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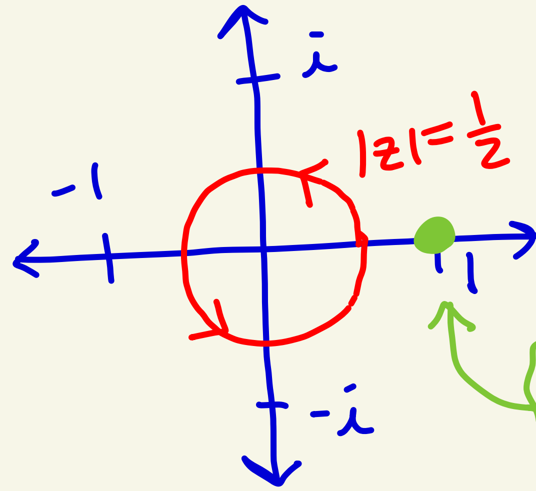
HW 5

Solutions

① (a)

$$\int_{|z|=\frac{1}{2}} \frac{dz}{(1-z)^3}$$

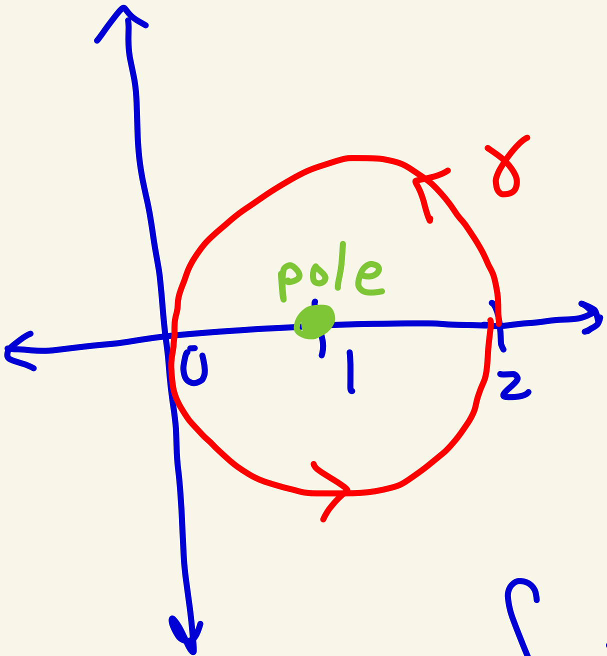


pole of $\frac{1}{(1-z)^3}$ at 1

The function $\frac{1}{(1-z)^3}$ is analytic on $\mathbb{C}-\{1\}$. So, $\frac{1}{(1-z)^3}$ is analytic on and inside the curve $|z|=\frac{1}{2}$

Thus by Cauchy's thm (math 4680) we know $\displaystyle\int_{|z|=\frac{1}{2}} \frac{dz}{(1-z)^3} = 0$

① (b) The function $f(z) = \frac{1}{(1-z)^3}$



is analytic on $\mathbb{C} - \{1\}$ and has a pole at 1.

Thus, by the residue thm

$$\int_\gamma \frac{dz}{(1-z)^3} = 2\pi i \, \mathrm{Res}(f;1)$$

Note that $f(z) = \frac{1}{(1-z)^3} = \frac{1}{(-1)^3 (z-1)^3} = \frac{-1}{(z-1)^3}$

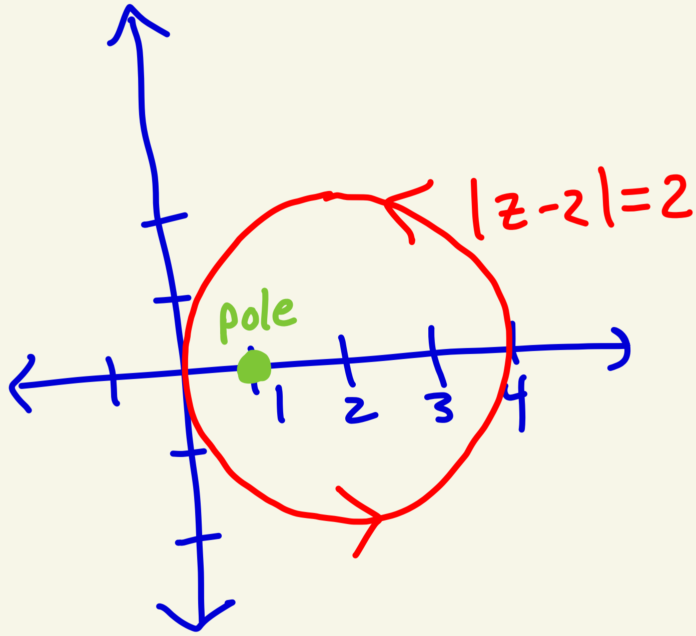
Laurent expansion of f at $z_0 = 1$ and the $\frac{1}{(z-1)}$ term is missing.

