Math 5800

9/20/21

- In HW 4 on step functions I added parts (d) and (e) to problem 7.

[In case you already downloaded this file.]

Theorem: [Thm 1.3.2 in WJ book]

Let f and g be step functions and $a, b \in \mathbb{R}$.

Then

① $af + bg$ is a step function

and ② $\int (af + bg) = a \int f + b \int g$

proof: Let $f = \sum_{j=1}^{n} c_j \cdot \chi_{I_j}$

and $g = \sum_{i=1}^{m} k_i \cdot \chi_{J_i}$ where

$c_1, c_2, \ldots, c_n, k_1, k_2, \ldots, k_m \in \mathbb{R}$

and $I_1, I_2, \ldots, I_n, J_1, J_2, \ldots, J_m$

are bounded intervals.

Then,

$$af+bg = a\left(\sum_{j=1}^{n} c_j \cdot \chi_{I_j}\right) + b\left(\sum_{i=1}^{m} k_i \cdot \chi_{J_i}\right)$$

$$= \sum_{j=1}^{n} (ac_j)\chi_{I_j} + \sum_{i=1}^{m} (bk_i) \cdot \chi_{J_i}$$

which is a step function. Furthermore,

$$\int af+bg =$$

$$= \int \left[\sum_{j=1}^{n} (ac_j)\chi_{I_j} + \sum_{i=1}^{m} (bk_i) \cdot \chi_{J_i}\right]$$

$$= \sum_{j=1}^{n} (ac_j) \cdot \ell(I_j) + \sum_{i=1}^{m} (bk_i) \cdot \ell(J_i)$$

$$= a\left[\sum_{j=1}^{n} c_j \cdot \ell(I_j)\right] + b\left[\sum_{i=1}^{m} k_i \cdot \ell(J_i)\right]$$

$$= a\int f + b\int g \qquad \blacksquare$$

Theorem [Thm 1.3.3 in WJ book]

Let f and g be step functions where $f(x) \geq g(x)$ for all $x \in \mathbb{R}$.
[Sometimes written as $f \geq g$]

Then, $\int f \geq \int g$.

proof:

By the previous theorem, $f-g$ is a step function.

Thus, $f - g = \sum_{j=1}^{n} c_j \cdot \chi_{I_j}$

where $c_j \in \mathbb{R}$ and the intervals $I_1, I_2, \ldots, I_n$ are disjoint and bounded.

by previous thm

Since $(f-g)(x) \geq 0$ for all x and the intervals are disjoint we know that $c_j \geq 0$ for all j.

Thus,

$$\int (f-g) = \sum_{j=1}^{n} \underbrace{c_j}_{\geq 0} \cdot \underbrace{\ell(I_j)}_{\geq 0} \geq 0$$

So, by the previous theorem

$$\int f - \int g = \int (f-g) \geq 0.$$

Thus,

$$\int f \geq \int g$$

$\blacksquare$

Def: [Weir book]

Let $\mathcal{R}$ denote the set of all subsets of $\mathbb{R}$ such that S can be written as

$$\boxed{S = I_1 \cup I_2 \cup \cdots \cup I_r} \quad (*)$$

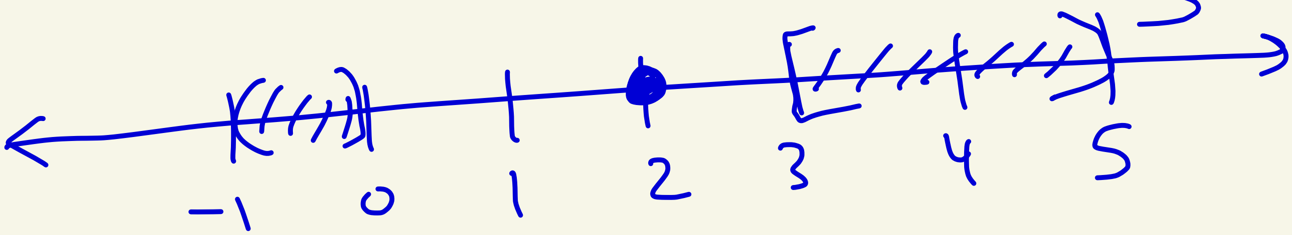
where $I_1, I_2, \ldots, I_r$ are disjoint bounded intervals.

