Math 5680

1/23/23

HW 0 Topic — Sequences

Def: A sequence $(z_n)_{n=1}^{\infty}$ is an ordered, infinite list of complex numbers.

Ex: Let $z_n = 3 + \frac{i}{n}$ where $n \geq 1$.

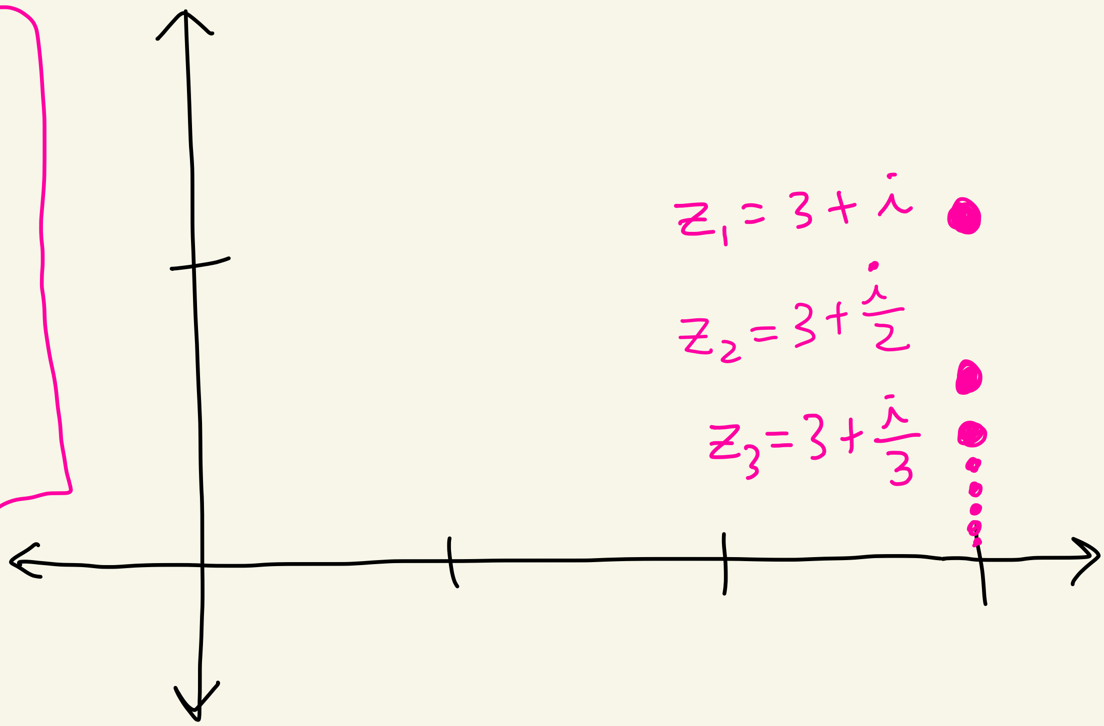
$z_1 = 3 + i$

$z_2 = 3 + \frac{i}{2}$

$z_3 = 3 + \frac{i}{3}$

$z_4 = 3 + \frac{i}{4}$

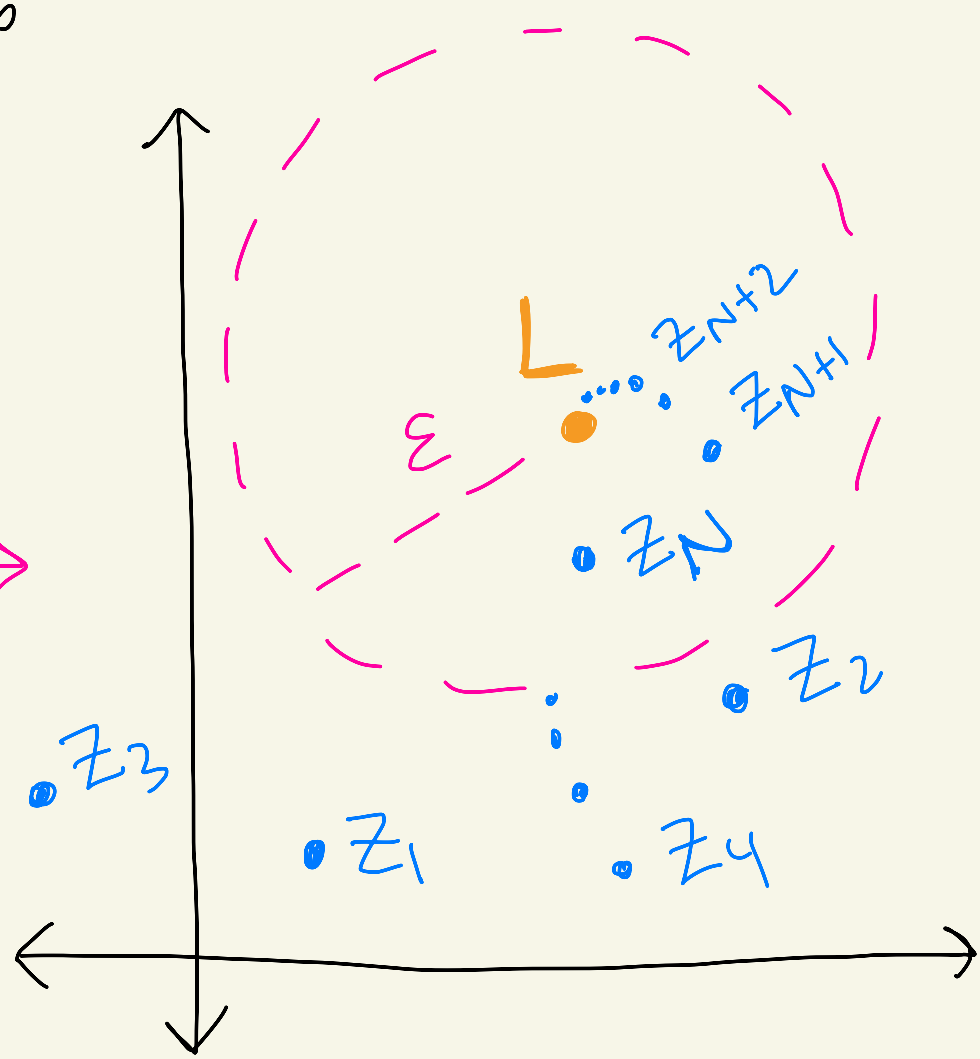
$\vdots \qquad \vdots$



Def: A sequence $(z_n)_{n=1}^{\infty}$ of complex numbers converges to $L \in \mathbb{C}$ if for every $\varepsilon > 0$ there exists $N > 0$ where if $n \geq N$ then $|z_n - L| < \varepsilon$