So, $b_{-1} = 0$. Thus,

$$\int_\gamma \frac{dz}{(1-z)^3} = 0$$

① (c)

Let $f(z) = \dfrac{e^z}{(1-z)^3}$



Then f is analytic on $\mathbb{C} - \{1\}$ where it has a pole at 1.

Note that

$$f(z) = \frac{e^z}{(1-z)^3} = \frac{e^z}{(-1)^3(z-1)^3} = \frac{-e^z}{(z-1)^3}$$

$$= \frac{\varphi(z)}{(z-1)^3} \quad \text{where } \varphi(z) = -e^z$$

is analytic at $z=1$ and $\varphi(1) \neq 0$.

This gives a pole of order 3 at $z_0 = 1$ and $\operatorname{Res}(f;1) = \dfrac{\varphi^{(3-1)}(1)}{(3-1)!}$

$\varphi'(z) = -e^z, \ \varphi''(z) = -e^z$

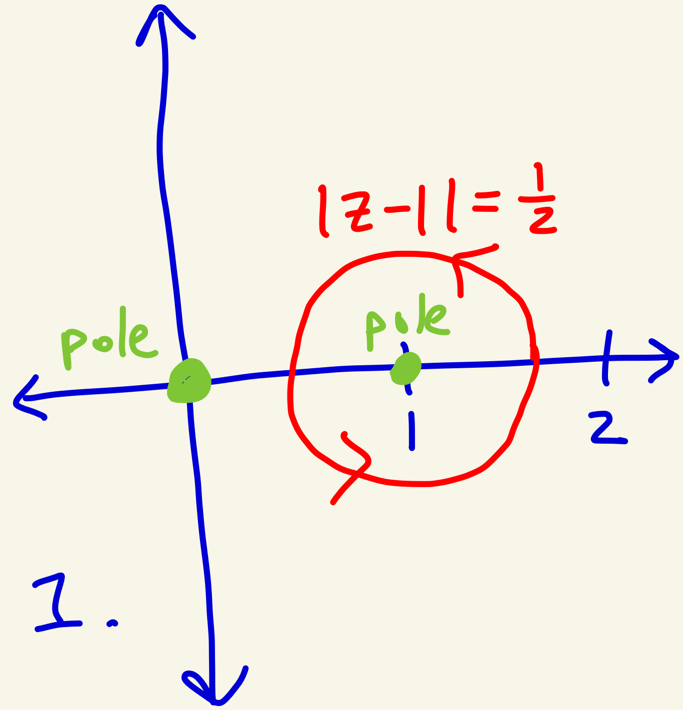
$$= \frac{-e^1}{2!} = -\frac{1}{2}e$$

Thus,

$$\int_{|z-2|=2} \frac{e^z}{(1-z)^3} = 2\pi i\left(-\frac{1}{2}e\right) = -\pi i e$$

① (d)

Let $f(z) = \dfrac{e^z}{z(1-z)^3}$



Then f is analytic on $\mathbb{C} - \{0, 1\}$.

f has poles at 0 and 1.

We have that

$$f(z) = \frac{e^z}{z(-1)^3(z-1)^3} = \frac{-e^z/z}{(z-1)^3} = \frac{\varphi(z)}{(z-1)^3}$$

where $\varphi(z) = -e^z/z$ is analytic at 1 and $\varphi(1) \neq 0$. So we have a pole of order 3 at $z_0 = 1$.

