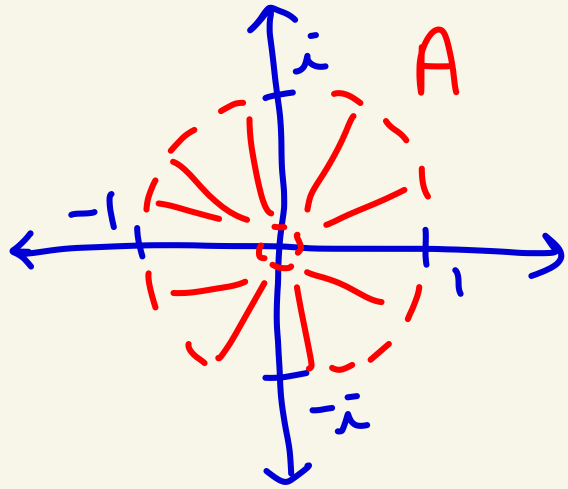
5680
HW 4 - Part 1
Solutions

① (a) Let $A = \{z \mid 0 < |z| < 1\}$



We want to expand $\frac{1}{z(z+1)}$ in A about $z_0 = 0$.

If $z \in A$, that is $0 < |z| < 1$, then

$$\frac{1}{z(z+1)} = \frac{1}{z} \cdot \frac{1}{1-(-z)} = \frac{1}{z}\left[1 + (-z) + (-z)^2 + \cdots\right]$$

need $|z| < 1$

residue

$$= \frac{1}{z} - 1 + z - z^2 + \cdots$$

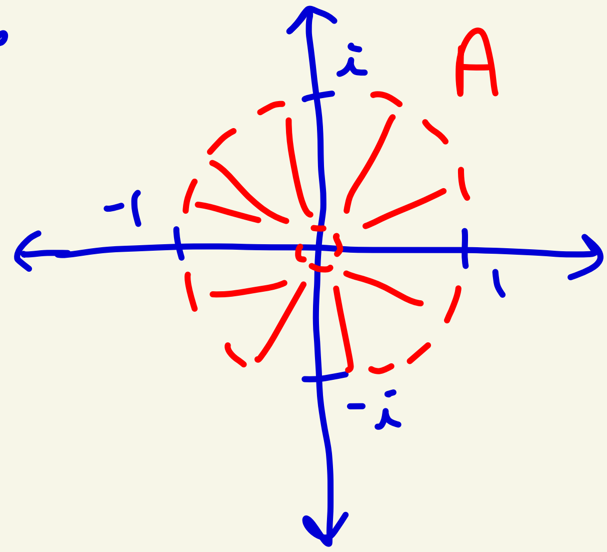
or in closed form we get

$$\frac{1}{z} \cdot \frac{1}{1-(-z)} = \frac{1}{z} \sum_{n=0}^{\infty} (-z)^n = \sum_{n=0}^{\infty} (-1)^n z^{n-1}$$

need $|z| < 1$

We have a pole of order 1, or simple pole. And $\operatorname{Res}(f; 0) = 1$.

① (b) $A = \{ z \mid 0 < |z| < 1 \}$



We want to expand f around $z_0 = 0$.

Let $z \in A$, that is $0 < |z| < 1$ then we have

$$\frac{z}{z+1} = z \cdot \frac{1}{1-(-z)} \underset{\text{need } |z|<1}{=} \underbrace{z \cdot \sum_{n=0}^{\infty} (-z)^n}_{z\left[1 - z + z^2 - z^3 + z^4 - \cdots\right]}$$

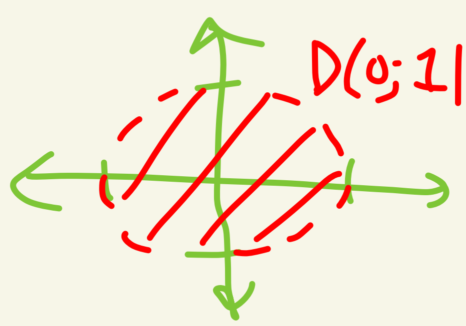
$$= \sum_{n=0}^{\infty} (-1)^n z^{n+1} = z - z^2 + z^3 - z^4 - \cdots$$

Here we have a removable singularity.

