Math 5680

4/5/28

Ex: $\boxed{\text{HW 4-Part 1 \#6}}$

Consider $f(z) = \dfrac{z+1}{z^3(z^2+1)}$

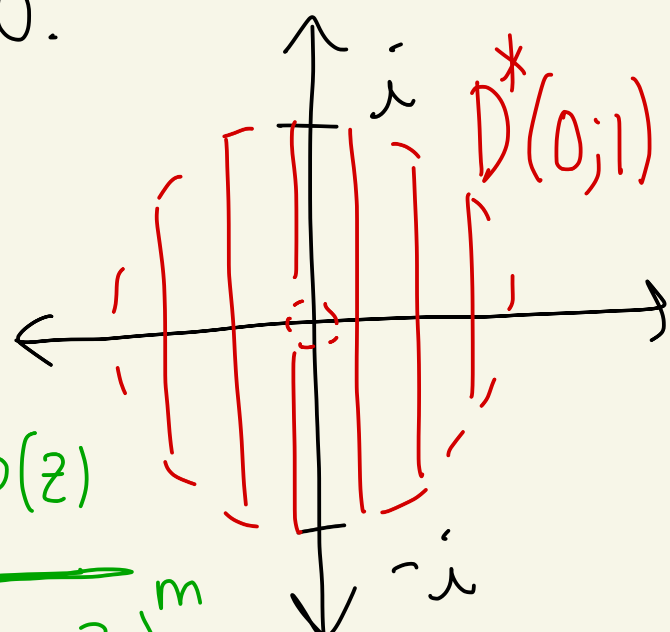
f has isolated singularities at $0, i, -i$

$z^3 = 0 \to z = 0$

$z^2+1 = 0 \to z = \pm i$

Let's look at $z_0 = 0$.

Let $z \in D^*(0;1)$.



Then,

$$f(z) = \frac{\left(\dfrac{z+1}{z^2+1}\right)}{z^3} \qquad \frac{\varphi(z)}{(z-z_0)^m}$$

Let $\varphi(z) = \dfrac{z+1}{z^2+1}$. We have that

φ is analytic at $z_0 = 0$

and $\varphi(0) = \frac{0+1}{0^2+1} = 1 \neq 0$.

So the theorem from Monday says that $z_0 = 0$ is a pole of order 3 and

$$\mathrm{Res}(f;0) = \frac{\varphi^{(3-1)}(0)}{(3-1)!} = \frac{\varphi^{(2)}(0)}{2!}$$

We have

$$\varphi'(z) = \frac{(1)(z^2+1) - (z+1)(2z)}{(z^2+1)^2}$$

$$= \frac{-z^2-2z+1}{(z^2+1)^2}$$

And

$$\varphi^{(2)}(z) = \frac{(-2z-2)(z^2+1)^2 - 2(z^2+1)(2z)(-z^2-2z+1)}{(z^2+1)^4}$$

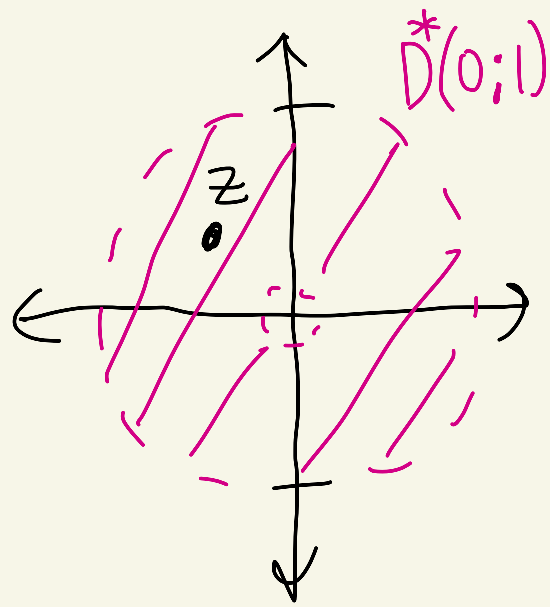
So,

$$\varphi^{(2)}(0) = \frac{(-2)(1)^2 - 2(1)(0)(1)}{1} = -2$$

Thus,

$$\text{Res}(f;0) = \frac{\varphi^{(2)}(0)}{2!} = \frac{-2}{2} = -1$$

Let's double check this by finding the Laurent series in $D^*(0;1)$



Let $z \in D^*(0;1)$

Then, $0 < |z| < 1$.

