c_1\phi^{-1}(w_1) + c_2\phi^{-1}(w_2)$$

This shows that ϕ^{-1} is linear.

(L3) Let V be a vector space over a field F, and let $T\colon V \to V$ be a linear operator. Suppose $T^2 = 0$. If λ is an eigenvalue of T, then prove that $\lambda = 0$.

Answer: Let $v \in V$ be a eigenvector of T with eigenvalue λ. This means that $v \neq 0$ and $T(v) = \lambda v$. But $T^2(v) = 0$ (since $T^2 = 0$), so

$$0 = T^2(v) = T(T(v)) = T(\lambda v) = \lambda\, T(v) = \lambda\lambda v = \lambda^2 v.$$

Therefore, $\lambda^2 v = 0$. Since $v \neq 0$, we conclude that $\lambda^2 = 0$ and so $\lambda = 0$.

Groups

(G1) Prove that the group of real numbers $\mathbb{R}$ is not a cyclic group under addition.
Answer: Suppose that $\mathbb{R}$ was cyclic. Then there exists $x \in \mathbb{R}$ with $x \neq 0$ and

$$\mathbb{R} = \langle x \rangle = \{\ldots, -3x, -2x, -x, 0, x, 2x, 3x, \ldots\}$$

But $\frac{x}{2} \in \mathbb{R}$ and $\frac{x}{2} \notin \langle x \rangle$. Contradiction. Thus $\mathbb{R}$ is not cyclic.

OR

All infinite cyclic groups are isomorphic to $\mathbb{Z}$ and thus are countable. $\mathbb{R}$ is uncountable. Thus $\mathbb{R}$ is not cyclic.

(G2) Let N and H be subgroups of a group G such that N is normal in G. Show that $H \cap N$ is normal in H.

Answer: Let $\pi : G \to G/N$ be the natural homomorphism with $\ker \pi = N$. Then π restricts to a homomorphism from H to G/N. Then the kernel of the restricted homomorphism, namely $N \cap H$, is normal in H.

OR

Let $h \in H$ and $n \in N \cap H$. Then, since $h \in G$ and $N \trianglelefteq G$, we have $h^{-1}nh \in h^{-1}Nh = N$. Since $h, n \in H$, we have $h^{-1}nh \in H$. Thus $h^{-1}nh \in H \cap N$ for all $n \in H \cap N$ and $h \in H$, that is, $H \cap N$ is a normal subgroup of H.

(G3) The **center** of a group G, denoted $Z(G)$, is defined to be the set of elements G which commute with all elements of G: that is,

$$Z(G) = \{x \in G \colon gx = xg \text{ for all } g \in G\}.$$

If H and K are groups, then prove that $Z(H \times K) = Z(H) \times Z(K)$ (i.e. the center of the direct product is equal to the direct product of the centers).

Answer: Let 1_H and 1_K denote the identity elements of H and K respectively.
($\subseteq$): Let $(x, y) \in Z(H \times K)$. We will prove that $(x, y) \in Z(H) \times Z(K)$ by proving that $x \in Z(H)$ and $y \in Z(K)$. First, for all $h \in H$, we have

$$(hx, y) = (h, 1_K)(x, y) = (x, y)(h, 1_K) = (xh, y),$$

so $hx = xh$. Therefore $x \in Z(H)$. A similar calculation with (x, y) and $(1_H, k)$ shows that $y \in Z(K)$. Therefore, $(x, y) \in Z(H) \times Z(K)$.
($\supseteq$): Let $(x, y) \in Z(H) \times Z(K)$, meaning that $x \in Z(H)$ and $y \in Z(K)$. Then, for all $(h, k) \in H \times K$, we have $hx = xh$ and $ky = yk$, so

$$(h, k)(x, y) = (hx, ky) = (xh, yk) = (x, y)(h, k).$$

Therefore, $(x, y) \in Z(H \times K)$.