With $\text{Res}(f;0) = 0$

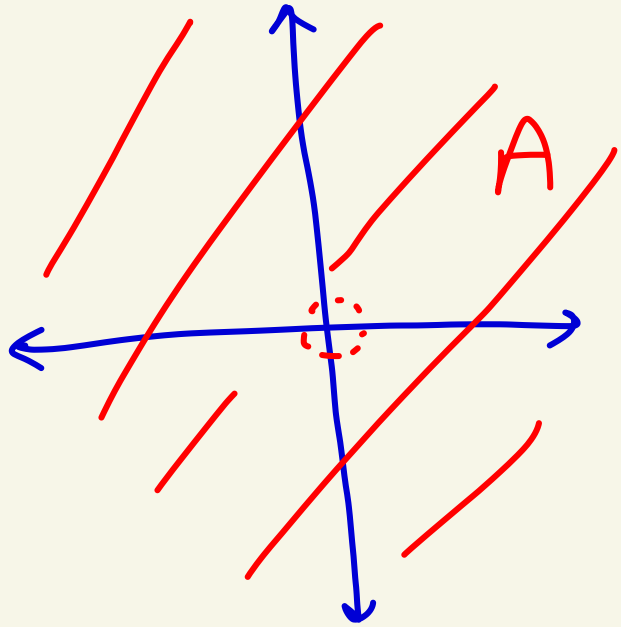
Note that the original function $\frac{z}{z+1}$ is well-defined at $z=0$. So this makes sense as it will then have a power series expansion there.

The above expansion makes sense on $D(0;1)$.



①(c)

$A = \{ z \mid 0 < |z| \} = \mathbb{C} - \{0\}$



We want to expand

$f(z) = \dfrac{3e^z}{z^2}$ about $z_0 = 0$

inside of A.

Suppose $z \in A$, ie $z \neq 0$, then

$$\frac{3e^z}{z^2} = \underbrace{\frac{3}{z^2} \sum_{n=0}^{\infty} \frac{z^n}{n!}}_{\frac{3}{z^2}\left[1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right]} = \sum_{n=0}^{\infty} \frac{3z^{n-2}}{n!}$$

residue

$$= \frac{3}{z^2} + \frac{3}{z} + \frac{3}{2} + \frac{1}{2} z + \cdots$$

We have a pole of order 2 and

$\mathrm{Res}(f; 0) = 3.$

② (a)