Given $S \in \mathcal{R}$ where $(*)$ is true, define

$$\ell(S) = \ell(I_1) + \ell(I_2) + \ldots + \ell(I_r)$$

Ex:

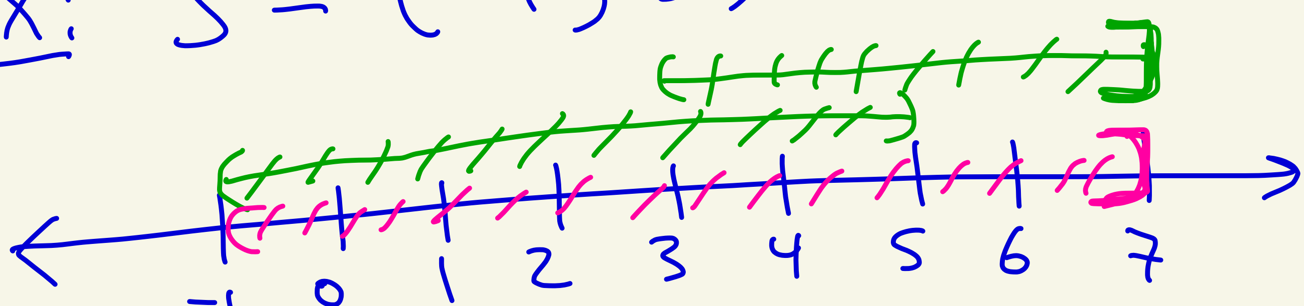
$S = [3,5) \cup [2,2] \cup (-1,0) \in \mathcal{R}$



$\ell(S) = \ell([3,5)) + \ell([2,2]) + \ell((-1,0))$

$= 2 + 0 + 1 = 3$

Ex: $S = (-1,5) \cup (3,7]$



So, $S = (-1,7] \in \mathcal{R}$

$\ell(S) = 8$

Note: Is $\ell(S)$ for $S \in \mathcal{R}$ well-defined?

Let $S \in \mathcal{R}$ with

$$S = I_1 \cup I_2 \cup \cdots \cup I_r$$

where $I_1, I_2, \ldots, I_r$ are disjoint bounded intervals.

By HW 4 problem 4, we have that

$$\chi_S = \chi_{I_1} + \chi_{I_2} + \cdots + \chi_{I_r}$$

and thus

$$\int \chi_S = \int \chi_{I_1} + \int \chi_{I_2} + \cdots + \int \chi_{I_r}$$

$$= \ell(I_1) + \ell(I_2) + \cdots + \ell(I_r)$$

So we could have defined

$$\ell(S) = \int \chi_S$$

Since

$$\chi_S = \chi_{I_1} + \chi_{I_2} + \dots + \chi_{I_r}$$

is a step function it doesn't matter how we decompose S into disjoint bounded intervals, we will always get the same value for $\int \chi_S$ by our theorems on step functions.

Hence $\ell(S)$ for $S \in \mathcal{R}$ is well-defined.

$\boxed{\text{Note}}$

HW 4 problem 7

(a) $S \in \mathcal{R}$ iff χ_S is a step function.

(b) If $S, T \in \mathcal{R}$, then $S \cup T$, $S \cap T$, $S - T$ are all in $\mathcal{R}$

(c) If $S, T \in \mathcal{R}$ and $S \subseteq T$, then $\ell(S) \leq \ell(T)$.

(d) If $A = \bigcup_{i=1}^{s} A_i$ where each A_i is a bounded interval.
Then $A \in \mathcal{R}$

(e)

(e) Let $I_1, I_2, \dots, I_r$ be disjoint bounded intervals. Suppose that there exist bounded intervals $J_1, J_2, \dots, J_t$ where

$$\bigcup_{j=1}^{r} I_j \subseteq \bigcup_{i=1}^{t} J_i$$

Then

$$\sum_{j=1}^{r} \ell(I_j) \leq \sum_{i=1}^{t} \ell(J_i)$$

Def: Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions where $f_n : \mathbb{R} \to \mathbb{R}$ for each $n \geq 1$.