Also,

$$\varphi(z) = -z^{-1}e^z$$

$$\varphi'(z) = z^{-2}e^z - z^{-1}e^z$$

$$\varphi''(z) = -2z^{-3}e^z + z^{-2}e^z + z^{-2}e^z - z^{-1}e^z$$

By the residue thm,

$$\int_{|z-1|=\frac{1}{2}} \frac{e^z}{z(1-z)^3}\,dz = 2\pi i \operatorname{Res}(f;1)$$

$$= 2\pi i \frac{\varphi^{(3-1)}(1)}{(3-1)!}$$

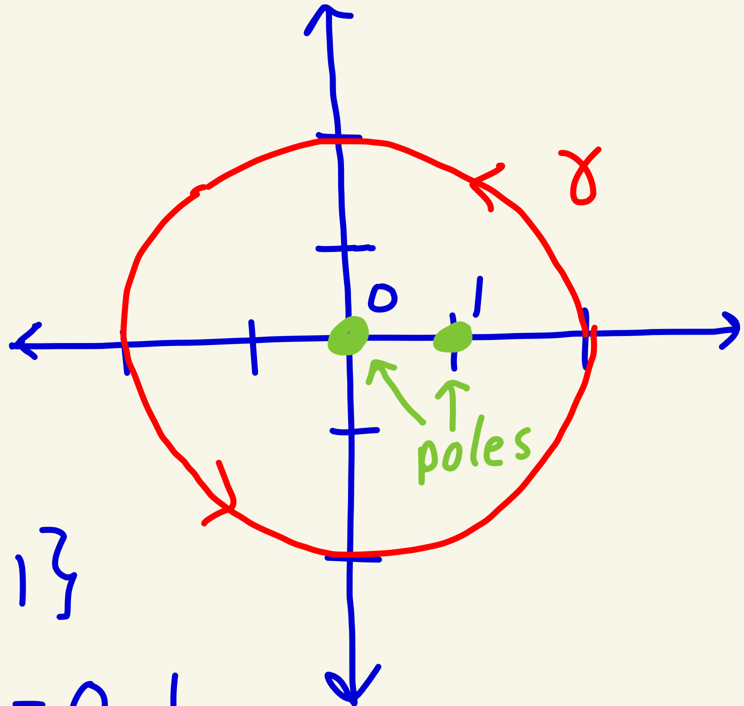
$$= \frac{2\pi i}{2!}\left[-2(1)^{-3}e^1 + (1)^{-2}e^1 + (1)^{-2}e^1 - (1)^{-1}e^1\right]$$

$$= \pi i\left[-2e + e + e - e\right]$$

$$= -e\pi i$$

$$\varphi''(z) = -2z^{-3}e^z + z^{-2}e^z + z^{-2}e^z - z^{-1}e^z$$

① (e)



The function

$$f(z) = \frac{e^z}{z^2(z-1)^3}$$

is analytic on $\mathbb{C} - \{0, 1\}$

It has poles at $z_0 = 0, 1$

By the residue theorem

$$\int_\gamma \frac{e^z \, dz}{z^2(z-1)^3} = 2\pi i \, \mathrm{Res}(f; 0) + 2\pi i \, \mathrm{Res}(f; 1)$$

Note that $f(z) = \dfrac{e^z/(z-1)^3}{z^2} = \dfrac{\varphi_1(z)}{z^2}$

where $\varphi_1(z) = \dfrac{e^z}{(z-1)^3}$ is analytic at

$z_0 = 0$ and $\varphi_1(0) \neq 0$. So we have a pole of order 2 at $z_0 = 0$ and so we have $\mathrm{Res}(f;0) = \dfrac{\varphi_1^{(2-1)}(0)}{(2-1)!} = \dfrac{\varphi_1'(0)}{1}$

$\varphi(z) = (z-1)^{-3} e^{z}$

$\varphi'(z) = -3(z-1)^{-4} e^{z} + (z-1)^{-3} e^{z}$

$= -3(0-1)^{-4} e^{0} + (0-1)^{-3} e^{0}$

$= -3 - 1 = \boxed{-4}$

Also, $f(z) = \dfrac{e^z/z^2}{(z-1)^3} = \dfrac{\varphi_2(z)}{(z-1)^3}$

where $\varphi_2(z) = e^z/z^2$ is analytic at $z_0 = 1$ and $\varphi_2(1) = e^1/1^2 = e \neq 0$.

So, we have a pole of order 3 at $z_0 = 1$. Also,

$\varphi_2(z) = z^{-2} e^{z}$

$\varphi_2'(z) = -2z^{-3} e^{z} + z^{-2} e^{z}$

$\varphi_2''(z) = +6z^{-4} e^{z} - 2z^{-3} e^{z} - 2z^{-3} e^{z} + z^{-2} e^{z}$

$= 6z^{-4} e^{z} - 4z^{-3} e^{z} + z^{-2} e^{z}$

So,

$$\operatorname{Res}(f;1) = \frac{\varphi_2^{(3-1)}(1)}{(3-1)!}$$

$$= \frac{6(1)^{-4}e^1 - 4(1)^{-3}e^1 + (1)^{-2}e^1}{2}$$

$$= \frac{6e - 4e + e}{2} = \boxed{\frac{3}{2}e}$$

Thus,

$$\int_\gamma \frac{e^z}{z^2(z-1)^3}\,dz = 2\pi i\left[-4 + \frac{3}{2}e\right]$$

$$= \boxed{-8\pi i + 3\pi i e}$$

② (a) Let $z = x + iy$.

