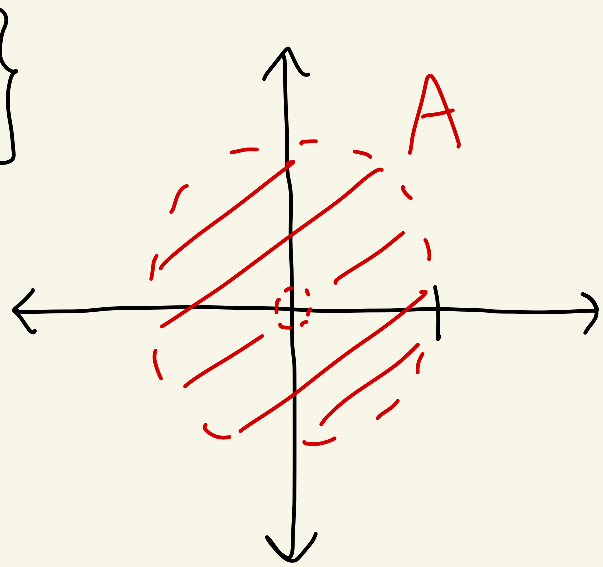
Math 5680
3/22/23

Ex: Let $f(z) = \dfrac{1}{z(z-1)}$

Let $A = \{ z \mid 0 < |z| < 1 \}$



f is analytic on A.

Let's find f's Laurent series on A.

Let $z \in A$.

Then $0 < |z| < 1$.

So,

$$\frac{1}{z(z-1)} = -\frac{1}{z}\left[\frac{1}{1-z}\right]$$

$$= -\frac{1}{z}\left[1 + z + z^2 + z^3 + \cdots\right]$$

$\boxed{|z|<1}$

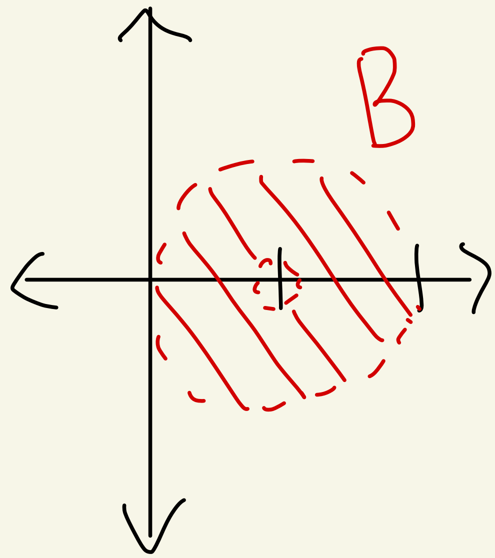
$$= -\frac{1}{z} - 1 - z - z^2 - z^3 - \cdots$$

$$\overbrace{\sum_{n=1}^{\infty} \frac{b_n}{z^n}} + \overbrace{\sum_{n=0}^{\infty} a_n z^n}$$

Let $B = \{ z \mid 0 < |z-1| < 1 \}$

Let $z \in B$

Then, $0 < |z-1| < 1$.



So,

$$\frac{1}{z(z-1)} = \frac{1}{(z-1)} \cdot \frac{1}{z}$$

$$= \frac{1}{(z-1)} \cdot \frac{1}{1+(z-1)}$$

$$= \frac{1}{(z-1)} \cdot \frac{1}{1-(-(z-1))}$$

$$\boxed{\begin{array}{l} |z-1|<1 \\ |-(z-1)|<1 \end{array}}$$

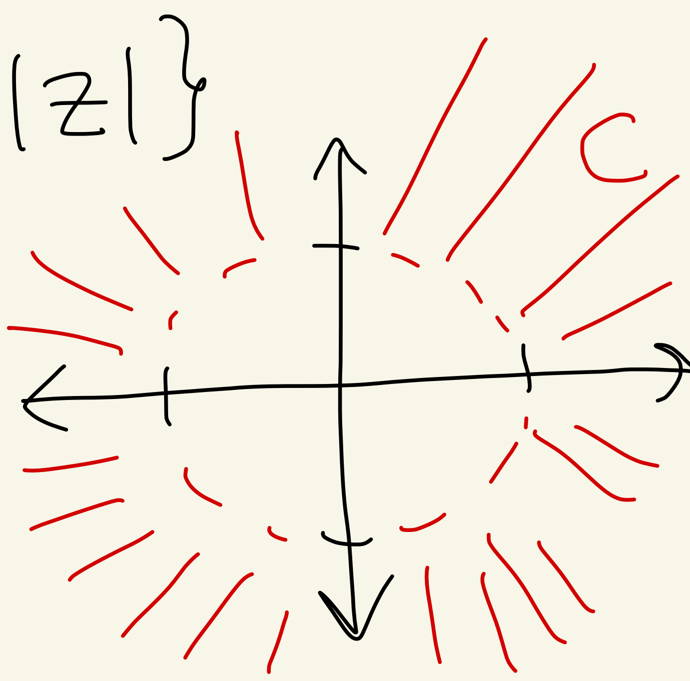
$$= \frac{1}{(z-1)} \cdot \left[1-(z-1)+(z-1)^2 - (z-1)^3+(z-1)^4-\dots\right]$$

$$= \underbrace{\frac{1}{(z-1)}}_{\sum_{n=1}^{\infty} \frac{b_n}{(z-1)^n}} + \underbrace{-1+(z-1)-(z-1)^2+\dots}_{\sum_{n=0}^{\infty} a_n (z-1)^n}$$

Let $C = \{z \mid 1<|z|\}$

f is analytic on C.

