Math 5800

11/29/21

HW 8 #2

In the solutions I had

"$f = f \cdot \chi_{[0,4]}$ almost everywhere

(except at $x = 0, 4$)"

It should say

"$f = f \cdot \chi_{[0,4]}$ everywhere"

I emailed about test 2

Plan:

M	W
Finish Topic 9	Topic 10
Topic 10	Questions / Review
	Final

Theorem: Let $f: \mathbb{R} \to \mathbb{R}$.
Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions where $\lim_{n\to\infty} f_n(x) = f(x)$ for almost all x in $\mathbb{R}$. Then f is measurable.

proof: We need to show that $\text{mid}\{-g, f, g\} \in L^1$ for all $g \in L^1$ with $g \geq 0$.

Let $g \in L^1$ with $g \geq 0$.
Let $h = \text{mid}\{-g, f, g\}$
Let $h_n = \text{mid}\{-g, f_n, g\}$.

We saw previously that

$$|h_n(x)| = |\text{mid}\{-g(x), f_n(x), g(x)\}|$$
$$\leq |g(x)|$$

for all $x \in \mathbb{R}$ and $n \geq 1$.

Since f_n is measurable for each $n \geq 1$, we know $h_n = \text{mid}\{-g, f_n, g\}$ is in L^1 for each $n \geq 1$.

By HW 9 #7, since $f_n \to f$ almost everywhere,

$$h_n = \text{mid}\{-g, f_n, g\} \longrightarrow h = \text{mid}\{-g, f, g\}$$

almost everywhere.

Thus,

- $(h_n)_{n=1}^{\infty}$ is a sequence of L^1 functions
- $h_n \to h$ almost everywhere
- $|h_n(x)| \leq g(x)$ for all $x \in \mathbb{R}$ and $n \geq 1$ where $g \in L^1$.

By the dominated convergence theorem, $h = \text{mid}\{-g, f, g\} \in L^1$.

Thus, f is measurable. $\blacksquare$

Theorem:

Let $f: \mathbb{R} \to \mathbb{R}$ and

$E = \{ x \in \mathbb{R} \mid f \text{ is discontinuous at } x \}$

If E has measure zero,

then f is measurable.

Proof: HW 9. $\blacksquare$

Topic 10 - Measurable sets

<u>Def:</u> Let $E \subseteq \mathbb{R}$.
We say that E is <u>measurable</u> if χ_E is a measurable function [ie, $\chi_E \in \widetilde{M}$].
We denote the set of measurable sets by M.

We say that E is <u>integrable</u> if χ_E is an integrable function [ie, $\chi_E \in L^1$].

Ex: Let $E = (-2, 5]$

$\chi_E = \chi_{(-2,5]}$ is a step function, and so $\chi_E \in L^1$.

Thus, $E = (-2, 5]$ is an integrable set.

Since $L^1 \subseteq \tilde{M}$, χ_E is also a measurable function, so $E = (-2, 5]$ is a measurable set.

So, $E \in M$.

<u>Ex:</u> $E = \mathbb{R}$

We showed previously that
$\chi_E \notin L^1$ but $\chi_E \in \tilde{M}$.
So, E is not integrable
but E is measurable.

<u>Theorem:</u> Let $E \subseteq \mathbb{R}$.
If E is integrable, then E is measurable.

<u>proof:</u> Suppose E is integrable.
Then $\chi_E \in L^1$.
Since $L^1 \subseteq \tilde{M}$, we have χ_E is measurable.
Thus, E is measurable. $\boxed{}$

Def:

Let $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$

where ∞ is thought of as a symbol with the following properties:

- $a < \infty$ for all $a \in \mathbb{R}$
- $a + \infty = \infty$ for all $a \in \mathbb{R}$
- $\infty + \infty = \infty$ for all $a \in \mathbb{R}$
- If $\sum_{n=1}^{\infty} a_n$ is an infinite sum with $a_n \in \mathbb{R}_\infty$ and $a_n \geq 0$ for all $n \geq 1$, then if $a_n = \infty$ then $\sum_{n=1}^{\infty} a_n = \infty$.