Suppose $z \neq -1$. Then

work on the top

$$\frac{z}{z+1} = \frac{-1+(z+1)}{z+1} = \frac{-1}{z+1} + 1$$

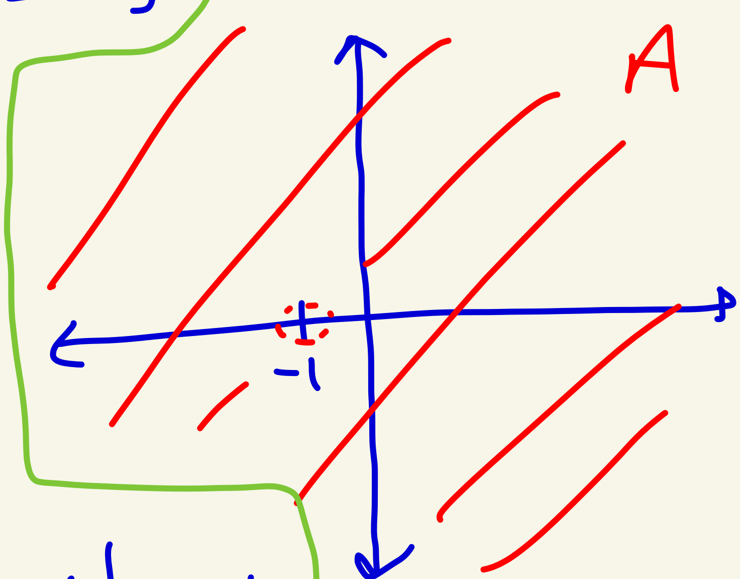
residue

bottom is already in the form $z-(-1)$

$$= \underbrace{\frac{-1}{z-(-1)} + 1}_{\text{this is the Laurent series}}$$

Our function equals this Laurent series for all $z \neq -1$. So let

$A = \mathbb{C} - \{-1\} = \{ z \mid 0 < |z+1| \}$

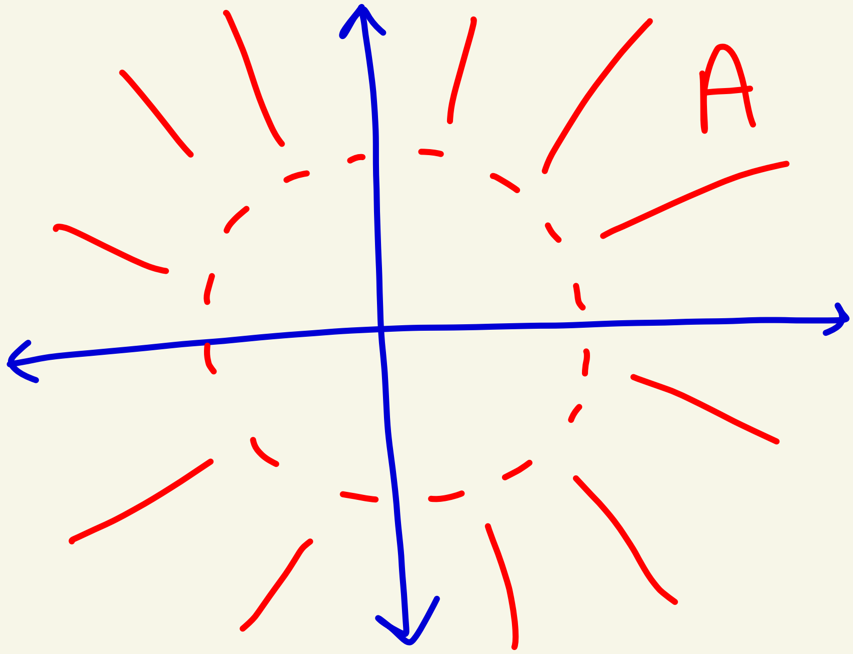


This is an isolated singularity. We have a simple pole, or a pole of order 1.

The residue is $\text{Res}(f; -1) = -1$

(3)

$A = \{ z \mid 1 < |z| \}$



We want to expand f inside of A, centered at $z_0 = 0$

Let $z \in A$, that is $1 < |z|$,

Then

$$\frac{1}{z(z+1)} = \frac{1}{z} \cdot \frac{1}{z\left(1+\frac{1}{z}\right)}$$

$$= \frac{1}{z^2} \cdot \frac{1}{1-\left(-\frac{1}{z}\right)}$$

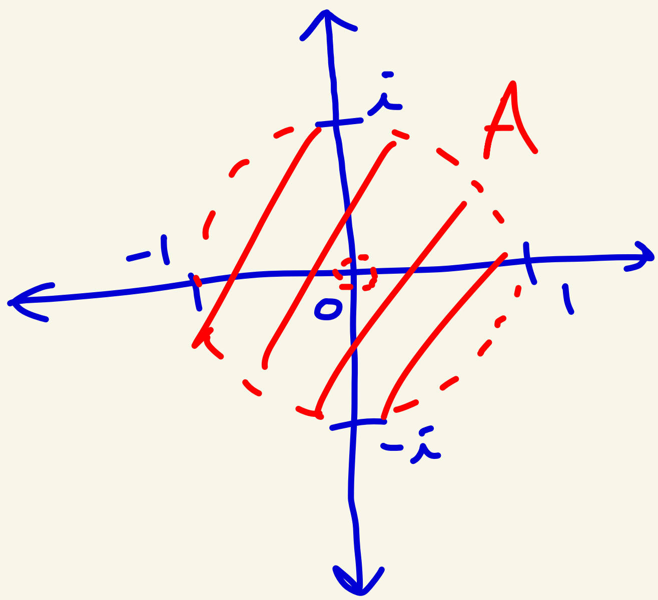
need $\left|-\frac{1}{z}\right| < 1$ or $1 < |z|$

$$= \underbrace{\frac{1}{z^2} \sum_{n=0}^{\infty} \left(-\frac{1}{z}\right)^n}_{\frac{1}{z^2}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right]} =$$

$$= \frac{1}{z^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{z^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+2}}$$

$$= \frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z^4} - \frac{1}{z^5} + \cdots$$

④ (a)



We want to expand

$$f(z) = \frac{1}{z(z-1)(z-2)}$$

inside of A.

