$$\boxed{\text{5680 Test 2 Solutions}}$$

pg 1

① (a)

$$z^3 e^z = z^3\left[1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right]$$

$$= z^3 + z^4 + \frac{z^5}{2!} + \frac{z^6}{3!} + \cdots$$

OR

$$z^3 e^z = z^3 \sum_{k=0}^{\infty} \frac{z^n}{n!} = \sum_{k=0}^{\infty} \frac{z^{n+3}}{n!}$$

Since $z^3 e^z$ is analytic on all of $\mathbb{C}$, by Taylors' theorem the radius of convergence is $R = \infty$.

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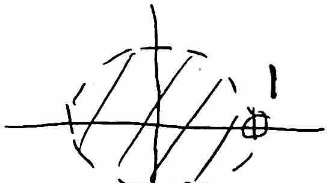
① (b) When $|z| < 1$ we have

$$\frac{\sin(2z)}{1-z} = \left(2z - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} - \cdots\right)\left(1 + z + z^2 + z^3 + \cdots\right)$$

$$= \left(2z - \frac{8z^3}{6} + \frac{32}{120}z^5 - \cdots\right)\left(1 + z + z^2 + z^3 + \cdots\right)$$

$$= 2z + 2z^2 + \left(2 - \frac{8}{6}\right)z^3 + \cdots$$

$$= 2z + 2z^2 + \frac{2}{3}z^3 + \cdots$$



$$\boxed{2 - \frac{8}{6} = \frac{12-8}{6} = \frac{4}{6} = \frac{2}{3}}$$

Since $\frac{\sin(2z)}{1-z}$ is analytic on $D(0;1)$, ~~it must con~~ the Taylor series must converge there by Taylors thm.

② (a)

$$f(z) = \log(z)$$
$$f'(z) = \frac{1}{z} = z^{-1}$$
$$f''(z) = -z^{-2}$$
$$f'''(z) = 2z^{-3}$$
$$f''''(z) = -3!\,z^{-4}$$
$$\vdots$$
$$f^{(k)}(z) = \frac{(-1)^{k+1}(k-1)!}{z^k}$$

$$f^{(0)}(-1+i) = \sqrt{2} + i\frac{3\pi}{4}$$

$$f^{(k)}(-1+i) = \frac{(-1)^{k+1}(k-1)!}{(-1+i)^k}$$

The taylor series is

$$\left(\sqrt{2}+i\frac{3\pi}{4}\right) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(k-1)!}{(-1+i)^k} \cdot \frac{1}{k!} z^k$$

$$= \left(\sqrt{2}+i\frac{3\pi}{4}\right) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(-1+i)^k \cdot k} z^k$$

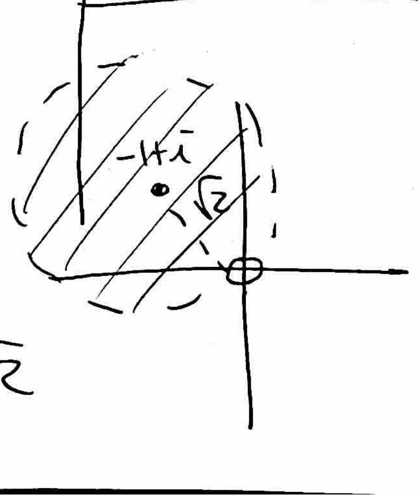
(b)

$$\lim_{k\to\infty} \left| \frac{(-1)^{k+2}}{(-1+i)^{k+1}(k+1)} z^{k+1} \cdot \frac{(-1+i)^k \cdot k}{(-1)^{k+1}} \frac{1}{z^k} \right|$$

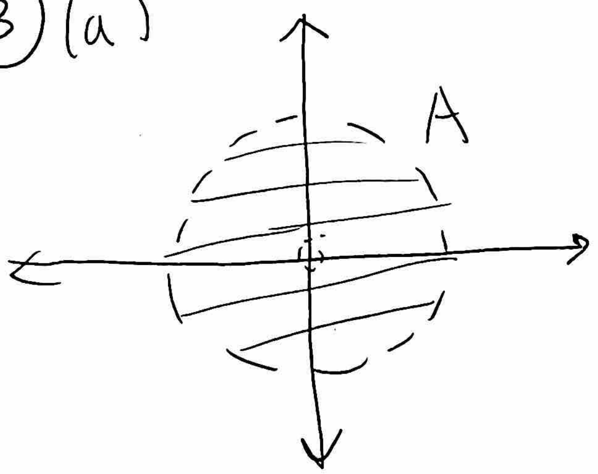
$$= \lim_{k\to\infty} \left| \frac{z\cdot k}{(-1+i)(k+1)} \right| = |z| \cdot \frac{1}{\sqrt{2}}$$

$$|z|\cdot\frac{1}{\sqrt{2}} < 1 \text{ iff } |z| < \sqrt{2}$$

The radius of convergence is $\sqrt{2}$



③ (a)



Let $z \in A$.

Then $0 < |z| < 1$.