If $(z_n)_{n=1}^{\infty}$ converges to L then we write $\lim_{n\to\infty} z_n = L$.

N depends on ε



Theorem [math 4680]

Let $(z_n)_{n=1}^{\infty}$ be a sequence of complex numbers and $L \in \mathbb{C}$.

Suppose $z_n = x_n + i y_n$ for $n \geq 1$ and $L = a + ib$.

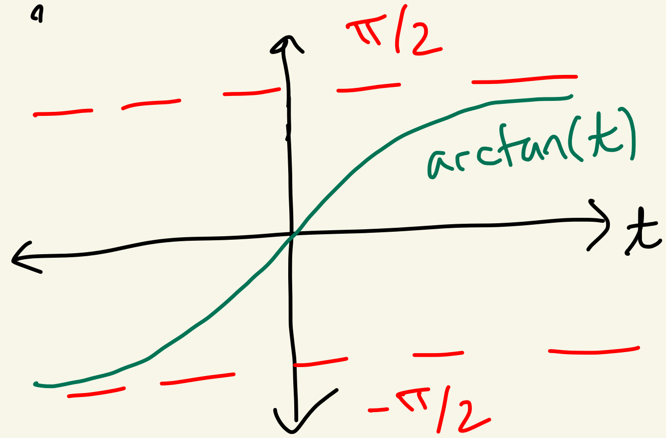
Then, $\lim\limits_{n\to\infty} z_n = L$ 4680 $\mathbb{C}$-limit

iff $\underbrace{\lim\limits_{n\to\infty} x_n = a \text{ and } \lim\limits_{n\to\infty} y_n = b.}_{\text{4650 / } \mathbb{R}\text{-limits}}$

Proof: See 4680 notes ▨

Ex: Let $z_n = \underbrace{\frac{2}{n}}_{x_n} + i\,\underbrace{\arctan(n)}_{y_n}$

Does z_n converge?