Then $\cos(z) = 0$

iff $\dfrac{e^{iz} + e^{-iz}}{2} = 0$

iff $e^{iz} + e^{-iz} = 0$

iff $e^{i(x+iy)} + e^{-i(x+iy)} = 0$

iff $e^{-y} e^{ix} + e^{y} e^{-ix} = 0$

iff $e^{-y}\left[\cos(x) + i\sin(x)\right] + e^{y}\left[\cos(-x) + i\sin(-x)\right] = 0$

$\sin(-x) = -\sin(x)$
$\cos(-x) = \cos(x)$

iff $(e^{-y} + e^{y})\cos(x) + i(e^{-y} - e^{y})\sin(x) = 0$

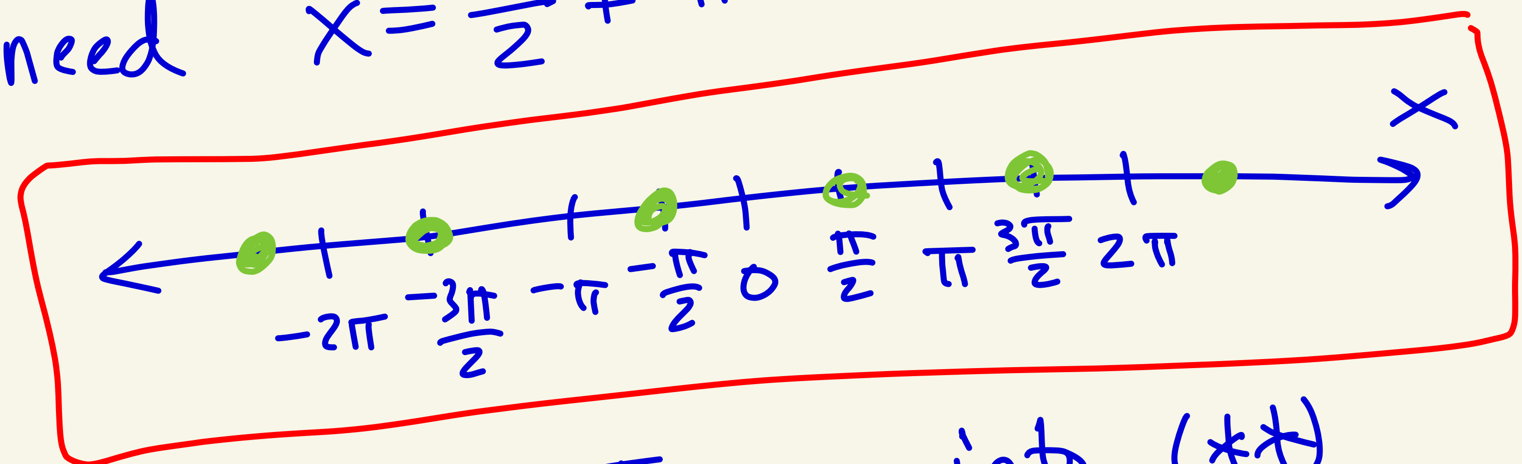
iff $\boxed{\begin{array}{l} (e^{-y} + e^{y})\cos(x) = 0 \quad (*) \\ \text{and } (e^{-y} - e^{y})\sin(x) = 0 \quad (**) \end{array}}$

In equation (*) either

$e^{-y}+e^{y}=0$ or $\cos(x)=0$

But $e^{-y}>0$ and $e^{y}>0$, so $e^{-y}+e^{y}\neq 0$.

Thus, for (*) to hold we need $x=\frac{\pi}{2}+\pi n$ where $n\in\mathbb{Z}$.



Now plug $x=\frac{\pi}{2}+\pi n$ into (**) to get

$$(e^{-y}-e^{y})\sin\left(\frac{\pi}{2}+\pi n\right)=0$$

Since $\sin\left(\frac{\pi}{2}+\pi n\right)\neq 0$ for all n this gives $e^{-y}-e^{y}=0$. multiply by e^{y} to get $1-e^{2y}=0$.

