Math 5800

10/25/21

# Recall:

$f \in L^1$ means: $f = g - h$
where $g, h \in L^0$

define $\underbrace{\int f}_{\substack{L^1 \\ \text{integral}}} = \underbrace{\int g}_{\substack{L^0 \\ \text{integral}}} - \underbrace{\int h}_{\substack{L^0 \\ \text{integral}}}$

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$L^0 \subseteq L^1$

**Theorem:** [WJ book Thm 1.5.3]

If $f \in L^1$ and $f(x) \geq 0$ for almost all $x$, then $\int f \geq 0$.

**proof:** Since $f \in L^1$ we know that $f = g - h$ where $g, h \in L^0$.

We know $f(x) \geq 0$ for almost all $x$.

Thus, $g(x) - h(x) \geq 0$ for almost all $x$.

So, $g(x) \geq h(x)$ for almost all $x$.

By our theorems on $L^0$ we have that $\underbrace{\int g}_{L^0 \text{ integrals}} \geq \underbrace{\int h}_{L^0 \text{ integrals}}$.

Thus, $\int g \geq \int h$.

So, $\int g - \int h \geq 0$.

Thus, $\int f \geq 0$. $\blacksquare$

<u>Corollary:</u> Let $f, g \in L^1$.
If $f(x) \geq g(x)$ for almost all $x$,
then $\int f \geq \int g$.

<u>proof:</u> Let $h = f - g$.
By a previous theorem, $h \in L^1$.
Since $f(x) \geq g(x)$ for almost all $x$,
we know that
$h(x) = f(x) - g(x) \geq 0$
for almost all $x$.
By the previous theorem $\int h \geq 0$.
We know $\int h = \int f - g \underset{\boxed{f, g \in L^1}}{=} \int f - \int g$.

Thus, $\int f - \int g \geq 0$.

So, $\int f \geq \int g$. ■

**Theorem:** Let $f \in L^0$.

Let $g: \mathbb{R} \to \mathbb{R}$ be any function.

If $f(x) = g(x)$ for almost all $x$, then the following is true:

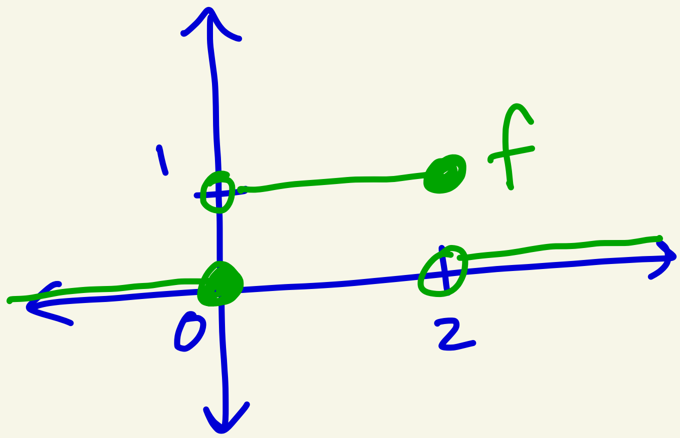
① $g \in L^0$

and ② $\int g = \int f$

**proof:** Homework. $\blacksquare$

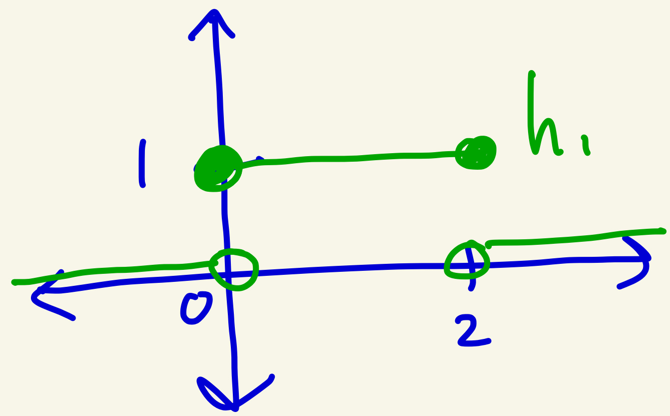
Ex: Let $f = \chi_{(0,2]}$.

Then, $f \in L^0$ and $\int f = 1 \cdot (2-0) = 2$




Let $h_1(x) = \chi_{[0,2]}$

Then, $h_1(x) = f(x)$
for $x \in \mathbb{R} - \{1\}$.