Synthesis

(S1) Prove that $\mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, ad - bc = 1 \right\}$ is a subgroup of $\mathrm{GL}(2,\mathbb{R})$. Then prove that $\mathrm{SL}(2,\mathbb{R})$ is a normal in $\mathrm{GL}(2,\mathbb{R})$.

<u>Answer</u>: First we should show that $\mathrm{SL}(2,\mathbb{R})$ is a subgroup of $\mathrm{GL}(2,\mathbb{R})$. Note that the identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has determinant 1. Thus $I \in \mathrm{SL}(2,\mathbb{R})$. Let $A, B \in \mathrm{SL}(2,\mathbb{R})$. Then $\det(AB^{-1}) = \det(A)\det(B)^{-1} = 1 \cdot 1 = 1$. So $AB^{-1} \in \mathrm{SL}(2,\mathbb{R})$. Thus $\mathrm{SL}(2,\mathbb{R})$ is a subgroup of $\mathrm{GL}(2,\mathbb{R})$.
Now we show that it is a normal subgroup. Let $A \in \mathrm{SL}(2,\mathbb{R})$ and $B \in \mathrm{GL}(2,\mathbb{R})$. Then $\det(B^{-1}AB) = \det(B^{-1})\det(A)\det(B) = \det(B^{-1}) \cdot 1 \cdot \det(B) = \det(B^{-1}B) = \det(I) = 1$. Thus $B^{-1}AB \in \mathrm{SL}(2,\mathbb{R})$.

OR

$\mathrm{SL}(2,\mathbb{R})$ is the kernel of the homomorphism $\det\colon \mathrm{GL}(2,\mathbb{R}) \to \mathbb{R}^*$, and the kernel of a homomorphism is a normal subgroup.

(S2) Suppose $A, B \in GL_2(\mathbb{R})$. Show that, if A and B are conjugate, then they have the same eigenvalues. Is the converse true?

<u>Answer</u>: If A and B are conjugate, then $A = SBS^{-1}$ for some $S \in GL_2(\mathbb{R})$. The eigenvalues of A and B are the roots of the characteristic polynomials of these matrices, so to prove the claim it suffices to show that A and B have the same characteristic polynomials:

$$\begin{aligned} \det(A - \lambda I) &= \det(SBS^{-1} - \lambda I) = \det(S(B - \lambda I)S^{-1}) \\ &= \det(S)\det(B - \lambda I)\det(S^{-1}) = \det(B - \lambda I) \end{aligned}$$

The matrices I and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ have the same characteristic polynomial, but they are not conjugate since the conjugacy class of I is $\{I\}$

(S3) Let $\mathbb{C}^*$ denote the group of non-zero complex numbers under multiplication. Define $\phi\colon \mathbb{C}^* \to \mathrm{GL}(2,\mathbb{R})$ by

$$\phi(a + bi) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

(where $a, b \in \mathbb{R}$). (a) Prove that ϕ is a group homomorphism. (b) Is ϕ injective? Is ϕ surjective? Prove your answers.

<u>Answer</u>: If $a + bi, c + di \in \mathbb{C}^*$ with $a, b, c, d \in \mathbb{R}$, then

$$\phi((a + bi)(c + di)) = \phi((ac - bd) + (ad + bc)i) = \begin{pmatrix} ac - bd & -ad - bc \\ ad + bc & ac - bd \end{pmatrix}$$

and

$$\phi(a+bi)\,\phi(c+di) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac-bd & -ad-bc \\ ad+bc & ac-bd \end{pmatrix}.$$

Therefore, $\phi((a+bi)(c+di)) = \phi(a+bi)\,\phi(c+di)$, so ϕ is a homomorphism.

ϕ is injective: if $\phi(a+bi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which implies $a = 1$ and $b = 0$, so $a + bi = 1$. Thus $\ker(\phi) = \{1\}$.

ϕ is not surjective: for instance, $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R})$ is not equal to $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ for some $a, b \in \mathbb{R}$.