In $\mathbb{R}$,

$$\lim_{n\to\infty} \frac{2}{n} = 0$$

and

$$\lim_{n\to\infty} \arctan(n) = \frac{\pi}{2}$$

From the theorem

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \frac{2}{n} + i \lim_{n\to\infty} \arctan(n)$$

$$= 0 + i\frac{\pi}{2} = i\frac{\pi}{2}$$

Def: A sequence $(z_n)_{n=1}^{\infty}$ of complex numbers is a Cauchy sequence if for every $\varepsilon > 0$ there exists $N > 0$ where if $n, m \geq N$ then $|z_n - z_m| < \varepsilon$

Theorem: A sequence $(z_n)_{n=1}^{\infty}$ of complex numbers is Cauchy iff there exists $L \in \mathbb{C}$ where $\lim_{n \to \infty} z_n = L$

[Says: Cauchy iff converges]

proof:

$(\Leftarrow)$ Suppose $(z_n)_{n=1}^{\infty}$ converges to $L \in \mathbb{C}$.

Let $\varepsilon > 0$.

Since $\lim_{n \to \infty} z_n = L$ we know there exists $N > 0$ where if $n \geq N$ then $|z_n - L| < \frac{\varepsilon}{2}$.

Hence, if $n, m \geq N$, then

$$
\begin{aligned}
|z_n - z_m| &= |z_n - L + L - z_m| \\
&\overset{\Delta}{\leq} |z_n - L| + |L - z_m| \\
&= |z_n - L| + |z_m - L| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}
$$

Thus, (z_n) is Cauchy.

$(\Leftarrow)$ Suppose $(z_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Let $z_n = x_n + i y_n$ for $n \geq 1$.

By HW 0 problem 3, since $(z_n)_{n=1}^{\infty}$ is Cauchy in $\mathbb{C}$ we know that

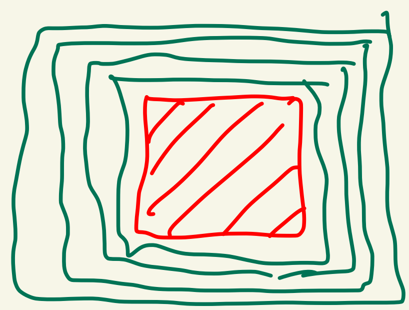
$(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are Cauchy in $\mathbb{R}$.

From Math 4650 (Analysis I) since $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are Cauchy by the completeness of $\mathbb{R}$ we know there exist $x \in \mathbb{R}$ and $y \in \mathbb{R}$ where $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} y_n = y$.

Let $L = x + iy$.

By our previous theorem

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n = x + iy = L$$



<u>Theorem:</u> Let $(z_n)_{n=1}^{\infty}$ and $(w_n)_{n=1}^{\infty}$ be sequences of complex numbers.

① Suppose $\lim_{n\to\infty} z_n = A$ and $\lim_{n\to\infty} w_n = B$

(a) If $\alpha, \beta \in \mathbb{C}$, then
$$\lim_{n\to\infty} (\alpha z_n + \beta w_n) = \alpha A + \beta B$$

(b) $\lim_{n\to\infty} (z_n w_n) = AB$

(c) If $w_n \neq 0$ for all n and $B \neq 0$, then $\lim_{n\to\infty} \frac{z_n}{w_n} = \frac{A}{B}$

② If $(z_n)_{n=1}^{\infty}$ converges, then its bounded, that is there exists $M > 0$ where $|z_n| \leq M$ for all $n \geq 1$.