Let $z \in \mathbb{C}$.



Then, $1 < |z|$.

Note

$$\frac{1}{z(z-1)} = -\frac{1}{z}\cdot\left[\frac{1}{1-z}\right]$$

can't expand since need $|z|<1$

$$= -\frac{1}{z}\cdot\frac{1}{z}\left[\frac{1}{\frac{1}{z}-1}\right]$$

$$= \left(-\frac{1}{z}\right)\left(-\frac{1}{z}\right)\left(\frac{1}{1-1/z}\right)$$

$\left|\frac{1}{z}\right|<1$ since $1<|z|$

$$= \frac{1}{z^2}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]$$

$$= \underbrace{\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\frac{1}{z^5}+\cdots}$$

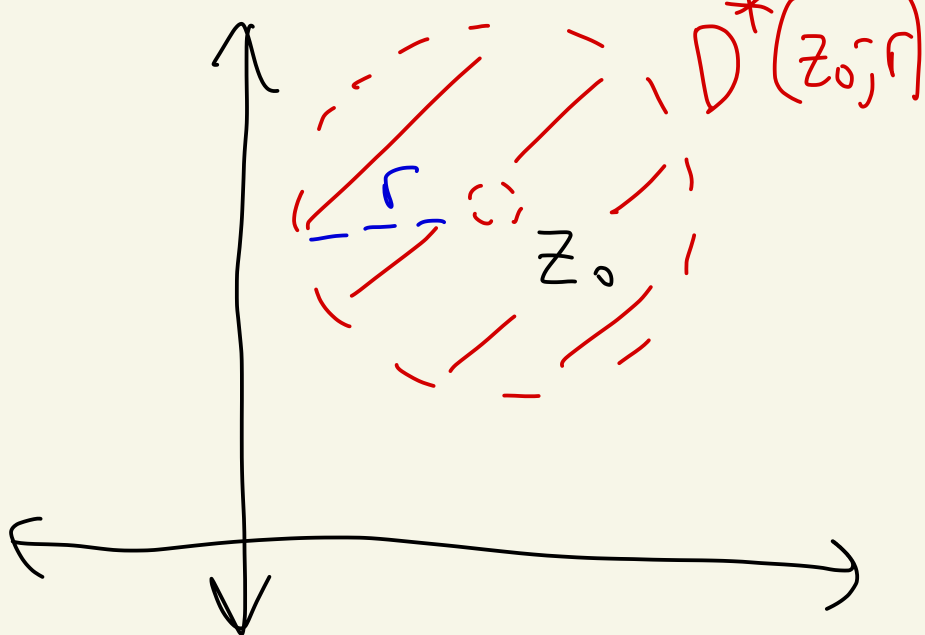
$$\sum_{n=1}^{\infty} \frac{b_n}{z^n} + \bigcirc \qquad \sum_{n=0}^{\infty} a_n z^n$$

<u>Def:</u> Let $z_0 \in \mathbb{C}$. We say that z_0 is an <u>isolated singularity</u> of f if

① f is not analytic at z_0

and ② f is analytic in some deleted r-neighborhood

$$D^*(z_0; r) = \{ z \mid 0 < |z - z_0| < r \}$$



If this is the case, then

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

for $z \in D^*(z_0; r)$ where the above is the Laurent series for f in $D^*(z_0; r)$.

Furthermore:

(A) If all but a finite number of the b_n's are zero, then z_0 is called a <u>pole</u> of f.

If k is the largest integer where $b_k \neq 0$, then z_0 is called a <u>pole of order k</u>.

A pole of order 1 is called

a <u>simple pole</u>.

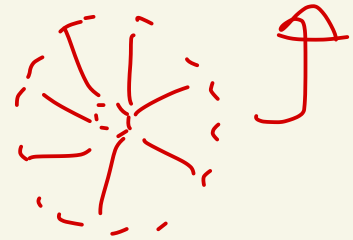
(B) If an infinite number of the b_n's are non-zero, then z_0 is called an <u>essential singularity</u>.

(C) We call b_1 the <u>residue</u> of f at z_0 and write $\text{Res}(f; z_0) = b_1$

(D) If all the b_n's are zero we say that z_0 is a <u>removable singularity</u>.

