Math 4740

1/24/22

I will use your calstatela email to mass email the class announcements.

Let me know if you want me to use a different email.

HW 1 Topic
Sets and Probability Spaces

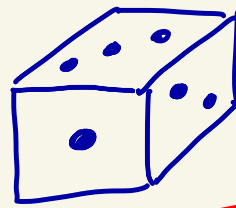
Def: A set is a collection of objects/elements.

If x is an element of a set S then we write $\underbrace{x \in S}_{\text{read: "x is in S"}}$.

If x is not an element of a set S then we write $\underbrace{x \notin S}_{\text{read: "x is not in S"}}$.

If S has a finite number of elements then the size of S is denoted by $|S|$.

<u>Ex:</u> Let's make a set that models rolling a six-sided die.



Let

$$S = \{1, 2, 3, 4, 5, 6\}$$

possible outcomes of rolling a 6-sided die

We have

$$3 \in S$$

$$8 \notin S$$

$$|S| = 6$$

later we will call S the sample space

<u>Note:</u> Order doesn't matter in a set. For example,

$$\{1, 2, 3, 4, 5, 6\} = \{2, 6, 5, 1, 3, 4\}$$

<u>Note:</u> Sets can't have duplicates

$\{1, 1, 5\}$ is not a set

General way to make a set

$\{$ description of elements in the set $|$ conditions the elements must satisfy to be in the set $\}$

read $|$: "where" "such that"

Some people use : instead of $|$

Ex: Let's make a set that models rolling two 6-sided dice, one green and one red.

$$S = \left\{ (g, r) \;\middle|\; \begin{matrix} g = 1,2,3,4,5,6 \\ r = 1,2,3,4,5,6 \end{matrix} \right\}$$

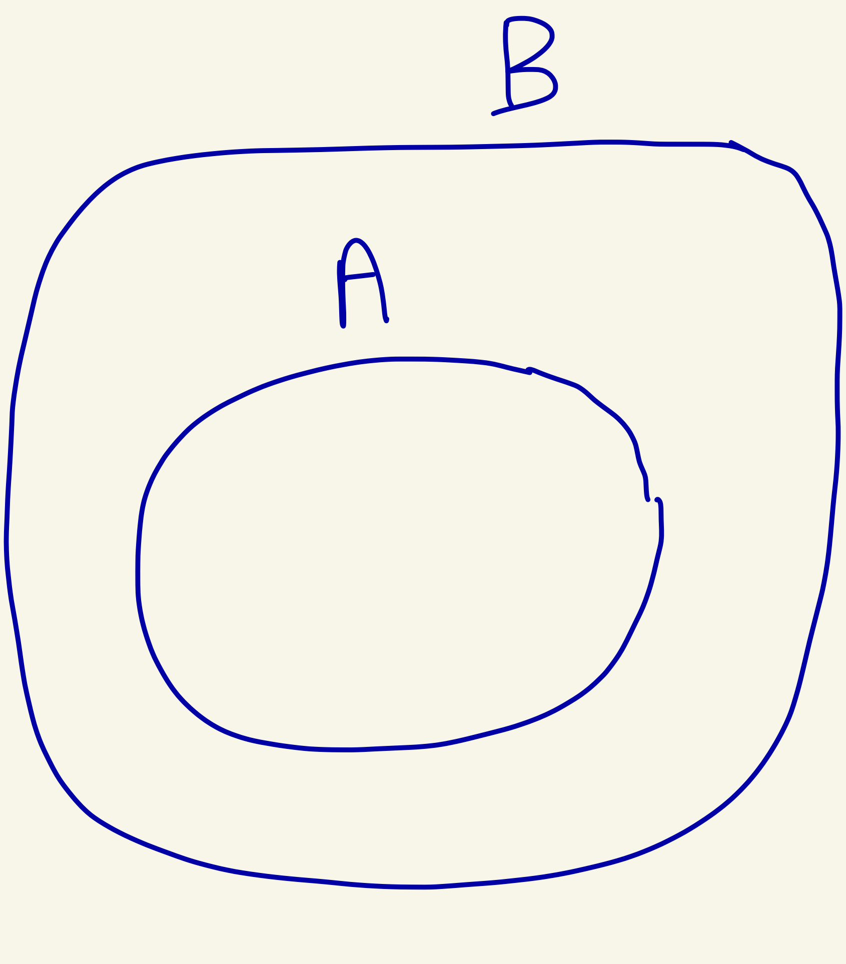
$$= \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$$
$$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),$$
$$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),$$
$$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),$$
$$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),$$
$$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$$

$(3,4) \leftarrow$ represents green die $= 3$, red die $= 4$

$(4,3) \leftarrow$ represents green die $= 4$, red die $= 3$

Note $|S| = 36$

<u>Def:</u> Let A and B be sets. We say that A is a <u>subset</u> of B if every element of A is also an element of B. We write $A \subseteq B$ if A is a subset of B.



Note:

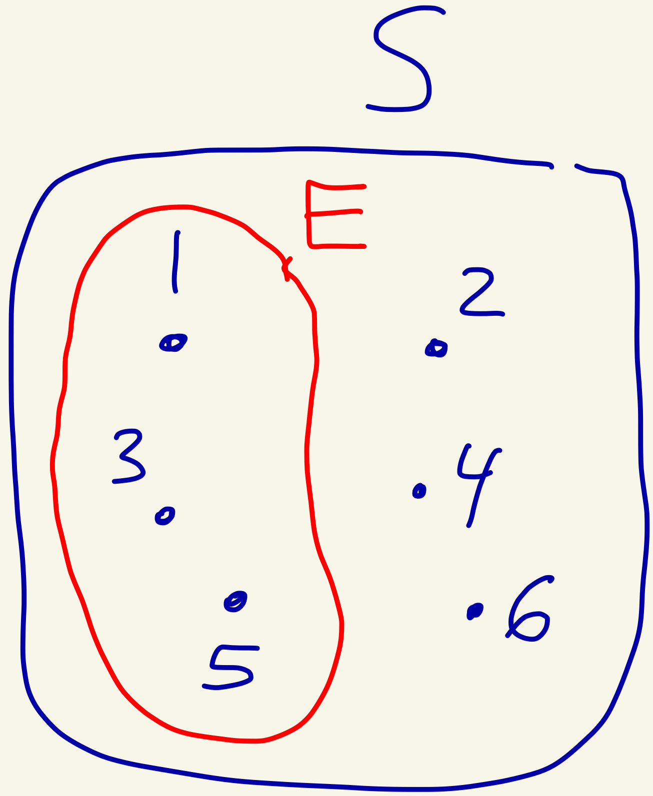
Some people write $A \subset B$

Ex: Consider rolling a 6-sided die.

$S = \{1, 2, 3, 4, 5, 6\}$ ← sample space

$E = \{1, 3, 5\}$

Then $E \subseteq S$.



Later we will call E an event. We will say that E "occured" if when we roll the die either 1, 3, or 5 comes up.

Ex: Suppose we roll two 6-sided dice, one green and one red.

$$\boxed{S = \left\{ (g,r) \;\middle|\; \begin{matrix} g = 1,2,3,4,5,6 \\ r = 1,2,3,4,5,6 \end{matrix} \right\}}$$ ← sample space

Let's make a subset where the dice add up to 7.

$$\boxed{E = \{ (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) \}}$$

Here $E \subseteq S$.

Later we will think of E as the event that the two dice add up to 7.

Note $|E| = 6$

$|S| = 36$

<u>Ex:</u> Suppose we flip a coin three times in a row and record each time if we get H = heads or T = tails.

Let's make a sample space to model this experiement.

sample space means all possible outcomes

$$S = \{ (H,H,H), (H,H,T), (H,T,H), (H,T,T), (T,H,H), (T,H,T), (T,T,H), (T,T,T) \}$$

Here (H, T, H) means:

1st flip = H
2nd flip = T
3rd flip = H

We use parenthesis to denote that order matters.

(Same example continued...)

$$E = \{(H,T,T), (T,H,T), (T,T,H)\}$$

This E would represent the event that exactly one H=head occured in the three flips.

Note

$$|S| = 8$$

$$|E| = 3$$

Math 4740

1/26/22

Note:

Tests will be taken on campus during class time unless the school stays online and we don't go back to in person.