And,

$$f(z) = \frac{1}{1+z^2} + \frac{1}{z+3}$$

$$\boxed{-\frac{27}{27} + \frac{1}{27} = \frac{-26}{27}}$$

$$= \frac{1}{1-(-z^2)} + \frac{1}{3+z}$$

$$= \frac{1}{1-(-z^2)} + \frac{1}{3} \cdot \frac{1}{1-\left(-\frac{z}{3}\right)}$$

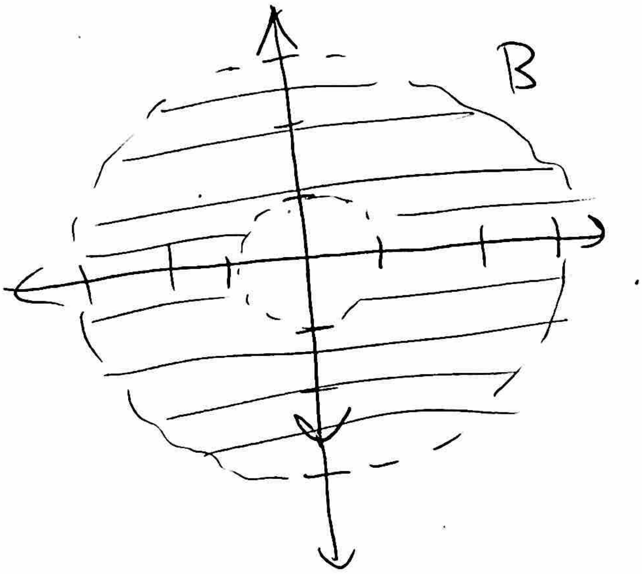
$$= \left(1 - z^2 + z^4 - z^6 + \dots\right) + \frac{1}{3}\left(1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \dots\right)$$

$$= \left(1 - z^2 + z^4 - z^6 + \dots\right) + \left(\frac{1}{3} - \frac{z}{3^2} + \frac{z^2}{3^3} - \frac{z^3}{3^4} + \dots\right)$$

$$= \left(1 + \frac{1}{3}\right) - \frac{1}{3^2} z + \left(-1 + \frac{1}{3^3}\right) z^2 - \frac{z^3}{3^4} + \dots$$

$$= 4/3 - \frac{1}{9} z - \frac{26}{27} z^2 - \frac{1}{3^4} z^3 + \dots$$

③ (b)



Let $z \in B$.

Then $1 < |z| < 3$.

So,

$$f(z) = \frac{1}{1+z^2} + \frac{1}{z+3} = \frac{1}{z^2}\left[\frac{1}{\frac{1}{z^2}+1}\right] + \frac{1}{3}\left[\frac{1}{\frac{z}{3}+1}\right]$$

$$= \frac{1}{z^2}\left[\frac{1}{1-\left(\frac{-1}{z^2}\right)}\right] + \frac{1}{3}\left[\frac{1}{1-\left(-\frac{z}{3}\right)}\right]$$

$$= \frac{1}{z^2}\left[1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} + \cdots\right] + \frac{1}{3}\left[1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \cdots\right).$$

$\boxed{\begin{array}{l} 1<|z| \\ \text{so, } \left|-\frac{1}{z^2}\right| = \frac{1}{|z|^2} < 1 \\ |z|<3 \\ \text{so, } \left|-\frac{z}{3}\right| < 1 \end{array}}$

$$= \left[\frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{z^6} - \frac{1}{z^8} + \cdots\right] + \left[\frac{1}{3} - \frac{z}{3^2} + \frac{z^2}{3^3} - \frac{z^3}{3^4} + \cdots\right]$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{z^{2k}} + \sum_{k=0}^{\infty} \frac{(-1)^k}{3^{k+1}} z^k$$

4 This is HW 4 - Part 2
Problem 1(d)

(5) $f(z) = \dfrac{1-z}{\sin(\pi z)}$

The singularities are when $\pi z = k\pi$ $k \in \mathbb{Z}$
ie where z is an integer.

<u>case 1:</u> Suppose $z_0 \neq 1$. and $z_0 = k$ where $k \in \mathbb{Z}$,
Then, set $g(z) = 1 - z$, $h(z) = \sin(\pi z)$.
Then, $g(k) \neq 0$, $h(k) = \sin(\pi k) = 0$,
$h'(k) = \pi \cos(\pi k) \neq 0$.
Thus, here we have a pole of order 1
and $\mathrm{Res}(f; k) = \dfrac{g(k)}{h'(k)} = \dfrac{1-k}{\pi\cos(\pi k)}$

<u>case 2:</u> Suppose $z_0 = 1$.
$g(z) = 1 - z$ has a zero of order 1.
$h(z) = \sin(z)$ has power series:

$$h(z) = 0 + \pi\cos(\pi)(z-1) + \dots$$

centered at $z_0 = 1$

$$\boxed{\begin{array}{l} h'(z) = \pi\cos(\pi k) \\ \pi\cos(\pi) = -\pi \neq 0 \end{array}}$$

So, here g & h both have zeros of order 1
so it's a removable singularity and $\mathrm{Res}(f, 1) = 0$

6 This is HW 4 - Part 1
Problem 11.