And,

$$f(z) = \frac{z+1}{z^3(z^2+1)}$$

$$= (1+z)\cdot\frac{1}{z^3}\cdot\frac{1}{1+z^2}$$

$$= (1+z)\cdot\frac{1}{z^3}\cdot\frac{1}{1-(-z^2)}$$

$$= (1+z)\cdot\frac{1}{z^3}\cdot\left[1-z^2+z^4-z^6+\dots\right]$$

$$\frac{1}{1-w} = 1+w+w^2+\dots$$

$$|w|<1$$

$$= (1+z)\left[\frac{1}{z^3}-\frac{1}{z}+z-z^3+\dots\right]$$

$$= \frac{1}{z^3}-\frac{1}{z}+z-z^3+\dots$$

$$+\frac{1}{z^2}-1+z^2-z^4+\dots$$

$$|z|<1$$

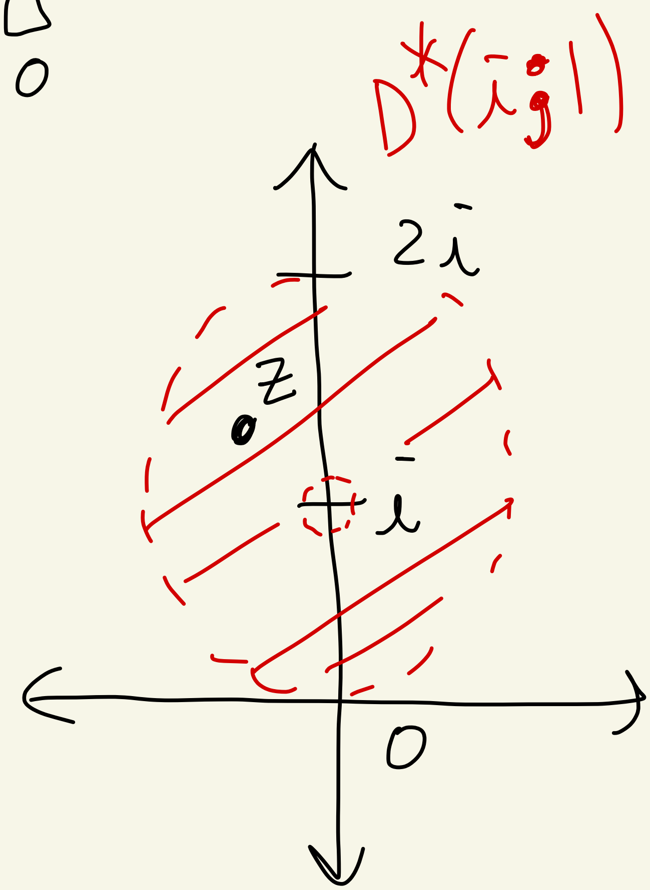
$$|z^2|<1$$

$$|-z^2|<1$$

$$= \frac{1}{z^3}+\frac{1}{z^2}-\underbrace{\frac{1}{z}}_{\uparrow}-1+z+z^2-z^3-z^4+\dots$$

So, $\mathrm{Res}(f;0) = -1.$

What if $z_0 = i$?



$$f(z) = \frac{z+1}{z^3(z^2+1)}$$

Let $z \in D^*(i;1)$

Then,

$$f(z) = \frac{z+1}{z^3(z+i)(z-i)}$$

$$= \frac{\left(\dfrac{z+1}{z^3(z+i)}\right)}{(z-i)}$$

$$\frac{\varphi(z)}{(z-z_0)^1}$$

Let $\varphi(z) = \dfrac{z+1}{z^3(z+i)}$

φ is analytic at i
and $\varphi(i) = \dfrac{i+1}{-i(2i)} \neq 0$

So, the theorem says that
f has a pole of order 1
at $z_0 = i$ and

$$\text{Res}(f; i) = \frac{\varphi^{(1-1)}(i)}{(1-1)!} = \frac{\varphi^{(0)}(i)}{0!}$$

$$= \varphi(i) = \frac{i+1}{-2i^2} = \frac{i+1}{2}$$

$$= \boxed{\frac{1}{2} + \frac{1}{2}i}$$

Theorem: (on simple poles)

Suppose f has an isolated singularity at z_0.

Then, z_0 is a simple pole of f iff $\lim_{z \to z_0} (z - z_0) f(z)$ exists

and is non-zero.