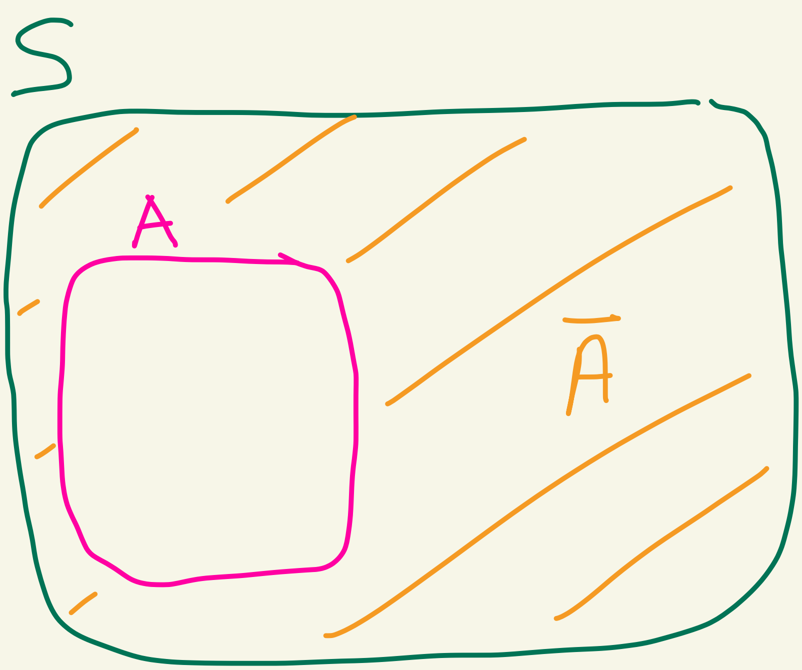
Note:

I put old student notes on the website after the schedule in case you want to get an idea of what we will do in the class

Def: Suppose S is some set and suppose $A \subseteq S$. The complement of A in S is defined to be

$$\underbrace{\overline{A} = \{ x \mid x \in S \text{ and } x \notin A \}}$$

read: $\overline{A}$ consists of all x where x is in S and x is not in A.



Two other notations for $\overline{A}$ are

$$A^c$$

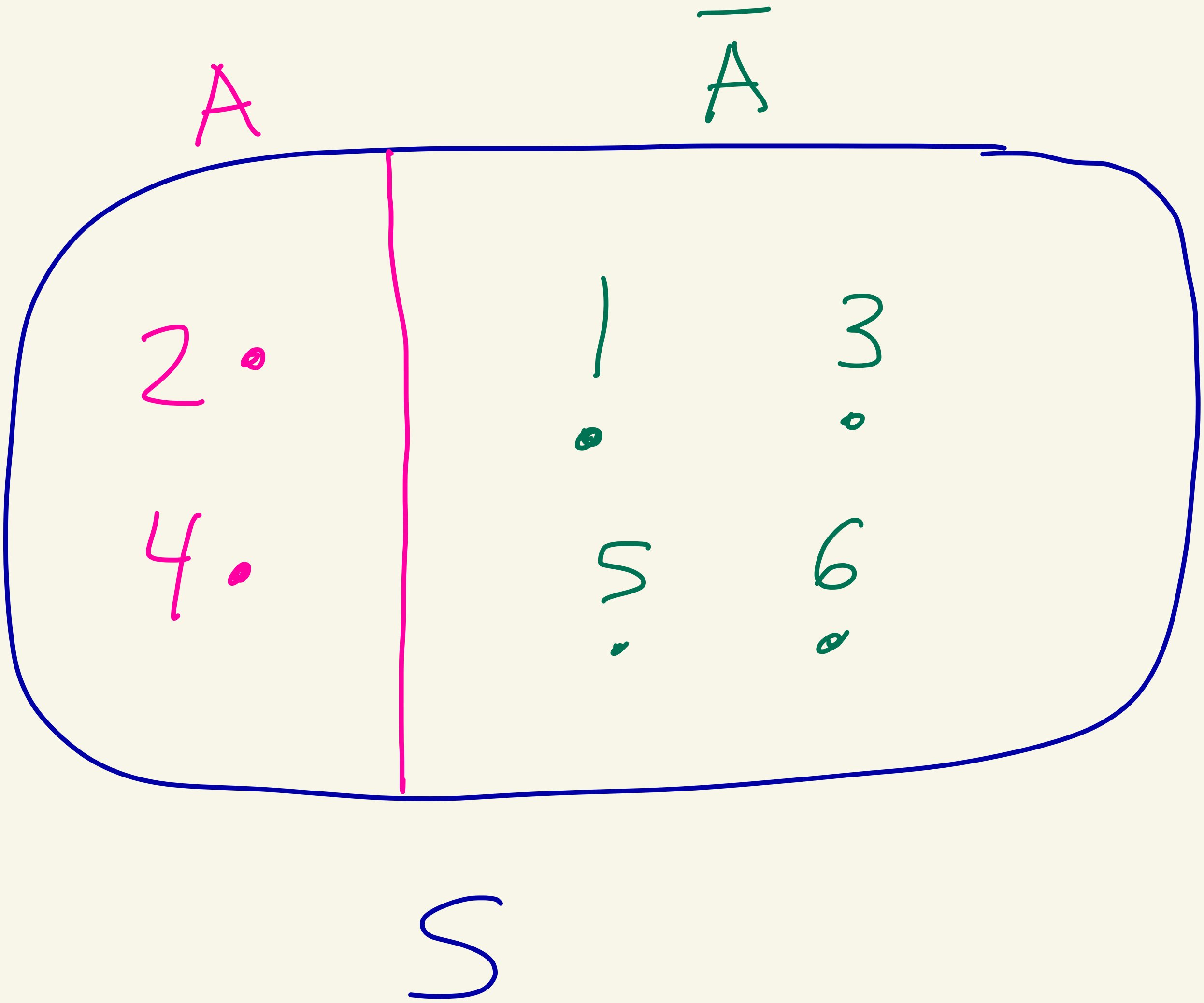
$$S - A$$

Ex: Let

$S = \{1, 2, 3, 4, 5, 6\}$

$A = \{4, 2\}$

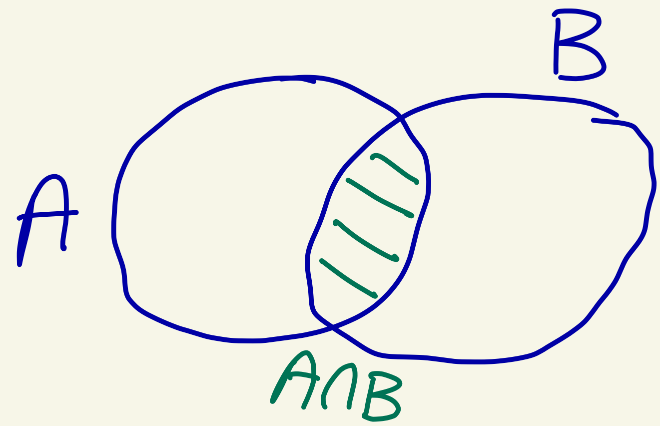
$\bar{A} = \{$



Def: Let A and B be sets.
The intersection of A and B is

$$\underbrace{A \cap B = \{ x \mid x \in A \text{ and } x \in B \}}$$

read: $A \cap B$ consists of all x where x is in A and x is in B

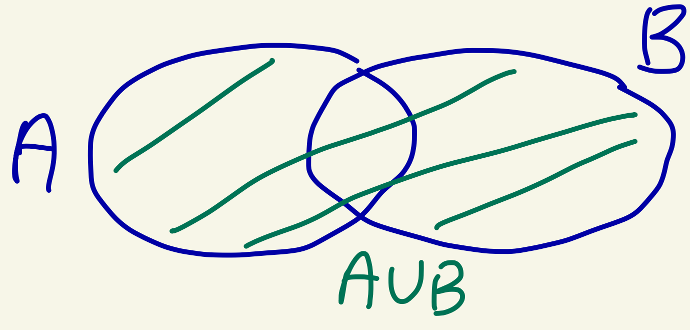


The union of A and B is

$$\underbrace{A \cup B = \{ x \mid x \in A \text{ or } x \in B \}}$$

read: $A \cup B$ consists of all x where x is in A or x is in B.