$\boxed{\text{Method 1}}$

We use partial fractions:

$$\frac{1}{z(z-1)(z-2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$$

$$1 = A(z-1)(z-2) + Bz(z-2) + Cz(z-1)$$

Plug in $z=0$: $1 = A(-1)(-2) + B(0) + C(0)$

$\frac{1}{2} = A$

Plug in $z=1$: $1 = A(0) + B(1)(-1) + C(0)$

$-1 = B$

plug in $z=2$: $1 = A(0) + B(0) + C(2)(1)$

$\frac{1}{2} = C$

Let $z \in A$. Then,

$$\frac{1}{z(z-1)(z-2)} = \frac{1}{2}\cdot\frac{1}{z} - \frac{1}{z-1} + \frac{1}{2}\cdot\frac{1}{z-2}$$

$$= \frac{1}{2}\cdot\frac{1}{z} + \frac{1}{1-z} - \frac{1}{2\cdot 2}\cdot\frac{1}{1-\frac{z}{2}} \qquad (*)$$

To expand $(*)$ we need $|z|<1$

We also need $\left|\frac{z}{2}\right|<1$ or $|z|<2$

Both of these are satisfied for $z \in A$ since then $|z|<1$.

Let $z \in A$. Then $(*)$ becomes

$$\frac{1}{2}\cdot\frac{1}{z} + \left[1+z+z^2+z^3+\ldots\right] - \frac{1}{4}\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\ldots\right]$$

$$= \frac{1/2}{z} + \left[1+z+z^2+z^3+\ldots\right] - \left[\frac{1}{2^2}+\frac{z}{2^3}+\frac{z^2}{2^4}+\ldots\right]$$

$$= \frac{1/2}{z} + \frac{3}{4} + \frac{7}{8}z + \frac{15}{16}z^2 + \ldots$$

$$= \frac{1/2}{z} + \sum_{n=0}^{\infty}\left(\frac{2^{n+2}-1}{2^{n+2}}\right)z^n$$

$\boxed{\text{Method 2}}$

As in method 1, let $z \in A$. Then we have both $|z|<1$ and $\left|\frac{z}{2}\right|<1$.

So,

$$\frac{1}{z(z-1)(z-2)} = \frac{1}{z}\cdot\left(\frac{-1}{1-z}\right)\left(\frac{-1/2}{1-\frac{z}{2}}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{z}\left[1+z+z^2+\cdots\right]\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\cdots\right]$$

$$= \frac{1}{2z}\left[1+\left(1+\frac{1}{2}\right)z+\left(1+\frac{1}{2}+\frac{1}{2^2}\right)z^2+\cdots\right]$$

$$= \frac{1}{2z}\left[1+\frac{3}{2}z+\frac{7}{4}z^2+\cdots\right]$$

$$= \frac{1/2}{z}+\frac{3}{4}+\frac{7}{8}z+\cdots$$

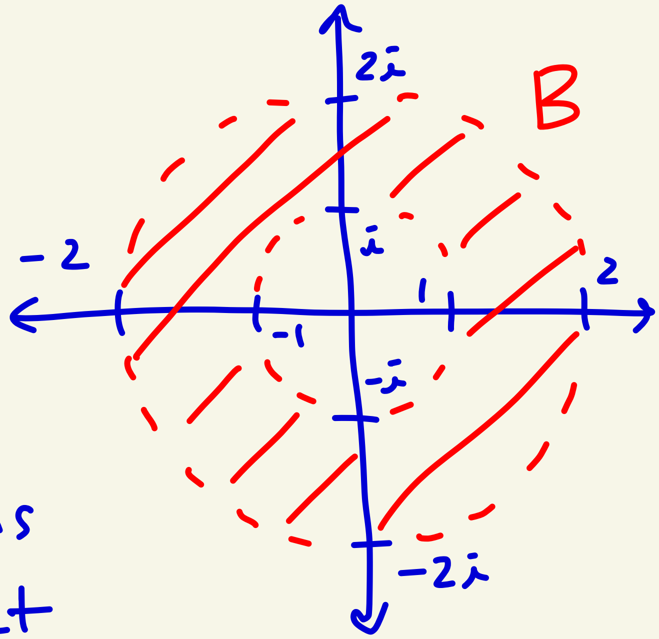
$$= \frac{1/2}{z}+\sum_{n=0}^{\infty}\left(\frac{2^{n+2}-1}{2^{n+2}}\right)z^n$$

(Same as method one)

④(b) Let's expand

$$f(z) = \frac{1}{z(z-1)(z-2)}$$

in B.



Let $z \in B$.

Using partial fractions like in 4(a) to get

$$\frac{1}{z(z-1)(z-2)} = \frac{1}{2}\cdot\frac{1}{z} - \frac{1}{z-1} + \frac{1}{2}\cdot\frac{1}{z-2}$$

$$= \frac{1}{2}\cdot\frac{1}{z} - \frac{1}{z(1-\frac{1}{z})} - \frac{1}{4}\cdot\frac{1}{1-\frac{z}{2}} \qquad (*)$$

If $z \in B$ then $1 < |z| < 2$.

