California State University – Los Angeles

Department of Mathematics

Master's Degree Comprehensive Examination

Analysis Spring 2025

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Do at least two (2) problems from Section 1 below, and at least three (3) problems from Section 2 below. All problems count equally. If you attempt more than two problems from Section 1, the best two will be used. If you attempt more than three problems from Section 2, the best three will be used. Be sure to show your work for all answers.

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only.
(3) Begin each problem on a new page.
(4) Assemble the problems you hand in in numerical order.

Exams are graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

SECTION 1 – Do two (2) problems from this section. If you attempt all three, then the best two will be used for your grade.

Spring 2025 #1. Use the definition of continuity to show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x^2$$

is continuous at $x = 1$.

Solution: We wish to prove that $f(x) = x^2$ is continuous at $x = 1$. By definition of continuity at $x = 1$, we must show:

$$\forall \varepsilon > 0, \quad \exists \delta > 0 \text{ such that } \left|x - 1\right| < \delta \implies \left|f(x) - f(1)\right| < \varepsilon.$$

Here, $f(1) = 1^2 = 1$, so we need to ensure:

$$\left|x^2 - 1\right| < \varepsilon \quad \text{whenever} \quad \left|x - 1\right| < \delta.$$

We begin by factoring the expression inside the absolute value:

$$|x^2 - 1| \;=\; |(x - 1)(x + 1)| \;=\; |x - 1| \,\cdot\, |x + 1|.$$

Our goal is to control this product by making $|x - 1|$ sufficiently small. Notice that if we require $|x - 1| < 1$, then:

$$|x + 1| \;=\; |(x - 1) + 2| \;\le\; |x - 1| + 2 \;<\; 1 + 2 \;=\; 3.$$

Hence, whenever $|x - 1| < 1$, we have

$$|x^2 - 1| \;=\; |x-1|\,|x+1| \;<\; |x-1| \cdot 3.$$

To make sure $|x^2 - 1| < \varepsilon$, it suffices to ensure $3\,|x-1| < \varepsilon$, or equivalently:

$$|x-1| < \frac{\varepsilon}{3}.$$

Thus, we can choose

$$\delta \;=\; \min\!\Big(1,\, \frac{\varepsilon}{3}\Big).$$

This choice of δ guarantees both $|x-1| < 1$ (to control $|x+1|$) and $3\,|x-1| < \varepsilon$. Therefore, whenever $|x-1| < \delta$, it follows that

$$|x^2 - 1| < \varepsilon.$$

Since our choice of δ depends only on ε and not on x, this completes the proof of continuity of $f(x) = x^2$ at $x = 1$ by the ε–δ definition.

(Note: This solution is AI-written but verified by a human committee member.)

Spring 2025 #2. Let $A = \{x \in \mathbb{Q} \mid 0 \leq x \leq 1\}$, where $\mathbb{Q}$ denotes the set of rational numbers. Is A compact? Prove that your answer is correct.

Solution. The set A is *not* compact. To see why, recall that A is compact if and only if it is closed and bounded by the Heine-Borel theorem. Since A is obviously bounded, it suffices to show it is not closed. Consider the sequence $\{x_n\}_{n=1}^{\infty}$ where x_n consists of the first n digits of $\sqrt{2}$.

For example, $x_1 = 1$, $x_2 = 1.4$, $x_3 = 1.41$, etc. Then it is clear that

$$\lim x_n = \sqrt{2}.$$

In particular, each $x_n \in \mathbb{Q}$, $x_n \neq 0$ for any n.

Next, define a sequence y_n by

$$y_n = \frac{1}{x_n}.$$

Since $x_n \in \mathbb{Q}$, then $y_n \in \mathbb{Q}$. By construction, we have

$$0 \leq y_n \leq 1,$$

so that $y_n \in A$ for every $n \in \mathbb{N}$. Moreover, we will have

$$\lim y_n = \frac{1}{\sqrt{2}},$$

which is not an element of A. Thus A is not closed, so it is not compact.