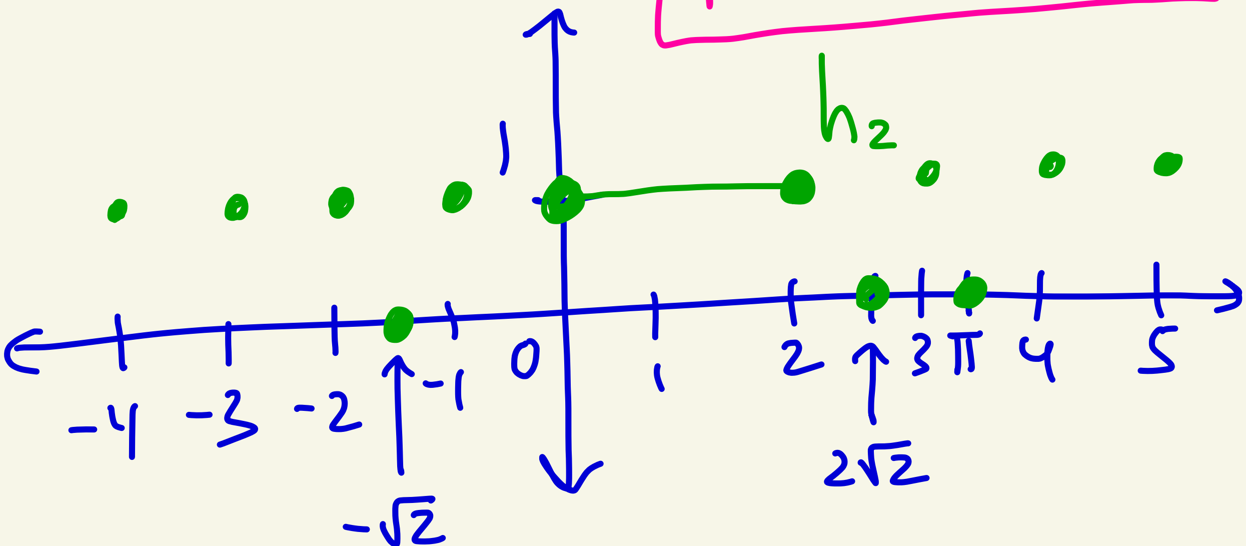

So, $h_1 = f$ almost everywhere.

So, $h_1 \in L^0$ and $\int h_1 = \int f = 2$.

Let

$$h_2(x) = \begin{cases} 1 & \text{if } x \in (0,2] \\ 1 & \text{if } x \in \mathbb{Q} \text{ and } x \notin (0,2] \\ 0 & \text{if } x \notin \mathbb{Q} \text{ and } x \notin (0,2] \end{cases}$$

partial picture of $h_2$




We have that $h_2(x) \neq f(x)$ iff

$$x \in \underbrace{\mathbb{Q} \cap \Big[ (-\infty, 0] \cup (2, \infty) \Big]}_{}$$

has measure zero since its a subset of $\mathbb{Q}$ which has measure zero.

So, $h_2(x) = f(x)$ for almost all $x$.

Thus, $h_2 \in L^0$ and $\int h_2 = \int f = 2$

$\boxtimes$

**Theorem:** Let $f \in L^1$.
Let $g : \mathbb{R} \to \mathbb{R}$ be any function.
If $f = g$ almost everywhere, then the following are true:

① $g \in L^1$

and ② $\int g = \int f$

**proof:** Since $f \in L^1$ we know that $f = a - b$ where $a, b \in L^0$.

And $\int f = \int a - \int b$.

Note that

$$g = f - f + g = a - b - f + g$$
$$= a - \underbrace{(b + f - g)}_{\text{We will show this is in } L^0}$$

$a$: We know is in $L^0$

Since $f=g$ almost everywhere, there exists an almost everywhere set $A$ where $f(x)=g(x)$ for all $x\in A$.

Thus, $f(x)-g(x)=0$ for all $x\in A$.

Thus, $b(x)+f(x)-g(x)=b(x)$ for all $x\in A$.

So, $b+f-g=b$ almost everywhere.

By the previous theorem, since $b\in L^0$ we know $b+f-g\in L^0$ and $\int(b+f-g)=\int b$.

Summarizing, both $a\in L^0$ and $b+f-g\in L^0$.

Thus, $g=a-(b+f-g)\in L^1$.

And, $\int g \overset{\text{def of } \int g}{=} \int a - \int(b+f-g) = \int a - \int b$

$= \int f.$

$\blacksquare$

**Theorem:** (Monotone convergence thm)

Let $(f_n)_{n=1}^{\infty}$ be a non-decreasing sequence of $L^1$ functions [ie, $f_n \in L^1$ for all $n \geq 1$]

Suppose that $\left(\int f_n\right)_{n=1}^{\infty}$ is a bounded sequence.

Then, $\lim_{n\to\infty} f_n(x)$ converges for almost all $x$.

Moreover, if $f: \mathbb{R} \to \mathbb{R}$ where
$f(x) = \lim_{n\to\infty} f_n(x)$ for almost all $x$,

then $f \in L^1$ and $\int f = \lim_{n\to\infty} \int f_n$

**proof:** Handout in email. $\blacksquare$

<u>Corollary:</u> Let $f \in L^1$ and $f \geq 0$ almost everywhere.
[ie, $f(x) \geq 0$ for almost all $x$]
If $\int f = 0$, then $\underbrace{f = 0 \text{ almost everywhere}}$

<u>means:</u> $f(x) = 0$ for almost all $x$

<u>proof:</u> Define $f_n = n \cdot f$ for $n \geq 1$.
So, $f_n(x) = n \cdot f(x)$ for all $x$.
By a thm in class, $f_n \in L^1$.
Then for all $x \in \mathbb{R}$ we have that

$$\underbrace{1 \cdot f(x)}_{f_1(x)} < \underbrace{2 \cdot f(x)}_{< \; f_2(x)} < \underbrace{3 \cdot f(x)}_{< \; f_3(x)} < \cdots \; < \cdots$$

So, $(f_n)_{n=1}^{\infty}$ is a non-decreasing sequence of $L^1$ functions.

Also, $\int f_n = \int n \cdot f = n \int f = n \cdot 0 = 0$.

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So the sequence $\left(\int f_n\right)_{n=1}^{\infty}$ is

$$0, \quad 0, \quad 0, \quad 0, \quad 0, \ \ldots$$

$$\int f_1 \quad \int f_2 \quad \int f_3 \quad \int f_4 \quad \int f_5 \ \ldots$$

So, $\left(\int f_n\right)_{n=1}^{\infty}$ is bounded.

By the monotone convergence theorem,

$$\lim_{n\to\infty} n \cdot f(x) = \lim_{n\to\infty} f_n(x)$$

converge for almost all $x$.

Note if $y \in \mathbb{R}$ and $y \neq 0$, then

$$\lim_{n\to\infty} n \cdot y = \infty.$$

Thus, the only times that $\lim\limits_{n\to\infty} n \cdot f(x)$ converge is when $f(x) = 0$.

So, $f(x) = 0$ for almost all $x$. $\blacksquare$