In math "or" can mean both



<u>Def:</u> The <u>empty set</u> is the set with no elements.

It's denoted by ϕ or $\{ \}$.

Ex: Let S be the sample space we made for flipping a coin three times in a row.

$$S = \{(H,H,H), (H,H,T), (H,T,H), (H,T,T), (T,H,H), (T,H,T), (T,T,H), (T,T,T)\}$$

Let

$$A = \{(H,H,T), (H,H,H), (T,T,H)\}$$

$$B = \{(T,T,T), (T,T,H), (H,H,H), (T,H,T)\}$$

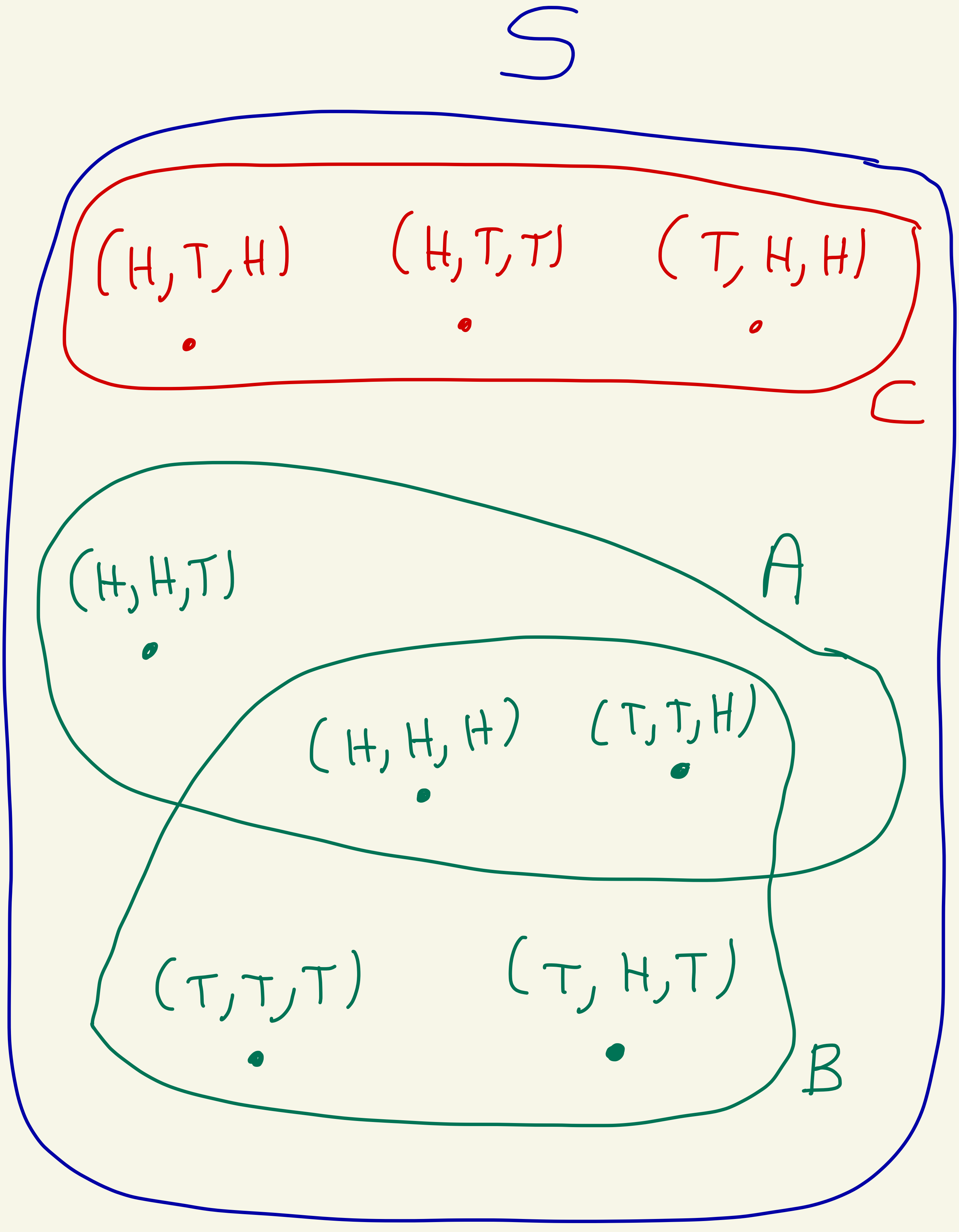
$$C = \{(H,T,H), (H,T,T), (T,H,H)\}$$

Then

$$A \cup B = \{(H,H,T), (H,H,H), (T,T,H), (T,T,T), (T,H,T)\}$$

$$A \cap B = \{(H,H,H), (T,T,H)\}$$

$$A \cap C = \emptyset \qquad B \cap C = \emptyset$$


S
(H, T, H)
(H, T, T)
(T, H, H)
C
(H, H, T)
A
(H, H, H)
(T, T, H)
(T, T, T)
(T, H, T)
B

Def: We say that two sets X and Y are disjoint if $X \cap Y = \phi$

Ex: In the previous example,

- $A \cap C = \phi$ so A and C were disjoint
- $B \cap C = \phi$ so B and C were disjoint

<u>Def:</u> Let $A_1, A_2, \ldots, A_n$ be sets.

Define

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

$$= \underbrace{\{ x \mid x \in A_1 \text{ and } x \in A_2 \text{ and } \cdots \text{ and } x \in A_n \}}_{\text{the } x\text{'s that are in } \underline{\text{all}} \text{ the } A_i\text{'s}}$$

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

$$= \underbrace{\{ x \mid x \text{ is in at least one of the sets } A_1, A_2, \ldots, A_n \}}_{\text{put all the } A_1, A_2, \ldots, A_n \text{ together into one set.}}$$