Think of it as

$$f_1, f_2, f_3, f_4, \ldots$$

We say that $(f_n)_{n=1}^{\infty}$ is **non-decreasing** if

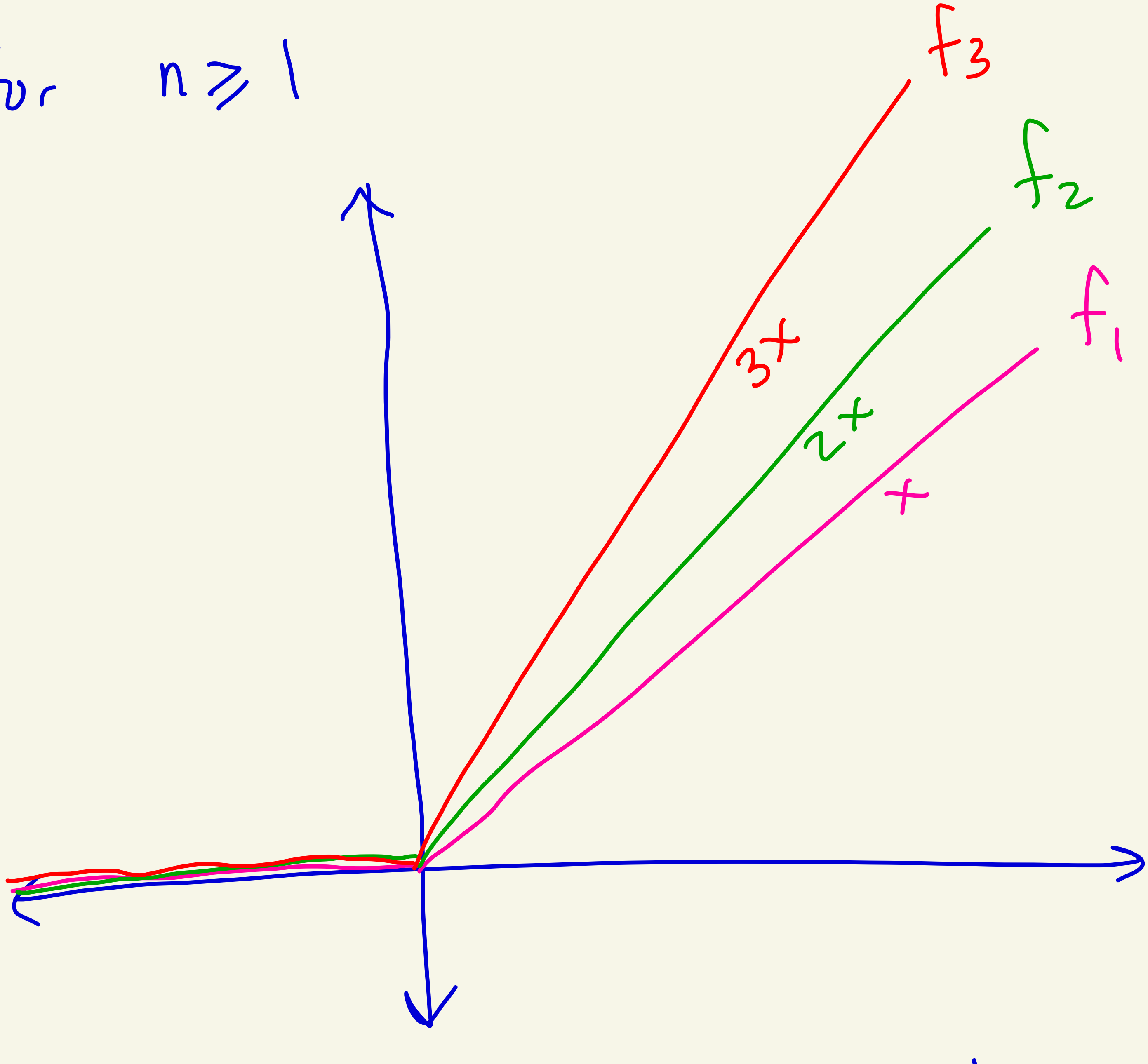
$$f_n(x) \leq f_{n+1}(x)$$

for all $n \geq 1$ and all $x \in \mathbb{R}$.