In this case,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{for } z \in D^*(z_0; r)$$



Define

$$\tilde{f}(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{for } z \in D(z_0; r)$$

Then, $f(z) = \tilde{f}(z)$ for $z \in D^*(z_0; r)$

but $\tilde{f}$ is also defined at z_0, as

$$\tilde{f}(z_0) = a_0 + a_1(z_0 - z_0) + a_2(z_0 - z_0)^2 + \cdots$$

$$= a_0$$

$\tilde{f}$ is analytic on $D(z_0; r)$ since the power series converges there.

So, $\tilde{f}$ extends f to be an analytic function on $D(z_0;r)$.

<u>Ex:</u> Let

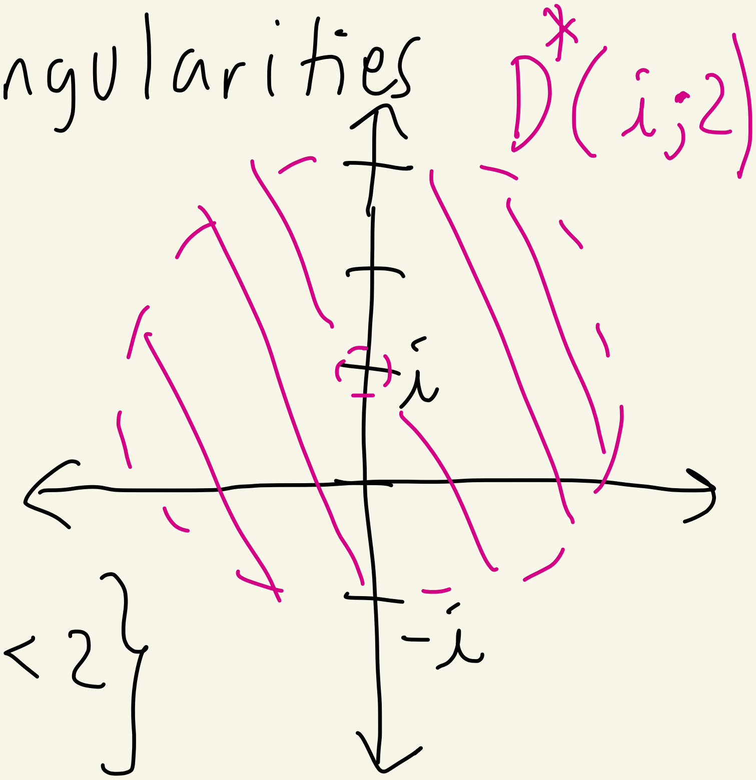
$$f(z) = \frac{z}{(z-i)(z^2+1)}$$

$$\boxed{\begin{aligned} z^2+1&=0 \\ (z+i)(z-i)&=0 \end{aligned}}$$

$$= \frac{z}{(z-i)^2(z+i)}$$

f has isolated singularities at i and $-i$.

Consider



$$D^*(i;2) = \{z \mid 0 < |z-i| < 2\}$$

Let $z \in D^*(i;2)$.

Then,

$$f(z) = \frac{z}{(z-i)(z^2+1)}$$

$$= \frac{z}{(z-i)^2(z+i)}$$

need to deal with

$$= \frac{i+(z-i)}{(z-i)^2} \cdot \frac{1}{(z+i)}$$

$$= \frac{i+(z-i)}{(z-i)^2} \cdot \frac{1}{(2i+z-i)}$$

$$= \frac{i+(z-i)}{(z-i)^2} \cdot \frac{1}{2i} \cdot \left(\frac{1}{1-\left(-\frac{(z-i)}{2i}\right)} \right)$$

$$\frac{1}{1-w} = 1+w+w^2+w^3+\dots$$
$$|w|<1$$

$$\stackrel{\downarrow}{=} \frac{i+(z-i)}{(z-i)^2} \cdot \frac{1}{2i}\left[1 - \frac{z-i}{2i} + \frac{(z-i)^2}{(2i)^2} - \frac{(z-i)^3}{(2i)^3} + \dots\right]$$

$$\left|\frac{z-i}{2i}\right| = \frac{|z-i|}{2} < 1$$
since $|z-i|<2$

$$= \left[i+(z-i)\right]\left[\frac{1}{2i(z-i)^2} - \frac{1}{(2i)^2(z-i)} + \frac{1}{(2i)^3} - \frac{(z-i)}{(2i)^4} + \frac{(z-i)^2}{(2i)^5} - \dots\right]$$

$$\frac{1}{i} = -i$$

$$\frac{1}{2i} = -\frac{1}{2}i$$

$$= \frac{1/2}{(z-i)^2} + \frac{\frac{i}{4} - \frac{1}{2}i}{(z-i)} + \left(\frac{i}{(2i)^3} - \frac{1}{(2i)^2}\right)$$

$$\frac{i}{4} - \frac{1}{2}i = -\frac{1}{4}i$$

$$+\left(-\frac{i}{(2i)^4}+\frac{1}{(2i)^3}\right)(z-i)$$

$$+\cdots$$

f has a pole of order 2

$$\mathrm{Res}(f;i)=\frac{-i}{4}=b_1$$