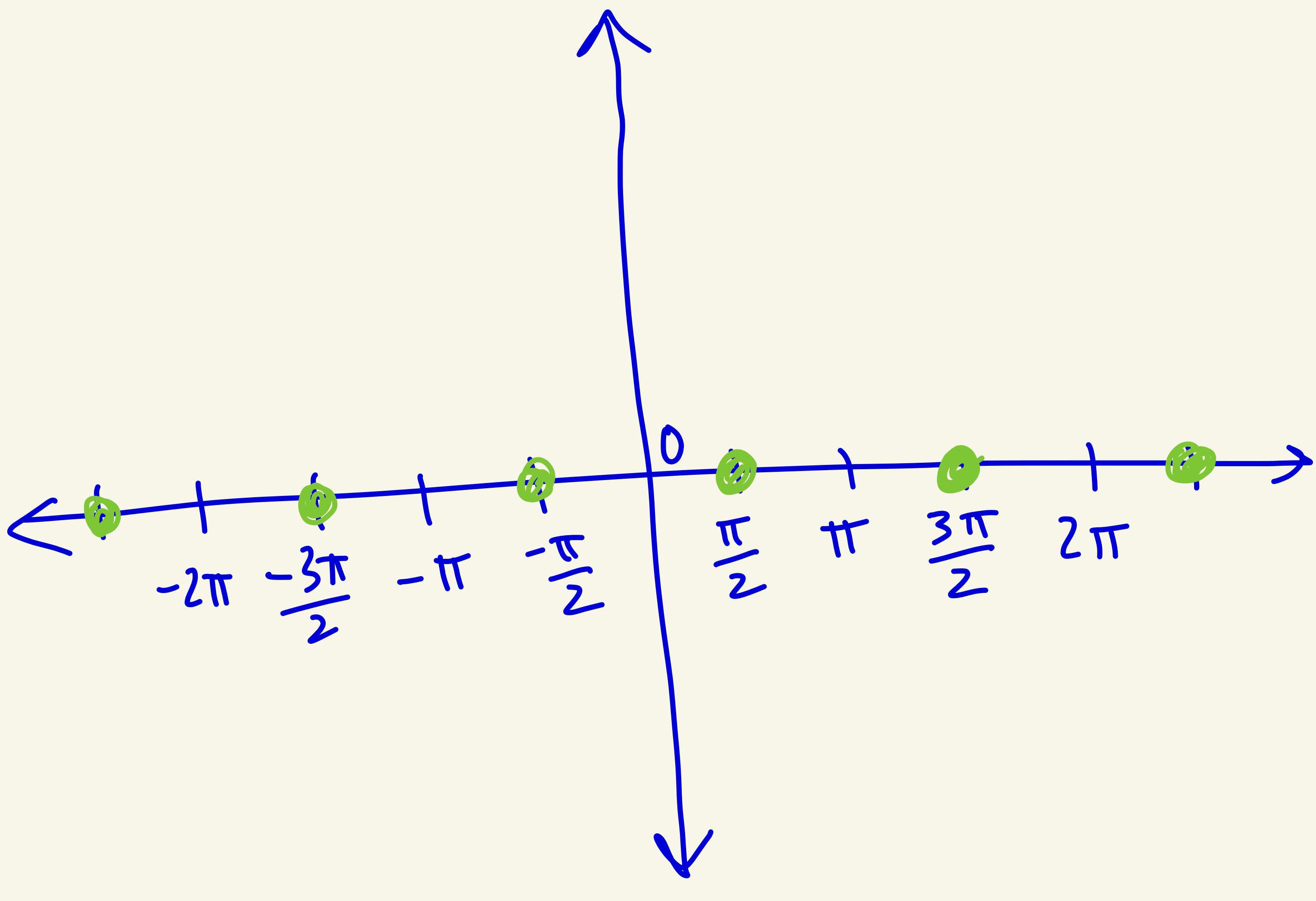
So, $e^{2y} = 1$.

Thus, $y = 0$.

