Math 5800

10/27/21

Test 2 - HW 6 & HW 7

Monday Nov. 15

<u>Corollary:</u> Let $f, g \in L^1$.

Then:

① $|f-g| \in L^1$

② If $\int |f-g| = 0$, then

$f = g$ almost everywhere.

<u>proof:</u>

① Since $f, g \in L^1$ we know

$f - g \in L^1$.

By HW 9 #5(b),

$|f-g| \in L^1$.

$$\boxed{\underline{\text{Note:}}\ |f-g|(x) = |f(x)-g(x)|}$$

(2) $|f-g| \geq 0$ on all of $\mathbb{R}$ and $|f-g| \in L^1$.

Suppose $\int |f-g| = 0$.

From Monday, this implies that $|f-g| = 0$ almost everywhere.

So, $|f(x)-g(x)| = 0$ for almost all x.

Recall: $|A-B| = 0$ iff $A = B$

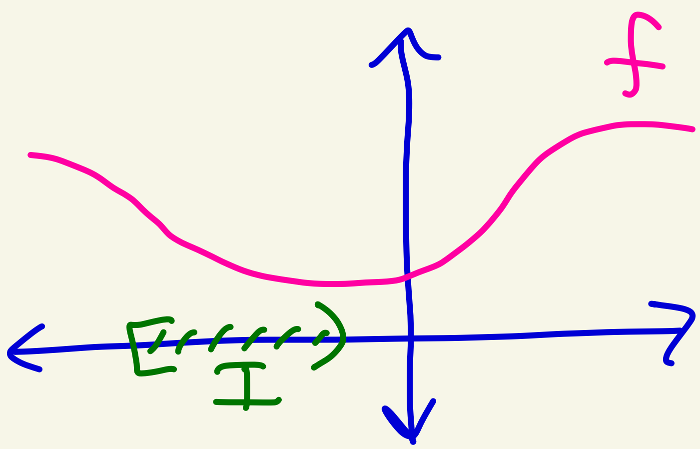
So, $f(x) = g(x)$ for almost all x. $\blacksquare$

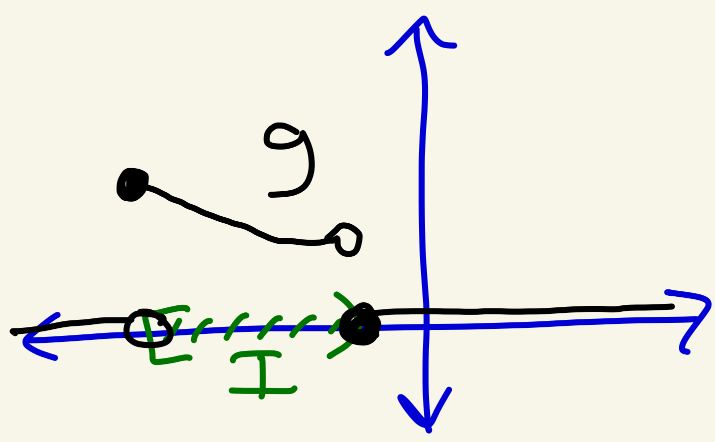
Notation: The Lebesgue integral is over the entire real line. So if $f \in L^1$, then we sometimes write

$$\int f = \int_{\mathbb{R}} f = \int_{-\infty}^{\infty} f(x)\,dx$$

Def: Let $I \subseteq \mathbb{R}$ be any interval (possibly unbounded). Let $f: \mathbb{R} \to \mathbb{R}$. Define $g = \chi_I \cdot f$, that is

$$g(x) = \begin{cases} f(x) & \text{if } x \in I \\ 0 & \text{if } x \notin I \end{cases}$$





If $g \in L^1$, then we define

$$\int_I f = \underbrace{\int g}_{\text{Lebesgue integral over all of } \mathbb{R}}$$

and we say that $f \in L^1(I)$ and f is <u>integrable on I</u>

We define

$$L^1(I) = \{ f : \mathbb{R} \to \mathbb{R} \mid f \cdot \chi_I \in L^1 \}$$

Note:

$f \cdot \chi_{(a,b)}$, $f \cdot \chi_{[a,b)}$,

$f \cdot \chi_{(a,b]}$, $f \cdot \chi_{[a,b]}$

are all equal to each other almost everywhere. So if one of them is in L^1 then they all are and

$$\int_{(a,b)} f = \int_{[a,b)} f = \int_{(a,b]} f = \int_{[a,b]} f$$

We can denote these integrals

by $\int_a^b f$ or $\int_a^b f(x)\,dx$

Notation:

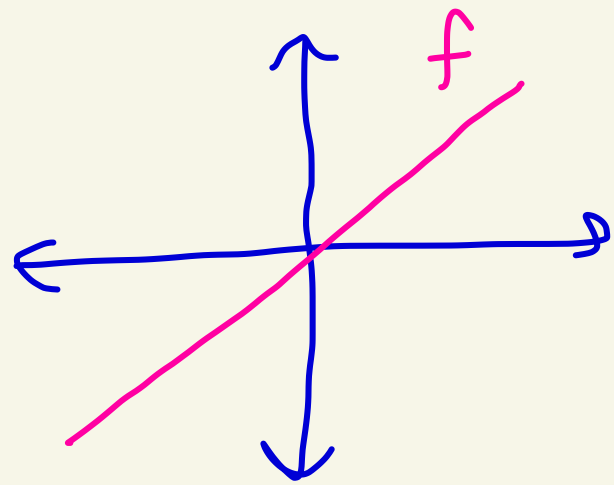
$\int_{[a,\infty)} f$ and $\int_{(a,\infty)} f$ are sometimes

written as $\int_a^\infty f$

$\int_{(-\infty,a]} f$ and $\int_{(-\infty,a)} f$ are sometimes

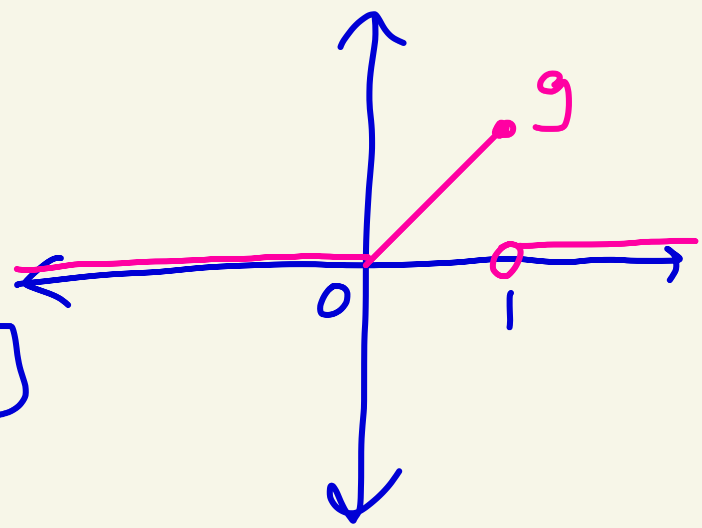
written as $\int_{-\infty}^a f$

Ex: Let $f:\mathbb{R}\to\mathbb{R}$ be $f(x)=x$ for all x.



Let's show that $f\in L^1([0,1])$.

Let $g=\chi_{[0,1]}\cdot f$



Then,

$$g(x)=\begin{cases} f(x) & \text{if } x\in[0,1] \\ 0 & \text{otherwise}\end{cases}$$

We saw earlier in the class that $g\in L^0\subseteq L^1$ and $\int g=\frac{1}{2}$.

So, $f\in L^1([0,1])$ and

$$\int_{[0,1]} f=\int_0^1 f=\int g=\frac{1}{2}$$

Theorem: Let $a \leq c \leq b$ where $a, b, c \in \mathbb{R}$.

If $f \in L^1([a,c])$ and $f \in L^1([c,b])$, then $f \in L^1([a,b])$ and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

proof: HW 7 # 6

$\blacksquare$

HW 7

④ Let $f \in L^0$ and $g: \mathbb{R} \to \mathbb{R}$.
Suppose $f = g$ almost everywhere.
Then, ① $g \in L^0$
and ② $\int g = \int f$

proof:
We are given that there exists an almost everywhere set $A \subseteq \mathbb{R}$ where $f(x) = g(x)$ for all $x \in A$.
Since $f \in L^0$ there exists a non-decreasing sequence of step functions $(\varphi_n)_{n=1}^{\infty}$ where $\lim_{n\to\infty} \varphi_n(x) = f(x)$ for all $x \in B$
where B is almost everywhere set and

$\lim_{n\to\infty} \int \varphi_n$ converges, and

$$\int f = \lim_{n\to\infty} \int \varphi_n .$$

Since A and B are almost everywhere sets, $A \cap B$ is an almost everywhere set.

If $x \in A \cap B$, then

$$\lim_{n\to\infty} \varphi_n(x) \underset{x \in B}{=} f(x) \underset{x \in A}{=} g(x)$$

Thus, $(\varphi_n)_{n=1}^{\infty}$ is a non-decreasing sequence of step functions with $\varphi_n \to g$ almost everywhere.

Since $\left(\int \varphi_n\right)_{n=1}^{\infty}$ converges, $g \in L^0$ and

$$\int g = \lim_{n\to\infty} \int \varphi_n = \int f. \quad \blacksquare$$