□

Spring 2025 #3. Let $\{a_n\}$ be a convergent sequence of real numbers, and define a new sequence $\{b_n\}$ by

$$b_n = \frac{1}{n}a_n.$$

(a) Prove that $\{b_n\}$ converges.
(b) Suppose $\lim_{n\to\infty} a_n = L$. What is $\lim_{n\to\infty} b_n$? Justify your answer.

Solution:

(a) Prove that $\{b_n\}$ converges. Since $\{a_n\}$ is convergent, there exists some real number L such that

$$\lim_{n\to\infty} a_n = L.$$

We do not need the value of L explicitly to show that $\{b_n\}$ converges; only that $\{a_n\}$ is bounded. Convergence implies boundedness, so there exists an $M > 0$ such that

$$|a_n| \leq M \quad \text{for all } n \in \mathbb{N}.$$

Define $b_n = \frac{1}{n} a_n$. Then

$$|b_n| \; = \; \left| \frac{1}{n} a_n \right| \; = \; \frac{1}{n} \, |a_n| \; \leq \; \frac{1}{n} \, M.$$

As $n \to \infty$, we have $\frac{M}{n} \to 0$. Therefore, the sequence $\{b_n\}$ is squeezed between $-\frac{M}{n}$ and $\frac{M}{n}$, both of which converge to 0. By the Squeeze Theorem, $\{b_n\}$ converges, regardless of the specific limit of $\{a_n\}$.

(b) Suppose $\lim_{n\to\infty} a_n = L$. Find $\lim_{n\to\infty} b_n$. From part (a) we have established that b_n converges. Now we use the fact that

$$\lim_{n\to\infty} a_n = L.$$

Consider

$$b_n \; = \; \frac{1}{n} \, a_n.$$

We can rewrite the limit in terms of a_n and $1/n$:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \Big(\frac{1}{n} \, a_n \Big).$$

If $\lim_{n\to\infty} a_n = L$ and clearly $\lim_{n\to\infty} \frac{1}{n} = 0$, then using standard limit laws (product rule for limits) gives

$$\lim_{n\to\infty} \left(\frac{1}{n} \, a_n \right) = \big(\lim_{n\to\infty} \tfrac{1}{n} \big) \big(\lim_{n\to\infty} a_n \big) = 0 \cdot L = 0.$$

Hence,

$$\lim_{n\to\infty} b_n = 0.$$

Answer:

(a) $\{b_n\}$ converges because $|b_n| = \frac{1}{n}|a_n| \leq \frac{M}{n}$ for some bound M on $\{|a_n|\}$, and thus $b_n \to 0$ by the Squeeze Theorem.

(b) If $\lim_{n\to\infty} a_n = L$, then by limit laws, $\lim_{n\to\infty} b_n = 0 \cdot L = 0$.

(Note: This solution is AI-written but verified by a human committee member.)

SECTION 2 – Do three (3) problems from this section. If you attempt more than three, then the best three will be used for your grade.

Spring 2025 #4. Let X be a normed space with two equivalent norms $\|\cdot\|_1$ and $\|\cdot\|_2$ defined on it. Show that a sequence $\{x_n\}_{n=1}^{\infty}$ in X converges to $x \in X$ with respect to the norm $\|\cdot\|_1$ if and only if the sequence converges to x with respect to the norm $\|\cdot\|_2$.

Solution: By definition of the equivalence of norms $\|\cdot\|_1$ and $\|\cdot\|_2$, there exist positive constants α and β such that for every $y \in X$,

$$\alpha\|y\|_1 \;\leq\; \|y\|_2 \;\leq\; \beta\|y\|_1.$$

We will use these inequalities to show the equivalence of convergence under these norms.