Therefore we have $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1$.

Thus, $(*)$ gives

$$\frac{1}{z(z-1)(z-2)} = \frac{1}{2z} - \frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right]$$

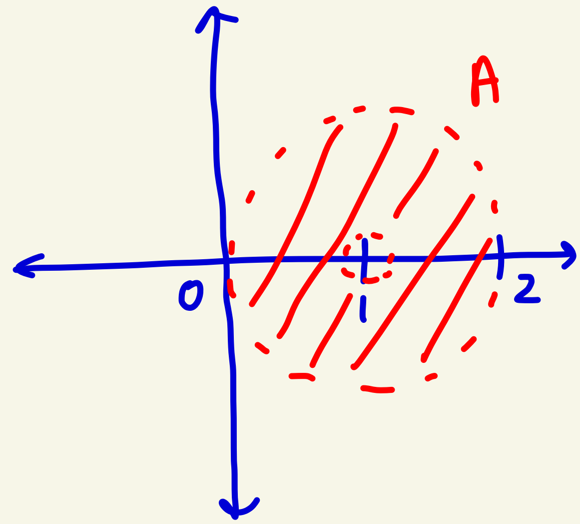
$$- \frac{1}{4}\left[1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right]$$

$$= \cdots - \frac{1}{z^4} - \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{2z} - \frac{1}{4} - \frac{z}{8} - \frac{z^2}{16} - \frac{z^3}{32} - \cdots$$

⑤ We want to expand

$$f(z) = \frac{1}{z^2(1-z)}$$ into a Laurent series in

$$A = \{ z \mid 0 < |z-1| < 1 \}.$$



We have that

$$f(z) = \frac{1}{z^2} \cdot \frac{-1}{z-1}$$

We need to expand $\frac{1}{z^2}$ about $z_0 = 1$.

The function $g(z) = \frac{1}{z^2}$ is analytic in A, so it has a power series in A of the form $\displaystyle\sum_{n=0}^{\infty} \frac{g^{(n)}(1)}{n!}(z-1)^n$.

We have that

$$g(z) = z^{-2}$$
$$g'(z) = -2z^{-3}$$
$$g''(z) = 3!\, z^{-4}$$
$$g'''(z) = -4!\, z^{-5}$$

and in general

$$g^{(n)}(z) = (-1)^n (n+1)!\, z^{-(n+2)}$$
$$g^{(n)}(1) = (-1)^n (n+1)!$$

Thus,

$$g(z) = \frac{1}{z^2} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{n!} (z-1)^n$$

$$(n+1)! = (n+1) \cdot n!$$

$$= \sum_{n=0}^{\infty} (-1)^n (n+1)(z-1)^n$$

So if $z \in A$, then

$$f(z) = \frac{-1}{z-1}\left[\sum_{n=0}^{\infty} (-1)^n (n+1)(z-1)^n\right]$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} (n+1)(z-1)^{n-1}$$

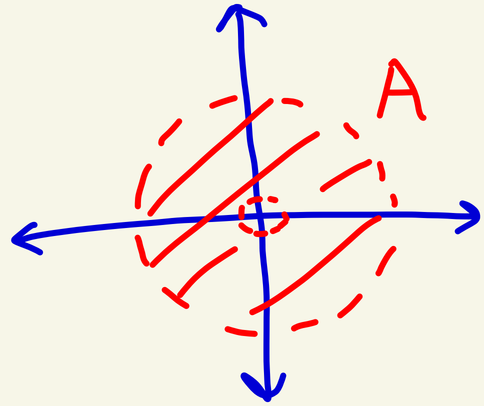
residue

$$= \frac{-1}{z-1} + 2 - 3(z-1) + 4(z-1)^2 - \ldots$$

We have a pole of order 1.