Ex: Let

$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

this could represent rolling a 12-sided die

dodecahedron

$A_1 = \{1, 2, 3\}$ $\quad A_4 = \{8, 3\}$

$A_2 = \{3, 4, 5\}$

$A_3 = \{5, 6, 7, 4\}$

Then,

$\bigcup_{i=1}^{4} A_i = A_1 \cup A_2 \cup A_3 \cup A_4$

$= \{1, 2, 3, 4, 5, 6, 7, 8\}$

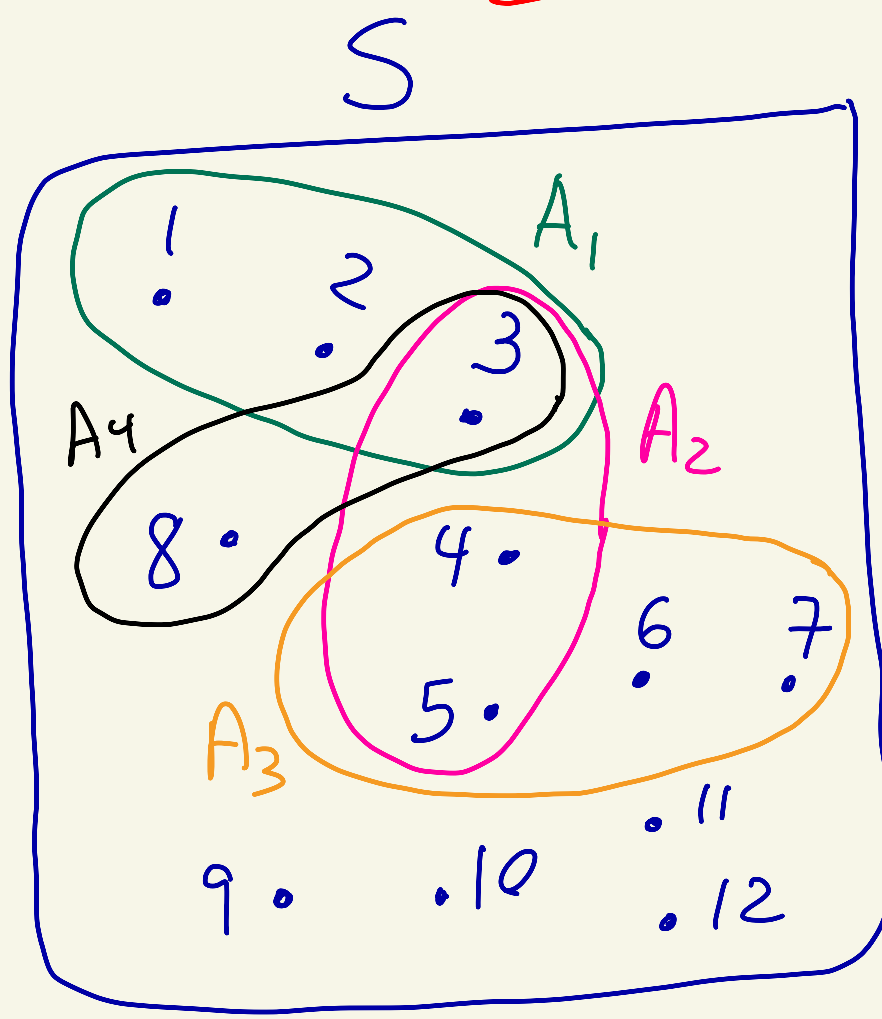
$A_1 \cup A_2 \cup A_4$

$= \{1, 2, 3, 4, 5, 8\}$

$\bigcap_{i=1}^{4} A_i = A_1 \cap A_2 \cap A_3 \cap A_4$

$= \emptyset$

$A_1 \cap A_2 \cap A_4 = \{3\}$



<u>Def:</u> Suppose we have an infinite number of sets $A_1, A_2, A_3, \ldots$

Define

$$\bigcap_{i=1}^{\infty} A_i = \{ x \mid x \text{ is in every one of the } A_i \}$$

$$\bigcup_{i=1}^{\infty} A_i = \{ x \mid x \text{ is in at least one of the } A_i \}$$

Ex: Let

$S = \mathbb{Z}$

$\mathbb{Z}$ is the set of integers

$= \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$

For $i \geq 1$, define

$$A_i = \left\{ n \;\middle|\; \begin{array}{l} n \text{ is an integer} \\ -i \leq n \leq i \end{array} \right\}$$

$$= \{-i, \ldots, 0, \ldots, i\}$$

For example,

$A_1 = \{-1, 0, 1\}$

$A_2 = \{-2, -1, 0, 1, 2\}$

$A_3 = \{-3, -2, -1, 0, 1, 2, 3\}$

$A_4 = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$

Then, $\displaystyle\bigcap_{i=1}^{\infty} A_i = \{-1, 0, 1\}$

$$\bigcup_{i=1}^{\infty} A_i = \mathbb{Z}$$

<u>Def:</u> Let A and B be two sets. The <u>Cartesian product</u> of A and B is

$$\underbrace{A \times B}_{\text{read: "A cross B"}} = \underbrace{\left\{ (a,b) \;\middle|\; \begin{array}{l} a \text{ is in } A \\ b \text{ is in } B \end{array} \right\}}_{\text{all elements of the form } (a,b) \text{ where } a\in A \text{ and } b\in B}$$

<u>Ex:</u> Let $A = \{H, T\}$
$B = \{1, 2, 3, 4\}$

Then,

$$A \times B = \{ (H,1), (H,2), (H,3), (H,4), (T,1), (T,2), (T,3), (T,4) \}$$

$$A \times A = \{ (H,H), (H,T), (T,H), (T,T) \}$$

$$B \times B = \{ (1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4) \}$$

Math 4740

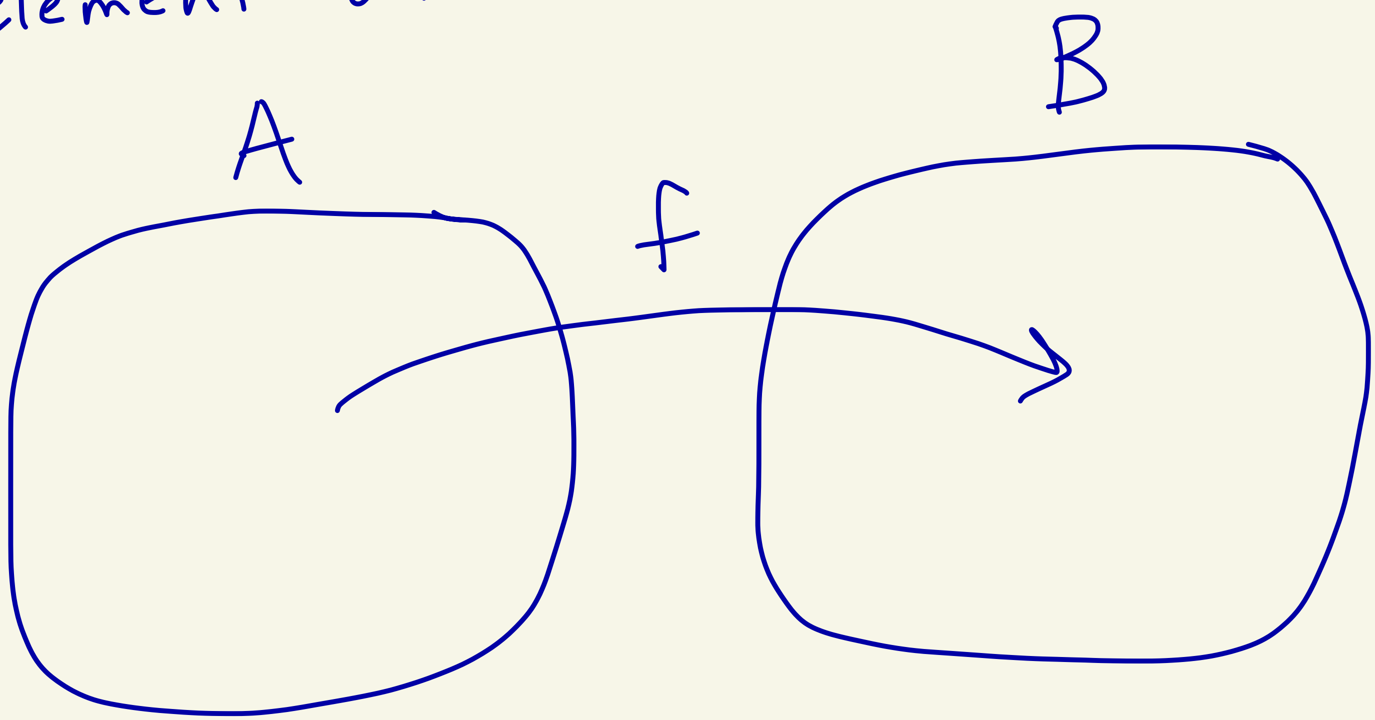
1/31/22

I sent out an email about office hours and also posted it on canvas under "Office Hours"
(Email / canvas has zoom link)

Monday: 1:45 - 2:45
(will become in person when we go back in Simpson Tower 317)

Tuesday: 12:30 - 2:00
(These will stay online)

<u>Def:</u> Let A and B be sets.
A <u>function</u> f from A to B,
notated $f: A \longrightarrow B$, is
a rule that assigns to each
element of A a distinct
element of B