(⇒) Suppose that $x_n \to x$ with respect to $\|\cdot\|_1$. This means that

$$\lim_{n\to\infty} \|x_n - x\|_1 = 0.$$

Using the inequality $\|y\|_2 \leq \beta\|y\|_1$, we have

$$\|x_n - x\|_2 \;\leq\; \beta\,\|x_n - x\|_1.$$

Taking the limit as $n \to \infty$ shows

$$\lim_{n\to\infty} \|x_n - x\|_2 \;\leq\; \beta \lim_{n\to\infty} \|x_n - x\|_1 = \beta \cdot 0 = 0.$$

Hence $\|x_n - x\|_2 \to 0$, which implies that $x_n \to x$ with respect to $\|\cdot\|_2$.

(⇐) Conversely, suppose that $x_n \to x$ with respect to $\|\cdot\|_2$. Then

$$\lim_{n\to\infty} \|x_n - x\|_2 = 0.$$

Using the inequality $\alpha\|y\|_1 \le \|y\|_2$, we obtain

$$\alpha \|x_n - x\|_1 \le \|x_n - x\|_2.$$

Taking the limit as $n \to \infty$ gives

$$\alpha \lim_{n\to\infty} \|x_n - x\|_1 \le \lim_{n\to\infty} \|x_n - x\|_2 = 0,$$

which implies

$$\lim_{n\to\infty} \|x_n - x\|_1 = 0.$$

Hence $x_n \to x$ with respect to $\|\cdot\|_1$.

Since we have shown both directions, the convergence of $\{x_n\}$ to x under $\|\cdot\|_1$ is equivalent to the convergence under $\|\cdot\|_2$.

(Note: This solution is AI-written but verified by a human committee member.)

Spring 2025 #5. Let ℓ^∞ be the set of all bounded sequences of real numbers. In other words,

$$\ell^\infty = \{(x_1, x_2, x_3, \dots) \mid \exists M \in \mathbb{R} \text{ such that } x_1, x_2, x_3, \dots \in [-M, M]\}.$$

Here $[-M, M] \subset \mathbb{R}$ denotes the closed interval from $-M$ to M.

Let W be the subset of ℓ^∞ consisting of all "eventually zero" sequences of real numbers. That is, $(x_1, x_2, x_3, \dots) \in W$ if and only if there is an N such that $x_k = 0$ for all $k \ge N$.

(a) Prove that W is a linear subspace of ℓ^∞.

Proof. First, observe that the zero vector in ℓ^∞ is the sequence

$$0 = (0, 0, 0, 0, \dots).$$

Note that this sequence is clearly "eventually zero", so that $0 \in W$.

Next, assume that

$$x = (x_1, x_2, x_3, \ldots)$$

and

$$y = (y_1, y_2, y_3, \ldots)$$

are elements of W. Then there exist $N_1, N_2 \in \mathbb{N}$ such that $x_k = 0$ if $k \geq N_1$, and $y_k = 0$ if $k \geq N_2$. By definition, we have

$$x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3, \ldots)$$

It is easy to see that

$$x_k + y_k = 0$$

whenever $k \geq \max\{N_1, N_2\}$. Thus, the sequence $x+y$ is also eventually zero, so that $x + y \in W$.

Finally, let $c \in \mathbb{R}$. Then

$$cx = (cx_1, cx_2, cx_3, \ldots).$$

Moreover, since $x_k = 0$ whenever $k \geq N_1$, then it must also be the case that $cx_k = 0$ whenever $k \geq N$. Thus, whenever $c \in \mathbb{R}$ and $x \in W$, then $cx \in W$. We can thus conclude that W is a subspace. □

(b) Recall that the standard metric on ℓ^∞ is given by

$$d((x_1, x_2, x_3, \ldots), (y_1, y_2, y_3, \ldots)) = \sup\{|x_i - y_i \mid i \in \mathbb{N}\}.$$

With respect to this metric, is W a closed linear subspace of ℓ^∞? Prove that your answer is correct.