$$\mathrm{Res}(f;1) = -1$$

⑥ Let $z \in A$. Then $0<|z|<1$. And,



$$\frac{z+1}{z^3(z^2+1)} = \frac{1+z}{z^3}\cdot\frac{1}{1-(-z^2)}$$

$$= \frac{1+z}{z^3}\left[1+(-z^2)+(-z^2)^2+(-z^2)^3+(-z^2)^4+\dots\right]$$

we have $|z|<1$
thus $|z|^2<1$
so $|z^2|<1$
so $|-z^2|<1$

$$= \frac{1+z}{z^3}\left[1-z^2+z^4-z^6+z^8-\dots\right]$$

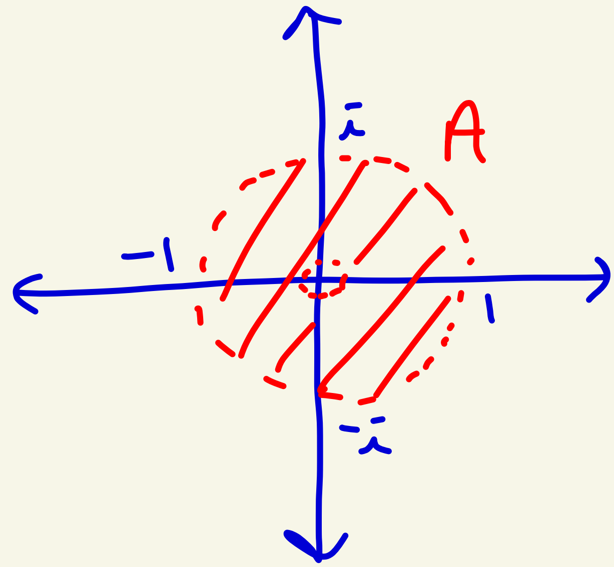
$$= (1+z)\left[\frac{1}{z^3}-\frac{1}{z}+z-z^3+z^5-\dots\right]$$

$$= \left[\frac{1}{z^3}-\frac{1}{z}+z-z^3+z^5-\dots\right]+\left[\frac{1}{z^2}-1+z^2-z^4+z^6-\dots\right]$$

$$= \boxed{\frac{1}{z^3}+\frac{1}{z^2}-\frac{1}{z}-1+z+z^2-z^3-z^4+z^5+z^6-\dots}$$

Also, $\operatorname{Res}(f;0) = -1$.

⑦ $A = \{ z \mid 0 < |z| < 1 \}$



Let $z \in A$.
Then, $|z| < 1$
and $z \neq 0$, so

$$f(z) = e^{1/z} \cdot \frac{1}{1-z}$$

since $z \neq 0$ and $|z|<1$

$$= \left[1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots \right]\left[1 + z + z^2 + z^3 + \cdots \right]$$

$$= \begin{array}{l}
1 + \frac{1}{z} + \frac{1}{2!}\cdot\frac{1}{z^2} + \frac{1}{3!}\cdot\frac{1}{z^3} + \cdots \\
z + 1 + \frac{1}{2!}\cdot\frac{1}{z} + \frac{1}{3!}\frac{1}{z^2} + \frac{1}{4!}\cdot\frac{1}{z^3} + \cdots \\
z^2 + z + \frac{1}{2!} + \frac{1}{3!}\frac{1}{z} + \frac{1}{4!}\frac{1}{z^2} + \frac{1}{5!}\cdot\frac{1}{z^3} + \cdots \\
z^3 + z^2 + \frac{1}{2!}z + \frac{1}{3!} + \frac{1}{4!}\frac{1}{z} + \frac{1}{5!}\frac{1}{z^2} + \frac{1}{6!}\frac{1}{z^3} + \cdots \\
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots
\end{array}$$

$$= \cdots + \left(\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots\right) \cdot \frac{1}{z^2}$$

$$+ \left(1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots\right) \cdot \frac{1}{z}$$

$$+ \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots\right) \cdot 1$$

$$+ \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots\right) z$$

$$+ \cdots$$

$$= \cdots + \underbrace{(e-2)}_{b_2} \cdot \frac{1}{z^2} + \underbrace{(e-1)}_{b_1} \cdot \frac{1}{z} + \underset{a_0}{e} + \underset{a_1}{e} z + e z^2 + \cdots$$

$$\boxed{\begin{aligned} e^z &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \\ e &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots \end{aligned}}$$

So,

$$b_2 = e - 2$$

$$b_1 = e - 1$$

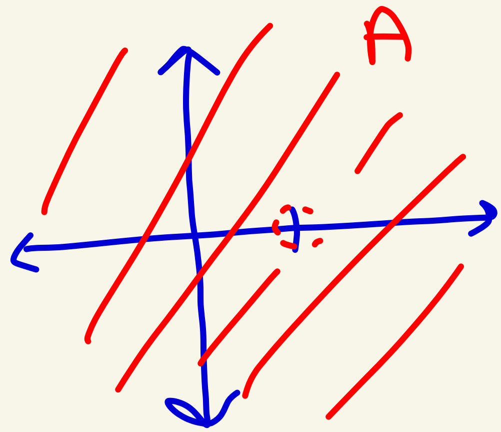
$$a_0 = e$$

$$a_1 = e$$

8 (a)

$$f(z) = \frac{1}{(1-z)^2} = \frac{1}{(z-1)^2}$$

This is the Laurent series for f centered at $z_0 = 1$ and valid on
$$A = \{ z \mid 0 < |z-1| \}$$



We have a pole of order 2.