Ex: Let

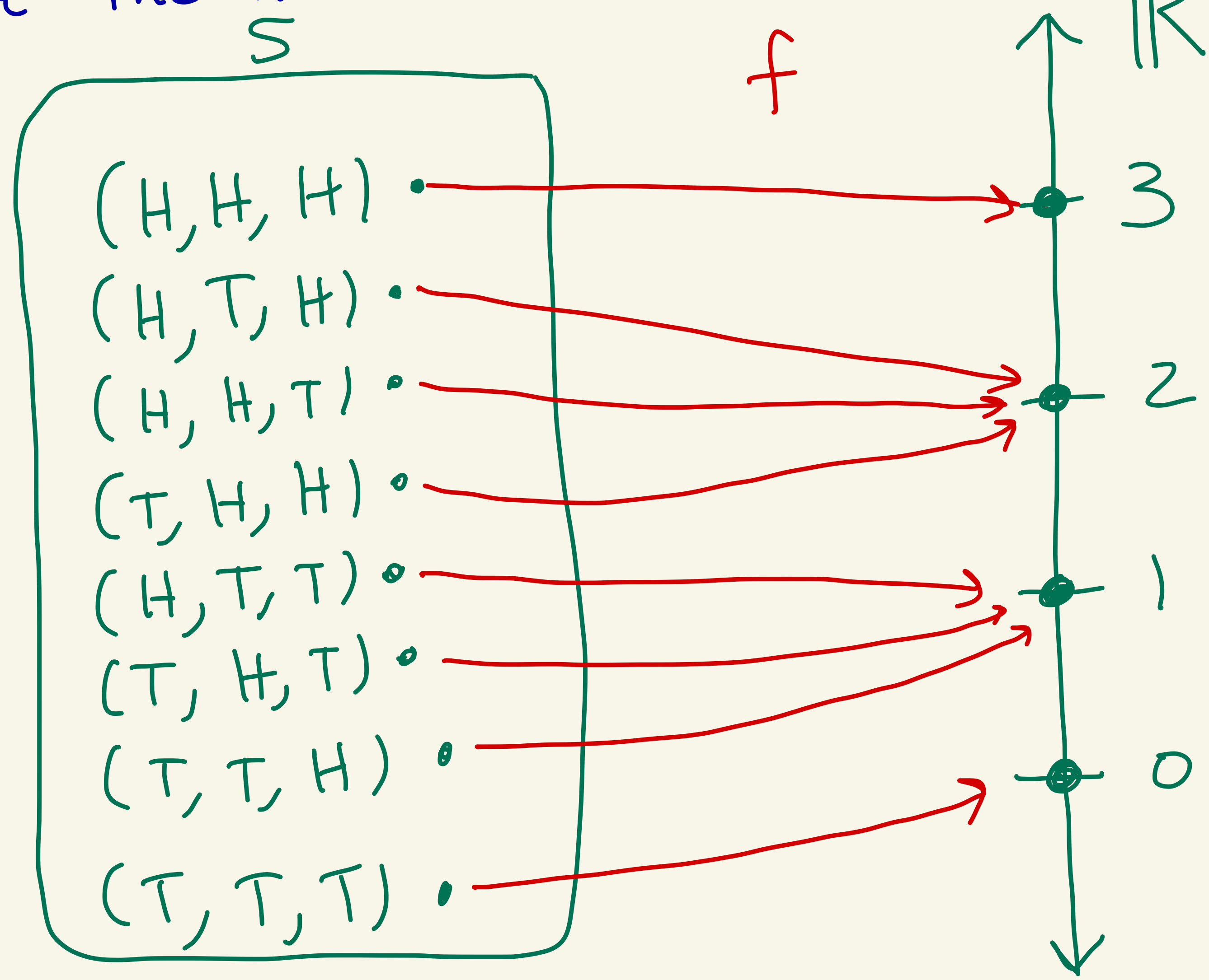
$S = \{(H,H,H), (H,T,H), (H,H,T), (H,T,T), (T,H,H), (T,T,H), (T,H,T), (T,T,T)\}$

be the sample space of flipping a coin 3 times.

Let $f : S \to \mathbb{R}$

[$\mathbb{R}$ means set of real numbers]

be the number of heads that occur.



For example, $f((H,T,H)) = 2$

Example of making a probability space

Suppose we want to model the experiment of throwing/rolling one 4-side die.

$$S = \{1, 2, 3, 4\}$$

sample space
all possible outcomes of rolling the die

Omega

$$\Omega = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{2,3,4\}, \{1,2,4\}, \{1,3,4\}, \{1,2,3,4\}\}$$

Ω is called the set of events
It's a set of subsets of S with special properties

Ω contains the sets that we measure the probability of

When S is finite we usually make Ω contain all the subsets of S.

What do these events mean?

ϕ ← represents that no number came up on the die

$\{3\}$ ← represents 3 came up on the die

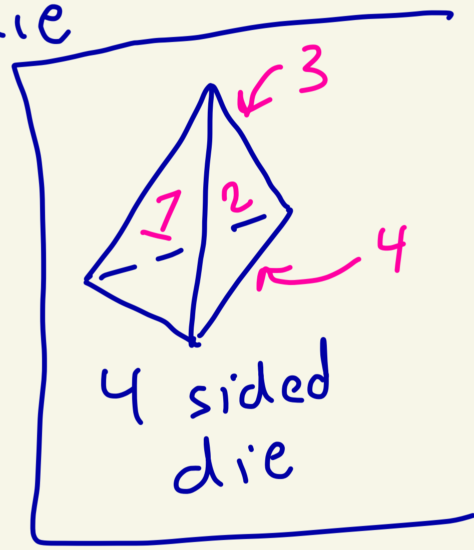
$\{1, 3\}$ ← represents 1 or 3 came up on the die

$\{2, 3, 4\}$ ← represents 2 or 3 or 4 came up on the die

$\{1, 2, 3, 4\}$ ← represents 1 or 2 or 3 or 4 came up on the die

Now we make the probability function $P : \Omega \to \mathbb{R}$.

On a normal 4-sided die each side is equally likely to occur.



First step is to assign the probability of each number/side individually.

$$P(\{1\}) = \frac{1}{4}$$
$$P(\{2\}) = \frac{1}{4}$$
$$P(\{3\}) = \frac{1}{4}$$
$$P(\{4\}) = \frac{1}{4}$$

each side is equally likely

Now we extend P across all the events by doing disjoint sums, for example, define

$$P(\{1,3\}) = P(\{1\}) + P(\{3\}) = \frac{1}{4} + \frac{1}{4} = 1/2$$

What's the probability of not rolling a 1?

$$P(\{2,3,4\}) = P(\{2\}) + P(\{3\}) + P(\{4\})$$

$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \boxed{\frac{3}{4}}$$

$\boxed{.75 \quad 75\%}$

We define

$$P(\emptyset) = 0$$

We have

$$P(\{1,2,3,4\}) = P(\{1\}) + P(\{2\}) + P(\{3\}) + P(\{4\})$$

$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \boxed{1}$$

$\boxed{100\%}$

Def: A probability space consists of two sets and a function (S, Ω, P). We call S the sample space of our experiment. The elements of S are called outcomes. Ω is a set of subsets of S. The elements of Ω are called events. $P: \Omega \to \mathbb{R}$ is a function where for each event E from Ω we get a probability $P(E)$ of the event E. Furthermore, the following axioms must be satisfied:

① S is an event in Ω

② If E is an event in Ω then $\overline{E}$ is an event in Ω

$\overline{E}$ means the comple-ment of E in S

(3) If $E_1, E_2, E_3, \dots$ is a finite or infinite sequence of events in Ω, then $\bigcup_i E_i$ is an event in Ω.

Andrey Kolmogorov gave this def in the 1930s.

(4) $0 \leq P(E) \leq 1$ for all events E in Ω

(5) $P(S) = 1$

(6) If $E_1, E_2, E_3, \dots$ is a finite or infinite sequence of events in Ω that are pair-wise disjoint [that is, $E_i \cap E_j = \phi$ if $i \neq j$]

then $P\left(\bigcup_i E_i\right) = \sum_i P(E_i)$

$\boxed{\text{end of definition}}$

disjoint means no overlap

Remark: A set Ω satisfying ①, ②, and ③ from the previous definition is called a <u>σ-algebra</u> or <u>σ-field</u>.

$\boxed{\begin{matrix}\sigma \\ \text{sigma}\end{matrix}}$

Remark: If Ω is a σ-algebra one can show that

(a) $\phi \in \Omega$

(b) If $E_1, E_2, E_3, \ldots$ are in Ω, then $\bigcap_i E_i$ is in Ω

pf: (a) $S \in \Omega$ by ①. Thus, by ② $\overline{S} = \phi$ is in Ω.

(b) Suppose $E_1, E_2, E_3, \ldots$ are in Ω.

By part ②, $\overline{E_1}, \overline{E_2}, \overline{E_3}, \ldots$ are in Ω.

By part ③, $\bigcup_i \overline{E_i}$ is in Ω.

By part ②, $\overline{\bigcup_i \overline{E_i}}$ is in Ω.

But,

$$\bigcap_i E_i = \overline{\bigcup_i \overline{E_i}}$$

$\blacksquare$

Math 4740

2 - 2 - 22

How to construct a $\boxed{\text{finite}}$ probability space

ie S is finite

Suppose S is a finite sample space that we want to make into a probability space.