Therefore the solutions to

$\cos(z) = 0$

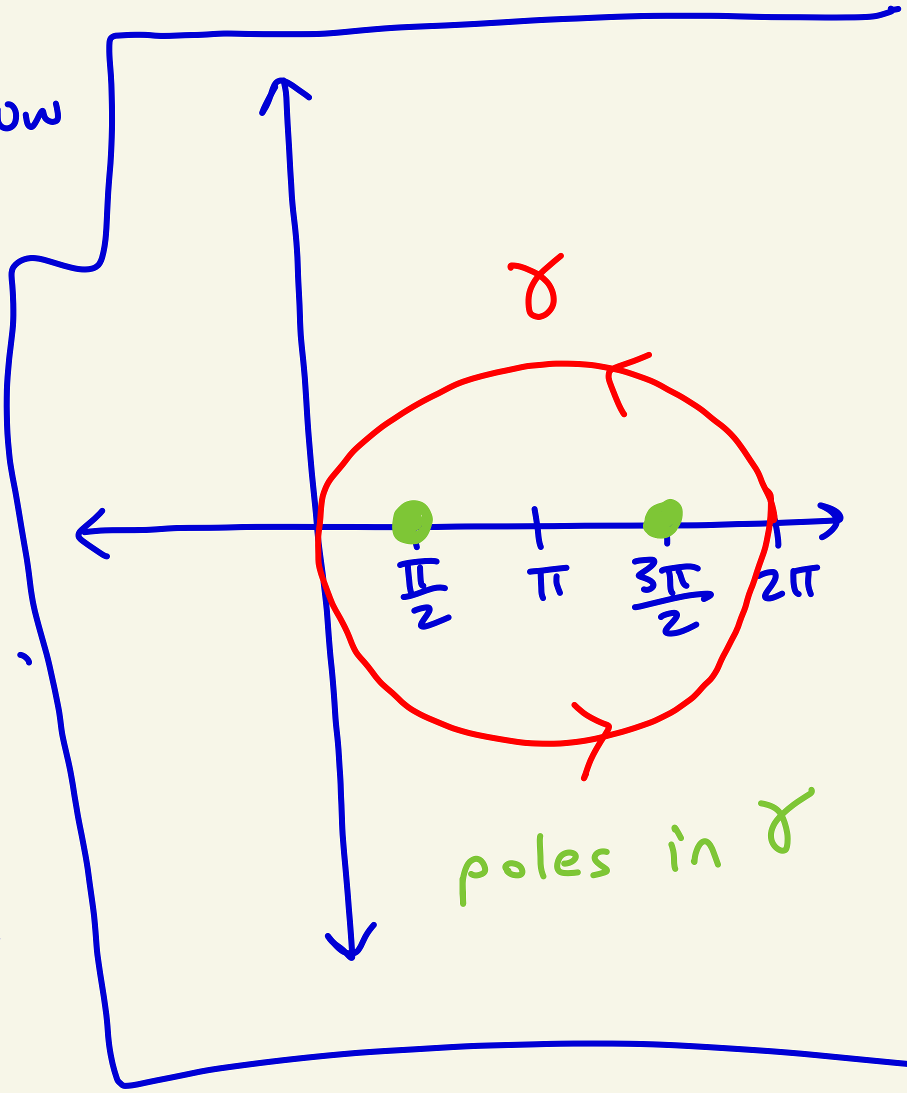
are $z = x + iy = \frac{\pi}{2} + \pi n$, $n \in \mathbb{Z}$.



② (b)

From 2(a) we know that $\cos(z) = 0$ iff $z = \frac{\pi}{2} + \pi n$ where $n \in \mathbb{Z}$.

Also, $\sin(z) \neq 0$ when $z = \frac{\pi}{2} + \pi n$.



Thus,

$$f(z) = \frac{\sin(z)}{\cos(z)}$$

has poles at $z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots$

So, $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ are the poles in γ.

Let $g(z) = \sin(z)$, $h(z) = \cos(z)$.

Then

$$g\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1 \neq 0$$

$$h\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

$$h'\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1 \neq 0$$

Thus, $\frac{\pi}{2}$ is a simple pole of f

and $\text{Res}\left(f; \frac{\pi}{2}\right) = \dfrac{g\left(\frac{\pi}{2}\right)}{h'\left(\frac{\pi}{2}\right)} = \dfrac{1}{-1} = -1$

Also

$$g\left(\frac{3\pi}{2}\right) = \sin\left(\frac{3\pi}{2}\right) = -1$$

$$h\left(\frac{3\pi}{2}\right) = \cos\left(\frac{3\pi}{2}\right) = 0$$

$$h'\left(\frac{3\pi}{2}\right) = -\sin\left(\frac{3\pi}{2}\right) = 1 \neq 0$$

Thus, $\frac{\pi}{2}$ is a simple pole of f

and $\text{Res}\left(f; \frac{3\pi}{2}\right) = \dfrac{g\left(\frac{3\pi}{2}\right)}{h'\left(\frac{3\pi}{2}\right)} = \dfrac{-1}{1} = -1$

