California State University – Los Angeles

Department of Mathematics

Master's Degree Comprehensive Examination

Analysis   Spring 2025

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Do at least two (2) problems from Section 1 below, and at least three (3) problems from Section 2 below. All problems count equally. If you attempt more than two problems from Section 1, the best two will be used. If you attempt more than three problems from Section 2, the best three will be used. Be sure to show your work for all answers.

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only.
(3) Begin each problem on a new page.
(4) Assemble the problems you hand in in numerical order.

Exams are graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

SECTION 1 – Do two (2) problems from this section. If you attempt all three, then the best two will be used for your grade.

Spring 2025 #1. Use the definition of continuity to show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x^2$$

is continuous at $x = 1$.

Spring 2025 #2. Let $A = \{x \in \mathbb{Q} \mid 0 \leq x \leq 1\}$, where $\mathbb{Q}$ denotes the set of rational numbers. Is A compact? Prove that your answer is correct.

Spring 2025 #3. Let $\{a_n\}$ be a convergent sequence of real numbers, and define a new sequence $\{b_n\}$ by

$$b_n = \frac{1}{n} a_n.$$

(a) Prove that $\{b_n\}$ converges.

(b) Suppose $\lim_{n\to\infty} a_n = L$. What is $\lim_{n\to\infty} b_n$? Justify your answer.

SECTION 2 – Do three (3) problems from this section. If you attempt more than three, then the best three will be used for your grade.

Spring 2025 #4. Let X be a normed space with two equivalent norms $\|\cdot\|_1$ and $\|\cdot\|_2$ defined on it. Show that a sequence $\{x_n\}_{n=1}^{\infty}$ in X converges to $x \in X$ with respect to the norm $\|\cdot\|_1$ if and only if the sequence converges to x with respect to the norm $\|\cdot\|_2$.

Spring 2025 #5. Let ℓ^∞ be the set of all bounded sequences of real numbers. In other words,

$$\ell^\infty = \{(x_1, x_2, x_3, \dots) \mid \exists M \in \mathbb{R} \text{ such that } x_1, x_2, x_3, \dots \in [-M, M]\}.$$

Here $[-M, M] \subset \mathbb{R}$ denotes the closed interval from $-M$ to M.

Let W be the subset of ℓ^∞ consisting of all "eventually zero" sequences of real numbers. That is, $(x_1, x_2, x_3, \dots) \in W$ if and only if there is an N such that $x_k = 0$ for all $k \geq N$.

(a) Prove that W is a linear subspace of ℓ^∞.

(b) Recall that the standard metric on ℓ^∞ is given by

$$d((x_1, x_2, x_3, \dots), (y_1, y_2, y_3, \dots)) = \sup\{|x_i - y_i| \mid i \in \mathbb{N}\}.$$

With respect to this metric, is W a closed linear subspace of ℓ^∞? Prove that your answer is correct.

Spring 2025 #6. Let $d(x, y) = \|x - y\|$ be the standard metric induced by a norm on a real vector space X, and define a new function ρ by:

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

(a) Prove that ρ is a metric on X.
(b) Assume that X contains more than one element. Can ρ be induced by a norm? Justify your answer.

Spring 2025 #7. Let $f(t) = |t|$ for $t \in [-\pi, \pi]$, and extend it to be 2π-periodic on $\mathbb{R}$.

(a) Prove that f is an even function.
(b) Find the Fourier series of $f(t)$ in trigonometric form.
Hint: Use (a).