8 (b) Since we can write $f(z) = \frac{\sin(z-1)}{z^2}$

as $\frac{\varphi(z)}{(z-0)^2}$ where $\varphi(z) = \sin(z-1)$

and $\varphi(z)$ is analytic at $z_0 = 0$ and

$\varphi(0) \neq 0$ we have that

f has a pole of order 2

at $z_0 = 0$ [From thm in class]

9

$$\frac{1}{e^z-1} = \frac{1}{\left(1+z+\frac{1}{2}z^2+\frac{1}{6}z^3+\frac{1}{24}z^4+\cdots\right)-1}$$

$$= \frac{1}{z+\frac{1}{2}z^2+\frac{1}{6}z^3+\frac{1}{24}z^4+\cdots}$$

$$= \frac{1}{z} - \frac{1}{2} + \frac{1}{12}z + \cdots$$

$$\begin{array}{r|l}
 & \frac{1}{z} - \frac{1}{2} + \frac{1}{12}z + \cdots \\
\hline
\left(z+\frac{1}{2}z^2+\frac{1}{6}z^3+\cdots\right) & 1 \\
 & -\left(1+\frac{1}{2}z+\frac{1}{6}z^2+\cdots\right) \\
\cline{2-2}
 & -\frac{1}{2}z - \frac{1}{6}z^2 - \cdots \\
 & -\left(-\frac{1}{2}z - \frac{1}{4}z^2 - \cdots\right) \\
\cline{2-2}
 & \frac{1}{12}z^2 + \cdots \\
 & -\left(\frac{1}{12}z^2 + \cdots\right) \\
\cline{2-2}
 & \vdots \qquad \vdots
\end{array}$$

So, we have a pole of order 1 with $\text{Res}(f;0)=1$

And $b_1 = 1$, $a_0 = -\frac{1}{2}$, $a_1 = \frac{1}{12}$.

(10) (a) Note that

$$f(z) = \frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{z-1} \cdot \frac{1}{z+1}$$

Also $g(z) = \frac{1}{z+1}$ is analytic at $z_0 = 1$

so it has a power series there.

Let's find it.

$$g(z) = (z+1)^{-1} \qquad g(1) = \frac{1}{2}$$

$$g'(z) = -(z+1)^{-2} \qquad g'(1) = -\frac{1}{2^2} = -\frac{1}{4}$$

$$g''(z) = 2(z+1)^{-3} \qquad g''(1) = \frac{2}{2^3} = \frac{1}{4}$$

$$\vdots \qquad \vdots$$

So near $z_0 = 1$ we have

$$g(z) = \frac{1}{z+1} = \sum_{n=0}^{\infty} \frac{g^{(n)}(1)}{n!}(z-1)^n$$

$$= \frac{1}{2} - \frac{1}{4}(z-1) + \frac{1}{8}(z-1)^2 + \cdots$$

So, near $z_0 = 1$ we get

$$f(z) = \frac{1}{z-1} \cdot \frac{1}{z+1}$$

$$= \frac{1}{z-1} \cdot \left[\frac{1}{2} - \frac{1}{4}(z+1) + \frac{1}{8}(z+1)^2 + \cdots \right]$$

$$= \frac{1/2}{z-1} - \frac{1}{4} + \frac{1}{8}(z-1) + \cdots$$

So, $\mathrm{Res}(f; 1) = 1/2$

⑩ (b) We use part (a) but also we expand z about $z_0 = 1$. You could use the formula

$$\sum \frac{g^{(n)}(1)}{n!}(z-1)^n$$

but in this case it's easy. We have

$$z = 1 + (z-1)$$

Thus, near $z_0 = 1$ we have

$$f(z) = \frac{z}{z^2-1} = z \cdot \frac{1}{z^2-1}$$

$$= \left[1+(z-1)\right]\left[\frac{1/2}{z-1} - \frac{1}{4} + \frac{1}{8}(z-1) + \cdots\right]$$

