① HW 8 – #1(b)

② HW 9 – #1(a)

③ HW 10 – #1(b)

④

$f(x) = x^2 + x + 1 \longrightarrow f(2) = 2^2 + 2 + 1 = 7$

$f'(x) = 2x + 1 \longrightarrow f'(2) = 2 \cdot 2 + 1 = 5$

$f''(x) = 2 \longrightarrow f''(2) = 2$

$f^{(k)}(x) = 0,\ k \geq 3 \longrightarrow f^{(k)}(2) = 0,\ k \geq 3$

power series expansion centered at $x_0 = 2$:

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2$$

$$= \boxed{7 + 5(x-2) + (x-2)^2 \qquad \text{radius } r = \infty}$$

⑤ $y'' - 2xy' - y = 0$

$y'(0) = 1,\ y(0) = 0$

power series centered at $x_0 = 0$

(a) We want a power series solution of the form:

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots$$

We have given:

$y(0) = -1$

$y'(0) = 1$

$y(0) = -1$

$y'(0) = 1$

Let's find $y''(0)$.

We have

$$y'' = 2xy' + y$$

So,

$$y''(0) = 2(0) \cdot y'(0) + y(0)$$
$$= 0 + (-1)$$
$$= -1$$

$\boxed{y''(0) = -1}$

Now let's find $y'''(0)$.

Differentiate $y'' = 2xy' + y$ to get

$$y''' = 2y' + 2xy'' + y' = 3y' + 2xy''$$

Thus,

$$y'''(0) = 3y'(0) + 2(0)\,y''(0)$$
$$= 3 \cdot 1 + 0$$
$$= 3$$

$y'''(0) = 3$

Thus,

$$y = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots$$
$$= -1 + x - \frac{1}{2!}x^2 + \frac{3}{3!}x^3 + \cdots$$
$$= -1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 + \cdots$$

(b) $y'' \underbrace{-2x}\, y' \underbrace{-}\, y = \underbrace{0}$

the coefficients are $-2x, -1, 0$ which are polynomials that have radii of convergence

$r = \infty$ each, thus the
solution has $r = \infty$
radius of convergence
and converges for all x.