Ex: Let

$$f_n(x) = \begin{cases} nx & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

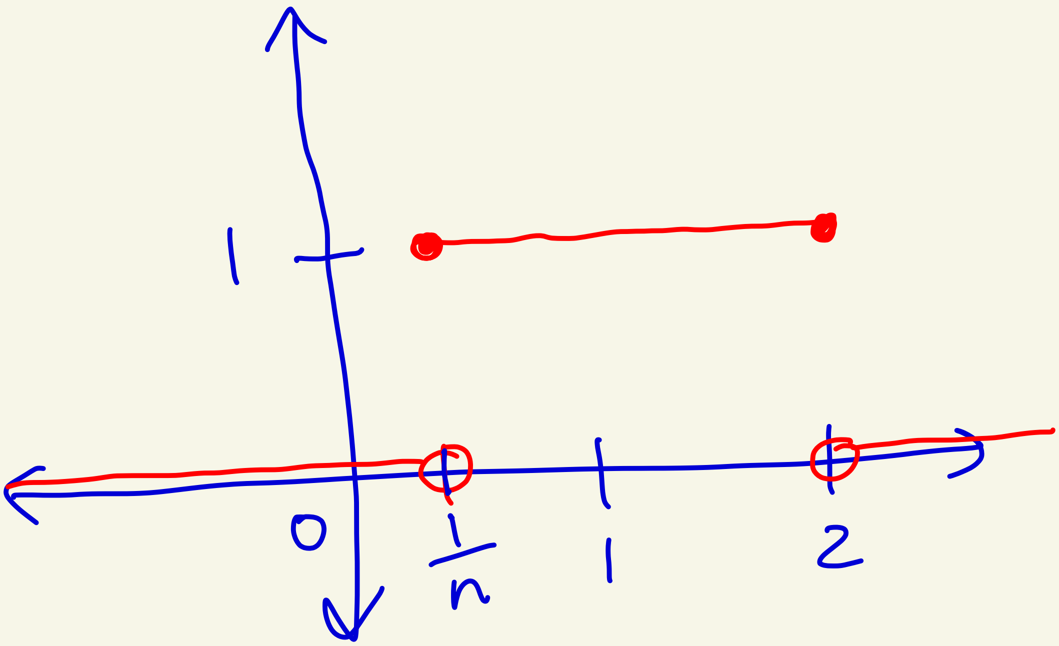
for $n \geq 1$



Then, $(f_n)_{n=1}^{\infty}$ is a non-decreasing sequence of functions.

Ex:

Let $\varphi_n = \chi_{[\frac{1}{n}, 2]}$



HW 4 - #2
If $S \subseteq T$, then $\chi_S(x) \leq \chi_T(x)$ for all x

Thus for all $x \in \mathbb{R}$ we have

$$\varphi_n(x) = \chi_{[\frac{1}{n}, 2]}(x) \leq \chi_{[\frac{1}{n+1}, 2]}(x) = \varphi_{n+1}(x)$$

$$\boxed{[\tfrac{1}{n}, 2] \subseteq [\tfrac{1}{n+1}, 2]}$$

for any $n \geq 1$.

Thus, $(\varphi_n)_{n=1}^{\infty}$ is a non-decreasing sequence of step functions.

Note,

$$\int \varphi_n = \int \chi_{[\frac{1}{n}, 2]}$$

$$= 1 \cdot \ell\left(\left[\frac{1}{n}, 2\right]\right)$$

$$= 2 - \frac{1}{n}$$

So, $$\lim_{n\to\infty} \int \varphi_n = \lim_{n\to\infty}\left(2 - \frac{1}{n}\right)$$

$$= 2 - 0$$

$$= 2 \qquad \blacksquare$$