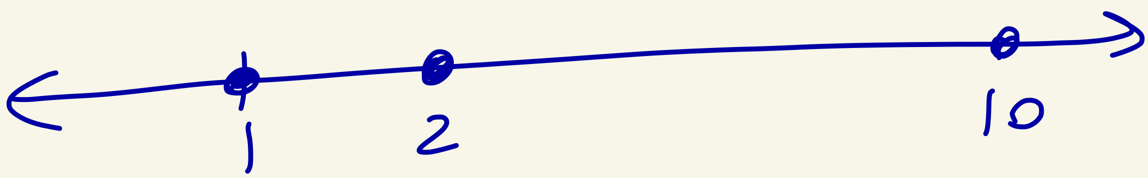
Def: The Lebesgue measure is defined as $\mu : M \to \mathbb{R}_\infty$

where

$$\mu(E) = \begin{cases} \int \chi_E & \text{if } E \text{ is integrable} \\ \infty & \text{if } E \text{ is not integrable} \end{cases}$$

μ generalizes "length"

Ex: Let $E = \{1, 2, 10\}$



$\chi_E = \chi_{[1,1]} + \chi_{[2,2]} + \chi_{[10,10]}$ is a step function, so $\chi_E \in L^1$.

Thus, $\mu(E) = \int \chi_E$

$$= \int \chi_{[1,1]} + \chi_{[2,2]} + \chi_{[10,10]}$$

$$= (1-1) + (2-2) + (10-10)$$

$$= 0$$

Ex: $E = (1, 10]$

$\chi_E = \chi_{(1,10]}$ is a step function, so $\chi_E \in L^1$. Thus, $\mu(E) = \int \chi_E$

$$= 10 - 1 = 9$$

Ex: $E = \mathbb{R}$

E is measurable from earlier.

$\chi_E \notin L^1$, ie E is not integrable.

Thus,

$$\mu(E) = \mu(\mathbb{R}) = \infty$$

Theorem: Let $E \subseteq \mathbb{R}$.

E has measure zero [Topic 3 def]

iff E is measurable and

$\mu(E) = 0$.

proof:

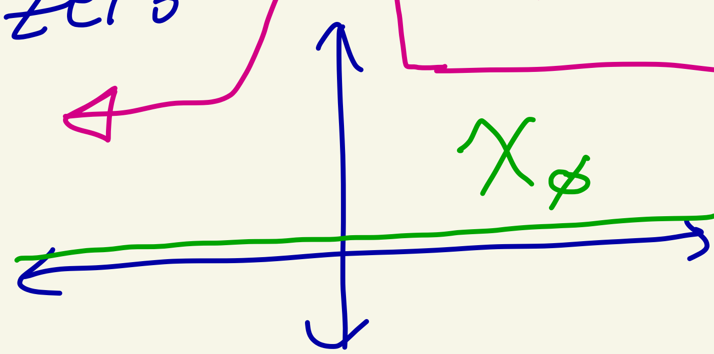
($\Rightarrow$) Suppose E has measure zero as defined in topic 3.

Then,

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

$\chi_E(x) = \chi_\emptyset(x)$ $\forall x \notin E$

is equal to the zero function $\chi_\emptyset$ almost everywhere.



Since $\chi_\phi \in L^1$ and $\chi_E = \chi_\phi$ almost everywhere, we know

$\chi_E \in L^1$ and $\int \chi_E = \int \chi_\phi = 0$

Thus, E is integrable and hence measurable.

And, $\mu(E) = \int \chi_E = \int \chi_\phi = 0.$

($\Leftarrow$) Now suppose E is a measurable set and $\mu(E) = 0$.

By def of μ, since $\mu(E) = 0$ we know E is integrable and $\mu(E) = \int \chi_E = 0.$

Since E is integrable, $\chi_E \in L^1$.

Thus, $\chi_E \in L^1$ and $\chi_E \geq 0$ [pg 15]

and $\int \chi_E = 0$.

By a theorem in class during topic 7, this implies that

$\chi_E(x) = 0$ for almost all x.

That is,

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

is 0 almost everywhere.

Thus, $\mathbb{R} - E$ is an almost everywhere set.

Thus, E has measure zero [Topic 3 def].

$\blacksquare$