Ω = power set of S

Define Ω to be the set that contains all the subsets of S [includes ϕ].

For each element $\omega \in S$ pick some real number n_ω with $0 \leq n_\omega \leq 1$ and define

$$P(\{\omega\}) = n_\omega$$

n_ω is probability of ω happening

At the same time pick these numbers so that

$$\underbrace{\sum_{\omega \in S}}_{\text{means sum over all } \omega \text{ in } S} n_\omega = 1.$$

Ex: $S = \{1,2,3,4\}$

$P(\{1\}) = n_1 = \frac{1}{4}$

$P(\{2\}) = n_2 = \frac{1}{4}$

$P(\{3\}) = n_3 = 1/4$

$P(\{4\}) = n_4 = 1/4$

$n_1 + n_2 + n_3 + n_4 = 1$

Now extend P to any set E in Ω.

Suppose $E = \{\omega_1, \omega_2, \ldots, \omega_n\}$

Define

$$P(E) = \underbrace{\sum_{i=1}^{n} P(\{\omega_i\})}$$

define $P(E)$ to be the sum of the probabilities of the elements of E

If $E = \phi$, define

$$P(\phi) = 0.$$

Theorem: The construction above creates a probability space (S, Ω, P).

proof: I'll put this proof on the website next to todays notes. $\boxed{\diagup\diagup}$

Ex: Suppose you have a six-sided die labeled 1, 2, 3, 4, 5, 6 and through experimentation you noticed it was a weighted die and the probabilities were roughly

number	probability
1	2/8
2	1/8
3	1/8
4	1/16
5	1/16
6	3/8

notice

$$\frac{2}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16}+\frac{3}{8} = 1$$

Let's make a probability space.

Define $S = \{1,2,3,4,5,6\}$

Define $\Omega = \{\text{all subsets of } S\}$

$= \{\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1,2\}, \{1,3\}, \ldots, \{1,2,3,4,5,6\}\}$

Ω = power set of S

Define $P: \Omega \to \mathbb{R}$ by

$P(\{1\}) = 2/8$ $P(\{4\}) = 1/16$

$P(\{2\}) = 1/8$ $P(\{5\}) = 1/16$

$P(\{3\}) = 1/8$ $P(\{6\}) = 3/8$

If E is an event in Ω we define $P(E) = \sum_{\omega \in E} P(\{\omega\})$

and $P(\phi) = 0$.

Note

$$P(S) = P(\{1\}) + P(\{2\}) + P(\{3\}) + P(\{4\}) + P(\{5\}) + P(\{6\})$$

$$= 2/8 + 1/8 + 1/8 + \frac{1}{16} + \frac{1}{16} + \frac{3}{8} = 1$$

What is the probability of rolling an even number?

$$\underbrace{P(\{2,4,6\})}_{\text{Probability of of rolling 2 or 4 or 6}} = P(\{2\}) + P(\{4\}) + P(\{6\})$$

$$= \frac{1}{8} + \frac{1}{16} + \frac{3}{8}$$

$$= 9/16 \approx 0.5625$$

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What is the probability of rolling 1 or 6 ?

$$P(\{1, 6\}) = P(\{1\}) + P(\{6\})$$

$$= \frac{2}{8} + \frac{3}{8} = \frac{5}{8}$$

Note:

You can construct a probability space when S is countably infinite, ie S is infinite and you can list the elements.

Suppose $S = \{w_1, w_2, w_3, w_4, \ldots\}$

infinitely many more

Define Ω to be the set of all subsets of S, ie the power set of S.

Define $P(\{w_i\})$ for each $w \in S$ so that $0 \leq P(\{w_i\}) \leq 1$ and $\sum_{i=1}^{\infty} P(\{w_i\}) = 1$.

If E is an event define
$$P(E) = \sum_{w \in E} P(\{w\})$$

$\boxed{\text{Theorem: This will be a probability space.}}$

Note:

Suppose (S, Ω, P) is a probability space and S is finite.

Suppose each outcome w in S is equally weighted, that is

$$P(\{w\}) = \frac{1}{|S|} \quad \text{for all } w \text{ in } S.$$

If this is the case, its easy to calculate the probability of an event E.

Suppose $E = \{w_1, w_2, \ldots, w_n\}$ has n elements.

Then,

$$P(E) = P(\{w_1\}) + P(\{w_2\}) + \ldots + P(\{w_n\})$$

$$= \frac{1}{|S|} + \frac{1}{|S|} + \ldots + \frac{1}{|S|}$$

$$= \frac{n}{|S|} = \frac{|E|}{|S|}.$$

So, $\boxed{P(E) = \frac{|E|}{|S|}}$

Ex: Suppose we do the experiment of rolling two 6-sided dice. Suppose these are normal dice so each side has equal chance of happening.

$(a,b) \leftarrow$ $\boxed{\text{denote } a \text{ on die 1 and } b \text{ on die 2}}$

$$S = \{ (1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \}$$

$\Omega = \{$all subsets of $S\}$

Each outcome is equally likely.

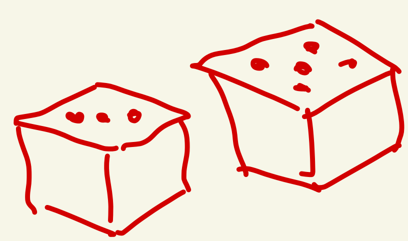
We have $|S| = 36$.

So, $P(\{(a,b)\}) = \frac{1}{36}$ for any a,b.

For example, $P(\{(3,5)\}) = \frac{1}{36}$

3 on first die

5 on 2nd die



Q: What is the probability that the sum of the dice equals 7?

Let E be the event that the sum of the dice is 7.

Then,

$$E = \{ \underset{\substack{\text{6 on} \\ \text{die 1}}}{6}, \underset{\substack{\text{1 on} \\ \text{die 2}}}{1}) \ldots \}$$

$E = \{ (6,1), (5,2), (4,3), (3,4), (2,5), (1,6) \}$

6 on die 1

1 on die 2

$6+1=7$

Since every outcome is equally weighted

$$P(E) = \frac{|E|}{|S|} = \frac{6}{36} = \frac{1}{6}$$

$$\approx 0.16\overline{6}$$

How to construct a finite probability space.

Let S be the finite sample space you want to make into a probability space.
Let Ω be the set that contains all subsets of S (including ϕ).

For each element ω of S define

$$\underbrace{P(\{\omega\}) = n_\omega}_{\text{assign a probability } n_\omega \text{ to each element } \omega \text{ in } S}$$

where $\underbrace{\sum_{\omega \in S}}_{\text{the sum of all the probabilities on the space must be 1.}} n_\omega = 1$, and $0 \leq n_\omega \leq 1$ for each n_ω.

Given an event E (that is a subset of S) where $E = \{\omega_1, \omega_2, \omega_3, \ldots, \omega_A\}$ define

$$P(E) = \sum_{i=1}^{A} P(\{\omega_i\}) = n_{\omega_1} + n_{\omega_2} + \ldots + n_{\omega_A}.$$

That is, the probability of E is the sum of the probabilities of the elements in E.

If $E = \phi$, then define $P(\phi) = 0$.

Then we have that

Prop: (S, Ω, P) constructed as above is a probability space. (9'')

proof: We first prove axioms (4), (5), (6).

axiom 4: Let E be an event from Ω.