Moreover, if z_0 is a simple pole then

$$\text{Res}(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

Ex: Let $f(z) = \frac{\cos(z)}{z}$ and $z_0 = 0$

$z_0 = 0$ is an isolated singularity.

And,

$$\lim_{z \to 0} \underset{(z-z_0)}{z} \cdot \frac{\cos(z)}{z} = \lim_{z\to 0} \cos(z)$$

$$= \cos(0)$$

$$= 1 \neq 0$$

So, we have a simple pole and

$\mathrm{Res}(f;0) = 1$.

Let's check:

If $z \neq 0$, then

$$\frac{\cos(z)}{z} = \frac{1}{z}\left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right]$$

$$= \underbrace{\frac{1}{z}}_{\text{pole of order } 1} - \frac{z}{2!} + \frac{z^3}{4!} - \frac{z^5}{6!} + \cdots$$

$\mathrm{Res}(f;0) = 1$

Theorem: Let g and h be analytic at z_0.
Suppose $g(z_0) \neq 0$, $h(z_0) = 0$, $h'(z_0) \neq 0$.
Then, $f(z) = \dfrac{g(z)}{h(z)}$ has a simple pole at z_0 and

$$\text{Res}(f; z_0) = \frac{g(z_0)}{h'(z_0)}$$

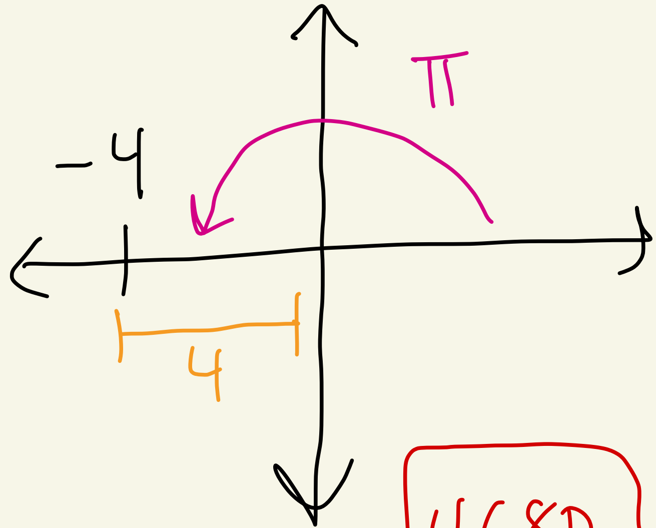
Ex: Let $f(z) = \dfrac{z}{z^4 + 4}$

Where are the singularities of f?

When $z^4 + 4 = 0$.

Solving:

$$z^4 = -4 = 4e^{\pi i}$$



roots:

$$z_k = 4^{1/4} e^{\left(\frac{\pi}{4} + \frac{2\pi k}{4}\right)i}$$

$k = 0, 1, 2, 3$

$$z_0 = \sqrt{2}\, e^{\frac{\pi}{4} i}$$

$$z_1 = \sqrt{2}\, e^{\frac{3\pi}{4} i}$$

$$z_2 = \sqrt{2}\, e^{\frac{5\pi}{4} i}$$

$$z_3 = \sqrt{2}\, e^{\frac{7\pi}{4} i}$$

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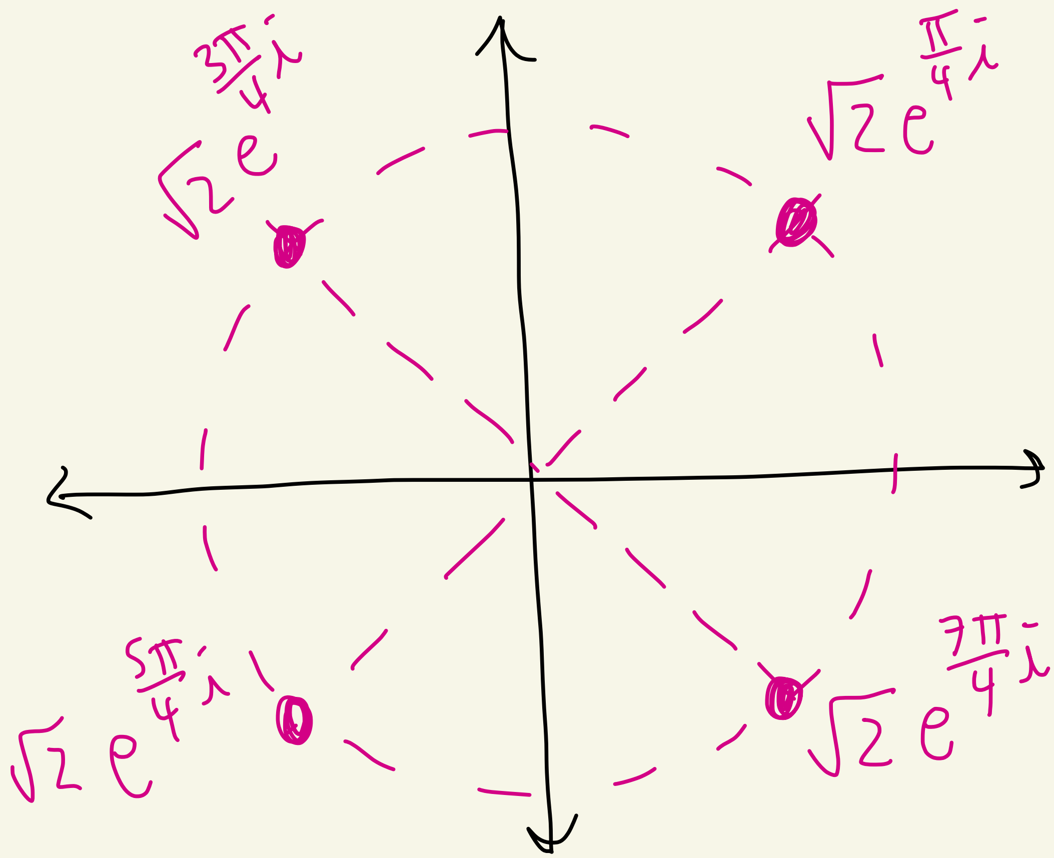
$$z^n = re^{i\theta}$$

$$z_k = r^{1/n} e^{\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)i}$$

$k = 0, 1, 2, \ldots, n-1$



$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$z_0 = \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right] = \sqrt{2}\left[\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right]$$

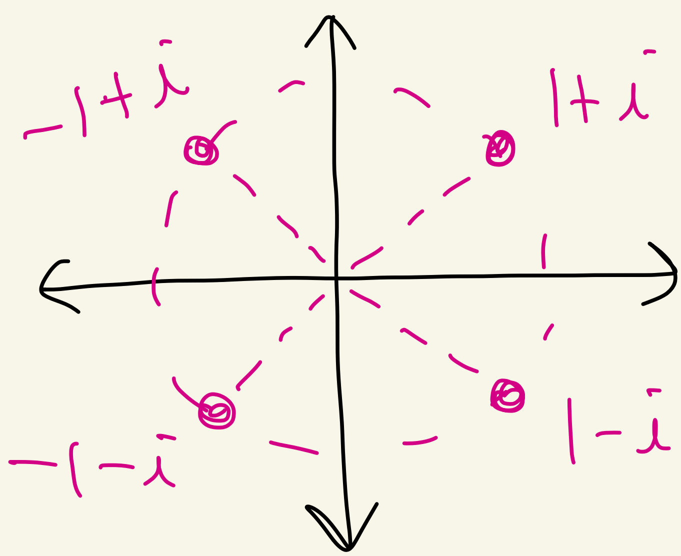
$$= 1 + i$$

$$z_1 = \sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right] = \sqrt{2}\left[-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right]$$

$$= -1 + i$$

$$z_2 = -1 - i$$

$$z_3 = 1 - i$$



You can verify these are all simple poles and find the residues. Let's check for one of them.

Let's check out $z_0 = 1 + i$.

$$f(z) = \frac{z}{z^4+4} = \frac{g(z)}{h(z)}$$

Then,

$g(1+i) = 1+i \neq 0$

$h(1+i) = 0$

$h'(1+i) = 4(1+i)^3 \neq 0$ $\leftarrow$ $h'(z) = 4z^3$

g and h are analytic everywhere and so are analytic at $1+i$.

The theorem says we have a simple pole and

$$\mathrm{Res}(f; 1+i) = \frac{g(1+i)}{h'(1+i)} = \frac{1+i}{4(1+i)^3}$$

$$= \boxed{\frac{1}{4} \cdot \frac{1}{(1+i)^2}}$$