Proof. The space W is not a closed linear subspace of ℓ^∞. To see why, consider the sequence x^m in W defined by

$$(x^m)_n = \begin{cases} \frac{1}{n} & \text{if } n \leq m \\ 0 & \text{if } n > m. \end{cases}$$

It is clear that x^m converges to the sequence

$$x = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right),$$

because

$$d(x^m, x) = \frac{1}{m} \to 0.$$

Note that $x \notin W$, as it is not "eventually zero". Thus W is not closed. $\square$

Spring 2025 #6. Let $d(x, y) = \|x - y\|$ be the standard metric induced by a norm on a real vector space X, and define a new function ρ by:

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

(a) Prove that ρ is a metric on X.
(b) Can ρ be induced by a norm? Justify your answer.

Solution:

(a) Prove that ρ is a metric on X. Recall that a function $\rho : X \times X \to \mathbb{R}$ is a metric if it satisfies the following for all $x, y, z \in X$:

(i) $\rho(x, y) \geq 0$ and $\rho(x, y) = 0$ if and only if $x = y$.
(ii) $\rho(x, y) = \rho(y, x)$ (symmetry).
(iii) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ (triangle inequality).

(i) Nonnegativity and Identity of Indiscernibles.

Since $d(x, y) = \|x - y\| \geq 0$, clearly

$$\rho(x, y) \;=\; \frac{d(x, y)}{1 + d(x, y)} \;\geq\; 0.$$

Furthermore, $\rho(x,y) = 0$ if and only if $d(x,y) = 0$, which occurs if and only if $x = y$. Thus,

$$\rho(x,x) = 0 \quad \text{and} \quad \rho(x,y) = 0 \implies x = y.$$

(ii) Symmetry.

Since $d(x,y) = d(y,x)$ for the usual norm-induced metric, we have

$$\rho(x,y) \;=\; \frac{\|x-y\|}{1+\|x-y\|} \;=\; \frac{\|y-x\|}{1+\|y-x\|} \;=\; \rho(y,x).$$

(iii) Triangle Inequality.

We need to show:

$$\rho(x,z) \;\le\; \rho(x,y) + \rho(y,z).$$

Using the definition of ρ, this is equivalent to:

$$\frac{d(x,z)}{1+d(x,z)} \;\le\; \frac{d(x,y)}{1+d(x,y)} \;+\; \frac{d(y,z)}{1+d(y,z)}.$$

Since $d(x,z) = \|x-z\|$, by the triangle inequality for the norm $\|\cdot\|$, we know

$$d(x,z) \;=\; \|x-z\| \;\le\; \|x-y\| + \|y-z\| \;=\; d(x,y) + d(y,z).$$

Thus

$$d(x,z) \;\le\; d(x,y) + d(y,z).$$

Define $A = d(x,y)$ and $B = d(y,z)$, so $d(x,z) \le A + B$. Then

$$\rho(x,z) = \frac{d(x,z)}{1+d(x,z)} \;\le\; \frac{A+B}{1+(A+B)} \;=\; \frac{A+B}{1+A+B}.$$

We wish to compare this with

$$\rho(x,y) + \rho(y,z) = \frac{A}{1+A} \;+\; \frac{B}{1+B}.$$

It is a known fact (and can be shown by direct algebraic manipulation or by a more general property of such "bounded metrics") that

$$\frac{A+B}{1+A+B} \le \frac{A}{1+A} + \frac{B}{1+B},$$

for all $A, B \ge 0$. A straightforward approach is to set up the difference

$$\left(\frac{A}{1+A} + \frac{B}{1+B}\right) - \frac{A+B}{1+A+B},$$

and verify it is nonnegative by putting everything over a common denominator. (See below for more details.)

Hence

$$\rho(x,z) = \frac{d(x,z)}{1+d(x,z)} \le \frac{A+B}{1+A+B} \le \frac{A}{1+A} + \frac{B}{1+B} = \rho(x,y) + \rho(y,z).$$

This completes the verification that ρ satisfies the triangle inequality.