Then $0 \leq \sum_{w \in E} P(\{w\}) \leq \sum_{w \in S} P(\{w\}) = 1$.

since $P(\{w\}) \geq 0$ for each w

there are possibly more things in S than in E

by how we defined P

axiom 5: $P(S) = \sum_{w \in S} P(\{w\}) = 1$ by the definition of P.

axiom 6: Let $E_1 = \{w_{11}, w_{12}, w_{13}, \ldots, w_{1n_1}\}$, $E_2 = \{w_{21}, w_{22}, w_{23}, \ldots, w_{2n_2}\}, \ldots, E_k = \{w_{k1}, w_{k2}, \ldots, w_{kn_k}\}$ where $E_i \cap E_j = \phi$ for $i \neq j$ (ie the sets are pairwise disjoint). So, none of the w's are equal to each other. Thus, if $E = E_1 \cup E_2 \cup \ldots \cup E_k$ then $E = \left\{ w_{ij} \,\middle|\, \begin{array}{l} 1 \leq i \leq k \\ \text{and for each } i \\ \text{we have } 1 \leq j \leq n_i \end{array} \right\}$

So, $P\left(\bigcup_{i=1}^{k} E_i\right) = \sum_{1 \leq i \leq k} \sum_{1 \leq j \leq n_i} P(\{w_{ij}\})$

$= \sum_{1 \leq i \leq k} P(E_i)$. ~~...~~

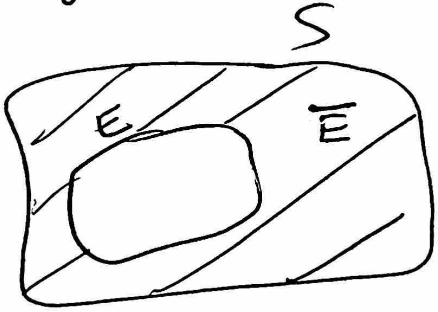
Recall that Ω is the set of all subsets of S. That is

$$\Omega = \{E \mid E \subseteq S\}$$

Note: The empty set ϕ is considered to be a subset of S. It can be logically defined so that is true.

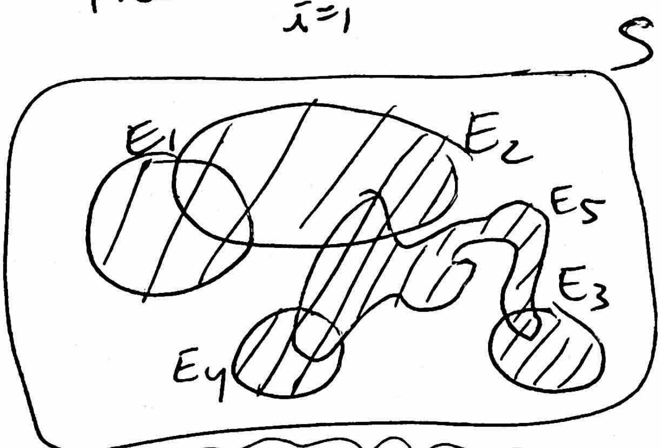
axiom ① S is a subset of S. So, S is in Ω.

axiom ② Suppose that E is a event in Ω. Then E is a subset of S. By definition $\bar{E}$ is a subset of S. So, $\bar{E}$ is an event in Ω.



axiom ③ Let $E_1, E_2, \ldots, E_k$ be events in Ω. (We assume we only have a finite number of E_i since S and Ω are both finite.)

Then $\bigcup_{i=1}^{k} E_i$ is a subset of S by definition.



Therefore, all the axioms are true. Hence (S, Ω, P) is a probability space.

/// is the union, i.e. $\bigcup_{i=1}^{k} E_i$.

Math 4740

2/7/22

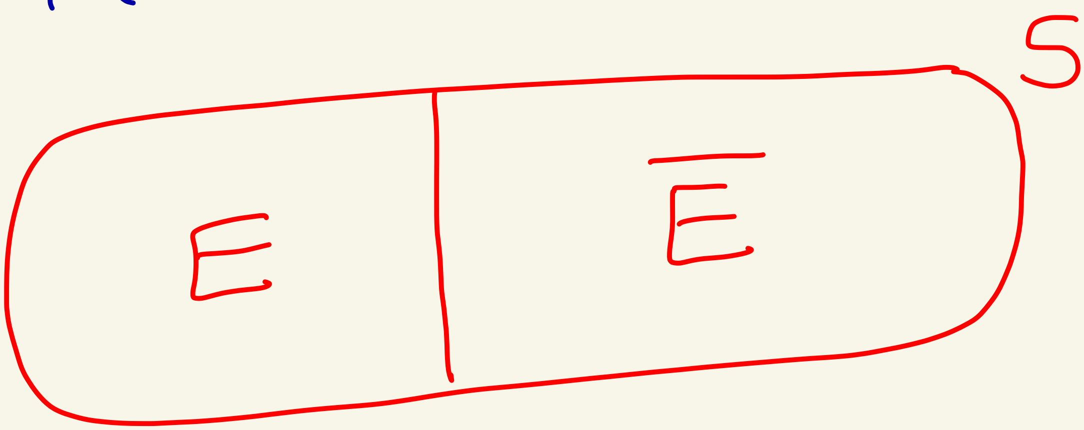
HW 1:

Wait to do problems 9 and 10 until we finish HW 2 topic

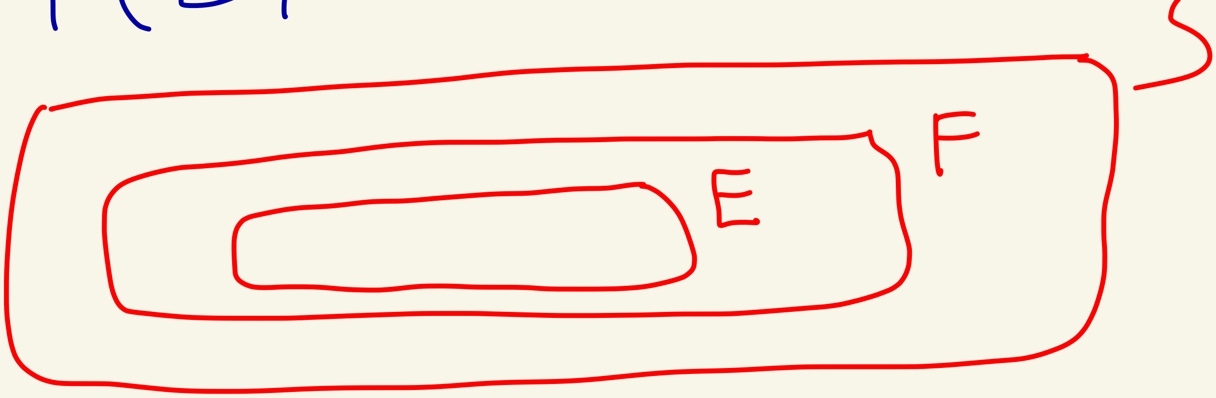
Theorem: Let (S, Ω, P) be a probability space. Let E and F be events.

Then

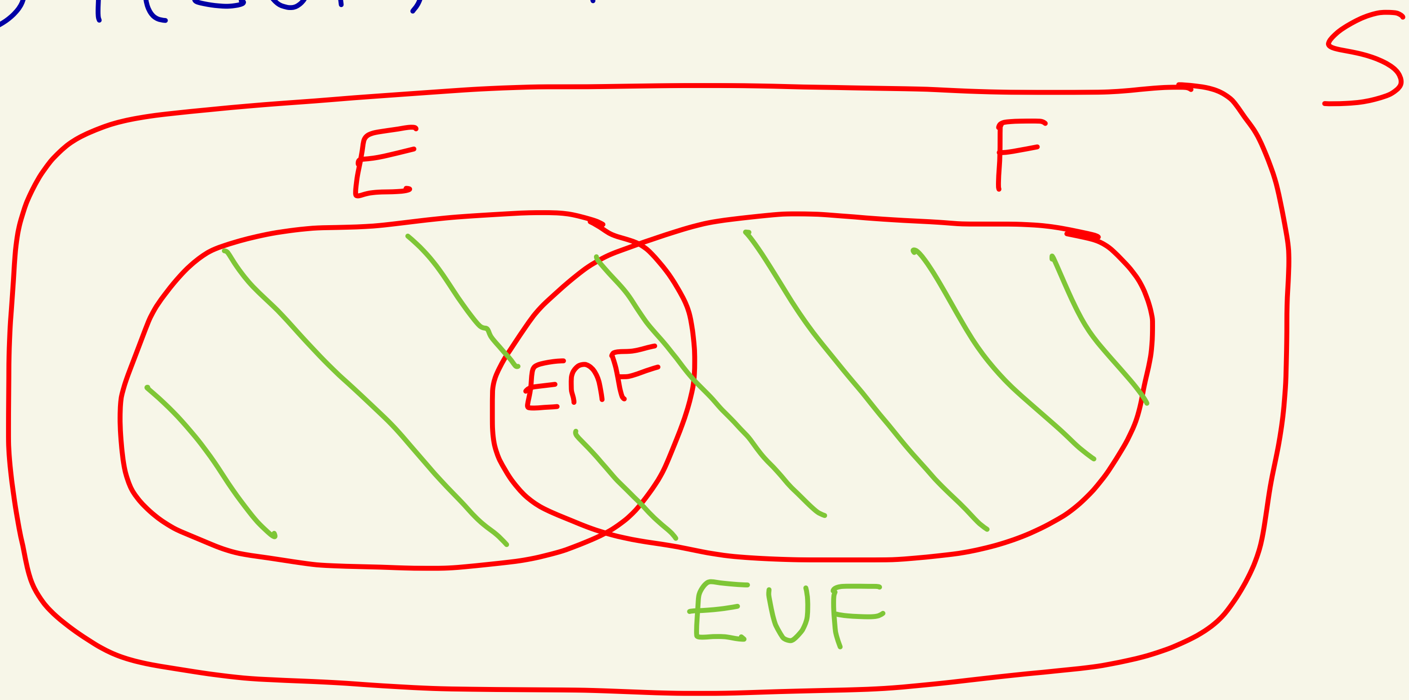
① $P(\overline{E}) = 1 - P(E)$



② If $E \subseteq F$, then $P(E) \leq P(F)$.