Thus,

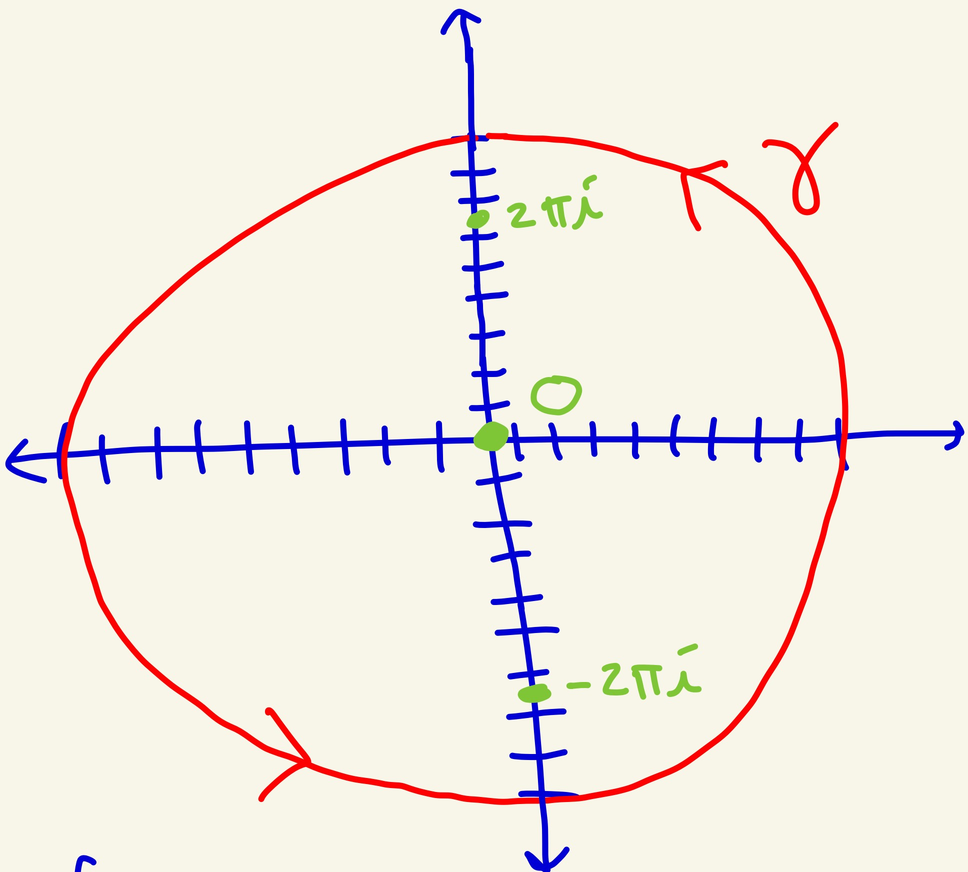
$$\int_\gamma \frac{\sin(z)}{\cos(z)}\,dz = 2\pi i\left[\operatorname{Res}\left(f;\frac{\pi}{2}\right) + \operatorname{Res}\left(f;\frac{3\pi}{2}\right)\right]$$

$$= 2\pi i\left[-1-1\right]$$

$$= \boxed{-4\pi i}$$

③ Note that $e^z - 1 = 0$ iff $e^z = 1$ iff $z = 0 + i2\pi n$ where $n \in \mathbb{Z}$.

The function $f(z) = \dfrac{1}{e^z - 1}$ has poles at $z = i2\pi n$, $n \in \mathbb{Z}$.



The poles of f that are inside γ are $-2\pi i$ and 0 and $2\pi i$.

Now, $f(z) = \dfrac{g(z)}{h(z)}$ where

$g(z) = 1$ and $h(z) = e^z - 1$.

We have $h'(z) = e^z$.