Therefore, ρ is a metric on X.

(b) Can ρ be induced by a norm? To be induced by a norm $\|\cdot\|_*$ on X, we would need

$$\rho(x,y) = \|x-y\|_*$$

for some norm $\|\cdot\|_*$. A standard requirement for a norm is homogeneity:

$$\|tv\|_* = |t|\,\|v\|_* \quad \text{for any scalar } t \text{ and vector } v.$$

If ρ were induced by a norm, we would expect a relationship of the form

$$\rho(tx,ty) = |t|\,\rho(x,y) \quad \text{for all } t \in \mathbb{R}.$$

However, from the definition

$$\rho(x,y) = \frac{\|x-y\|}{1+\|x-y\|},$$

if we scale x and y by $t \neq 0$, we get

$$\rho(tx,ty) = \frac{\|tx-ty\|}{1+\|tx-ty\|} = \frac{|t|\cdot\|x-y\|}{1+|t|\cdot\|x-y\|}.$$

In general,

$$\frac{|t| \cdot \|x-y\|}{1+|t| \cdot \|x-y\|} \neq |t| \cdot \frac{\|x-y\|}{1+\|x-y\|}.$$

To exhibit a specific counterexample: Because $X \neq 0$, we know that X contains a unit vector x. (To see that, take $0 \neq v \in X$, then let $x = v/||v||$.) Let $y = 0$ and $t = 2$. Substituting into the expressions above gives us

$$\frac{|t| \cdot \|x-y\|}{1+|t| \cdot \|x-y\|} = \frac{2}{1+2} \neq 2 \cdot \frac{1}{2} = |t| \cdot \frac{\|x-y\|}{1+\|x-y\|}.$$

This shows that ρ does *not* satisfy the necessary homogeneity for being induced by a norm.

(Note: This solution was AI-written but then edited by a human committee member.)

Answer:

(a) ρ is a metric because it is nonnegative, equals zero only when $x = y$, is symmetric, and satisfies the triangle inequality (by leveraging the fact that $d(x, y)$ itself is a metric).

(b) ρ cannot be induced by a norm since it does not scale linearly under scalar multiplication of vectors (fails the homogeneity condition).

Spring 2025 #7. Let $f(t) = |t|$ for $t \in [-\pi, \pi]$, and extend it to be 2π-periodic on $\mathbb{R}$.

(a) Prove that f is an even function.

(b) Find the Fourier series of $f(t)$ in trigonometric form.
Hint: Use (a).

Answer:

(a) By definition, $f(t)$ is an even function if

$$f(-t) = f(t).$$

This holds for $f(t) = |t|$ because

$$f(-t) = |-t| = |t| = f(t).$$

(b) Since $f(t)$ is even, then $f(t)\sin(nt)$ is odd, so that

$$\int_{-\pi}^{\pi} f(t)\sin(nt)\ dt = 0.$$

Thus, all the Fourier coefficients corresponding to the sine terms are zero.

For the coefficients corresponding to the cosine terms, we observe that

$$\begin{aligned} c_0 &= \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}}|t|\ dt \\ &= \frac{\pi^{\frac{3}{2}}}{\sqrt{2}} \end{aligned}$$

and, for $n \geq 1$,

$$\begin{aligned} c_n &= \int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}}|t|\cos(nt)\ dt \\ &= \frac{2\left(\pi n \sin\left(\pi n\right) + \cos\left(\pi n\right) - 1\right)}{\sqrt{\pi}\, n^2} \\ &= \frac{2\left((-1)^n - 1\right)}{\sqrt{\pi}\, n^2}. \end{aligned}$$

Thus, the Fourier series for $f(t)$ is

$$f(t) \sim \frac{\pi^{\frac{3}{2}}}{\sqrt{2}} + \sum_{n=1}^{\infty} \frac{2\left((-1)^n - 1\right)}{\sqrt{\pi}\, n^2}\cos(nt).$$