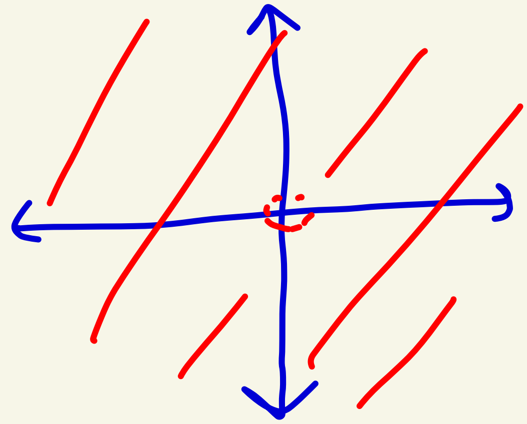
from problem 9(a)

$$= \frac{1/2}{z-1} + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{8} - \frac{1}{4}\right)(z-1) + \cdots$$

$$= \frac{1/2}{z-1} + \frac{1}{4} - \frac{1}{8}(z-1) + \cdots$$

$$\mathrm{Res}(f;1) = 1/2$$

10 (c) If $z \neq 0$, then



$$f(z) = \frac{e^z - 1}{z^2}$$

$$= \frac{-1 + e^z}{z^2}$$

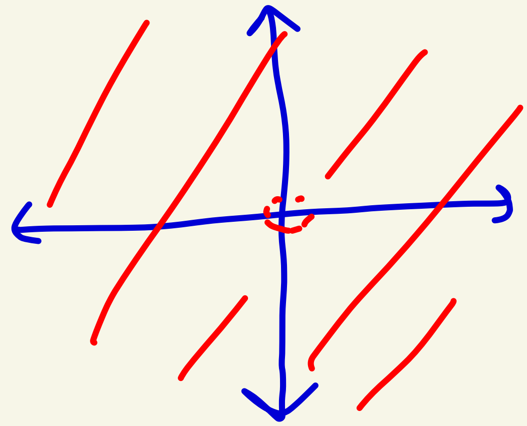
$$= \frac{\left(-1 + 1 + \frac{1}{1!}z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots\right)}{z^2}$$

$$= \frac{1}{z^2} \cdot \left[z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots \right]$$

$$= \frac{1}{z} + \frac{1}{2} + \frac{1}{6}z + \cdots$$

$$\mathrm{Res}(f;0) = 1$$

(10) (d) If $z \neq 0$, then



$$f(z) = \frac{e^z - 1}{z}$$

$$= \frac{-1 + e^z}{z}$$

$$= \frac{\left(-1 + 1 + \frac{1}{1!}z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots\right)}{z}$$

$$= \frac{1}{z} \cdot \left[z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots\right]$$

$$= 1 + \frac{1}{2}z + \frac{1}{6}z^2 + \cdots$$

$$\mathrm{Res}(f;0) = 0$$

This is a removable singularity. The original function f isn't defined at 0. But we can "fill it in" at $z=0$ by giving it the value of the power series at 0 which is 1.

11 Suppose that f is analytic at z_0 and has a zero of multiplicity k at z_0.

That is, $f(z) = (z-z_0)^k \varphi(z)$

where $\varphi(z)$ is analytic at z_0 and $\varphi(z_0) \neq 0$.

Then,

$$f'(z) = k(z-z_0)^{k-1}\varphi(z) + (z-z_0)^k \varphi'(z)$$

And so,

$$\frac{f'(z)}{f(z)} = \frac{k(z-z_0)^{k-1}\varphi(z) + (z-z_0)^k \varphi'(z)}{(z-z_0)^k \varphi(z)}$$

$$= \frac{k}{z-z_0} + \frac{\varphi'(z)}{\varphi(z)}$$

Note that since φ is analytic at z_0 and $\varphi(z_0) \neq 0$ we know that

$$\frac{\varphi'(z)}{\varphi(z)} = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

φ is analytic at z_0 and so φ' is analytic at z_0 by 4680. Thus, $\frac{\varphi'}{\varphi}$ is analytic at z_0. So by Taylors theorem we get this expansion

and converges near z_0.

Thus, near z_0 we have

$$\frac{f'(z)}{f(z)} = \frac{k}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

So, $\operatorname{Res}\left(\frac{f'}{f}; z_0\right) = k$