So, $g(-2\pi i) = 1$, $h(-2\pi i) = e^{-2\pi i} - 1 = 1 - 1 = 0$

$h'(-2\pi i) = e^{-2\pi i} = 1 \neq 0$

So, $\text{Res}(f; -2\pi i) = \dfrac{g(-2\pi i)}{h'(-2\pi i)} = \dfrac{1}{1} = \boxed{1}$

Also, $g(0) = 1$, $h(0) = e^0 - 1 = 1 - 1 = 0$

$h'(0) = e^0 = 1 \neq 0$

So, $\text{Res}(f; 0) = \dfrac{g(0)}{h'(0)} = \dfrac{1}{1} = \boxed{1}$

Also, $g(2\pi i) = 1$, $h(2\pi i) = e^{2\pi i} - 1 = 1 - 1 = 0$

$h'(2\pi i) = e^{2\pi i} = 1 \neq 0$

So, $\text{Res}(f; 2\pi i) = \dfrac{g(2\pi i)}{h'(2\pi i)} = \dfrac{1}{1} = \boxed{1}$

Thus,

$$\int_{\gamma} \frac{1}{e^z - 1}\, dz$$

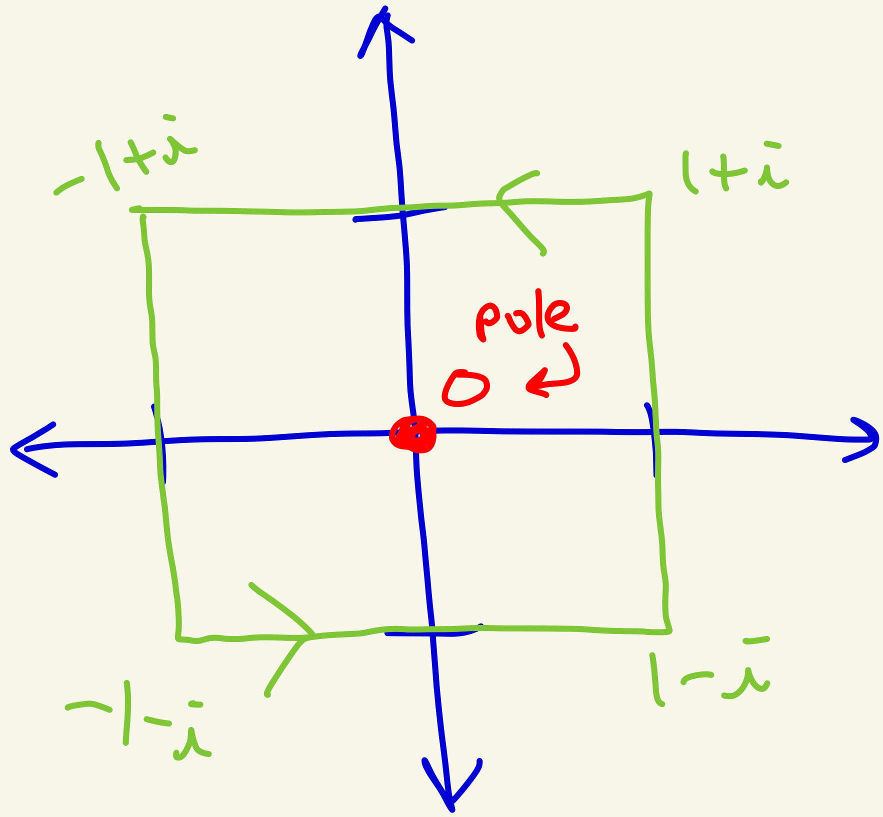
$$= 2\pi i \left[\operatorname{Res}(f; -2\pi i) + \operatorname{Res}(f; 0) + \operatorname{Res}(f; 2\pi i) \right]$$

$$= 2\pi i \left[1 + 1 + 1 \right] = \boxed{6\pi i}$$

④

Let $f(z) = \dfrac{e^{z^2}}{z^2}$

f is analytic on $\mathbb{C} - \{0\}$.



Then,

$f(z) = \dfrac{\varphi(z)}{z^2}$ where $\varphi(z) = e^{z^2}$

and $\varphi(0) = e^{0^2} = 1 \neq 0$

and φ is analytic at 0.

Thus, f has a pole of order 2 at $z_0 = 0$.

So, $$\operatorname{Res}(f;0) = \frac{\varphi^{(2-1)}(0)}{(2-1)!}$$

$$= \frac{\varphi'(0)}{1}$$

$\varphi(z) = e^{z^2}$

$\varphi'(z) = 2ze^{z^2}$

$$= \frac{2(0)e^{0^2}}{1} = 0.$$

Thus,

$$\int_\gamma \frac{e^{z^2}}{z^2}\,dz = 2\pi i \operatorname{Res}(f;0)$$

$$= 2\pi i \left[0\right]$$

$$= 0.$$