③ $P(E\cup F) = P(E) + P(F) - P(E\cap F)$

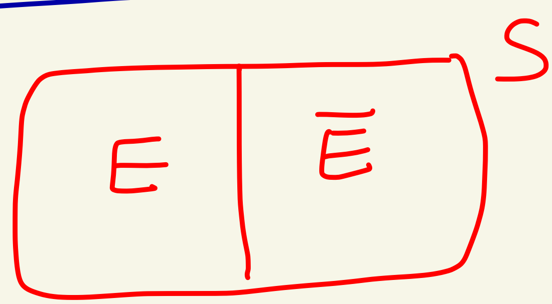


④ If E and F are disjoint, ie $E\cap F = \phi$, then

$P(E\cup F) = P(E) + P(F)$

axiom 6 of prob. space

Proof: Let's prove ①.



We know $S = E\cup\bar{E}$

and $E\cap\bar{E} = \phi$.

So, $1 \underset{\text{axiom 5}}{=} P(S) = P(E\cup\bar{E}) \underset{\text{axiom 6}}{=} P(E) + P(\bar{E})$.

So, $P(\bar{E}) = 1 - P(E)$.

I'll post the proofs of 2 and 3 under the notes for the day on the website for those interested.

Ex: Suppose we roll two 12-sided dice. [Each number on the die are equally likely]. What is the probability that at least one of the dice is 4, 5, 6, 7, 8, 9, 10, 11, or 12?

	die 1	die 2	
Examples:	$\boxed{3}$	$\boxed{7}$	have at least one of 4-12
	$\boxed{8}$	$\boxed{9}$	
	$\boxed{1}$	$\boxed{1}$	doesn't have a 4-12

$$S = \left\{ (a,b) \;\middle|\; \begin{matrix} a = 1,2,\ldots,12 \\ b = 1,2,\ldots,12 \end{matrix} \right\}$$

$$= \{ \underbrace{(1,1)}_{\substack{\text{die 1} = 1 \\ \text{die 2} = 1}}, \underbrace{(5,9)}_{\substack{\text{die 1} = 5 \\ \text{die 2} = 9}}, \underbrace{(10,11)}_{\substack{\text{die 1} = 10 \\ \text{die 2} = 11}}, \ldots \}$$

lots more

So, $|S| = 12 \cdot 12 = 12^2 = 144$

Let E be the event that at least one of the dice is either 4, 5, 6, 7, 8, 9, 10, 11, or 12.

lots more

$$E = \{ (4,4), (6,1), (12,2), (11,12), \ldots \}$$

Too hard to count E.

Let's count $\overline{E}$ which is the event that neither of the dice are 4, 5, 6, 7, 8, 9, 10, 11, or 12.

So, $\overline{E}$ is the event that both dice are in the range 1, 2, 3.

So,

$$\overline{E} = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$$

We have $|\overline{E}| = 9$.

Since each outcome is equally likely with usual 12-sided dice we have

$$P(\overline{E}) = \frac{|\overline{E}|}{|S|} = \frac{9}{144} = \frac{1}{16}$$

Thus,

$$P(E) = 1 - P(\overline{E}) = 1 - \frac{1}{16}$$

$$= \boxed{\frac{15}{16}}$$

$$\approx \boxed{0.9375}$$

Thm: $P(E) = 1 - P(\overline{E})$

HW 2 Topic – Counting & Probability

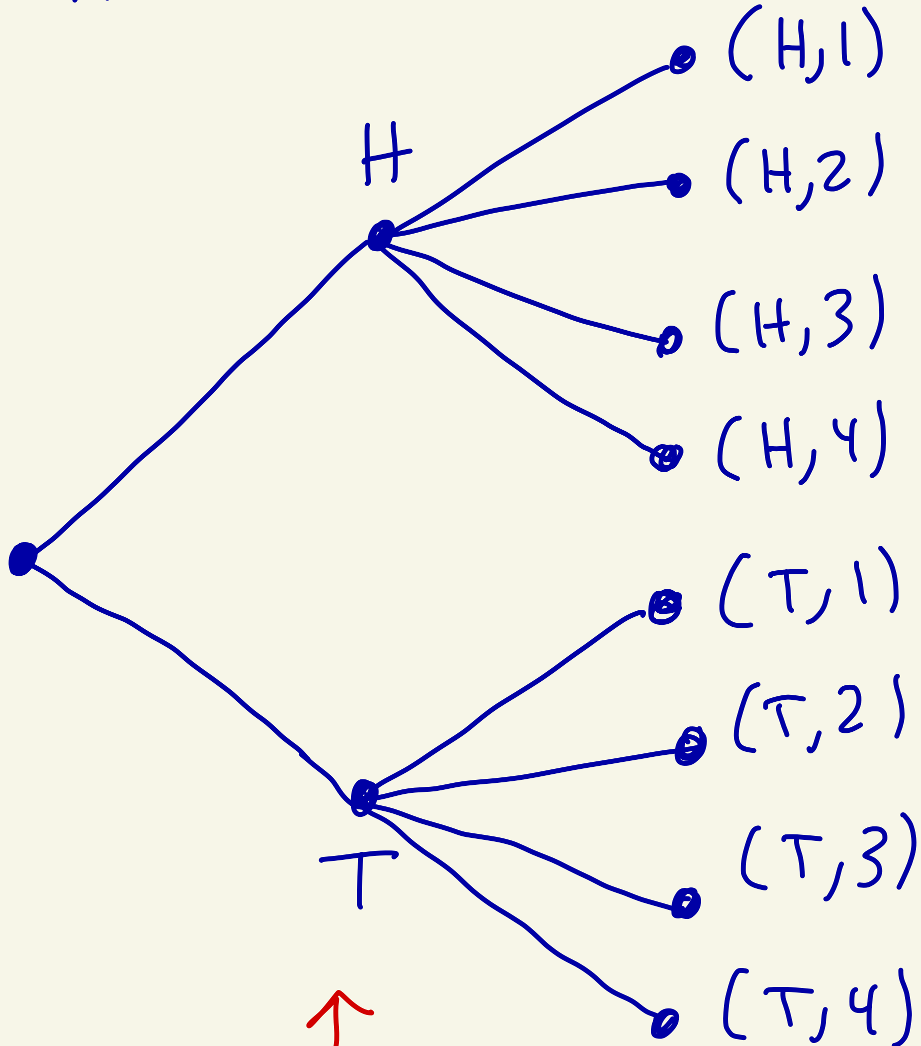
Basic counting principle

If r experiments that are to be performed are such that the first one may result in n_1 possible outcomes; and if for each of these n_1 possible outcomes, there are n_2 possible outcomes for the second experiment; and if for each of the possible outcomes of the first two experiments there are n_3 possible outcomes of the third experiment; and if, ..., then there are

$n_1 n_2 \cdots n_r$ possible outcomes

for the r experiments.

Ex: Suppose we toss a coin and then roll a 4-sided die. How many possible outcomes are there when we do this?



$n_1 \cdot n_2$

$= 2 \cdot 4$

$= 8$

total possible outcomes

$n_1 = 2$

outcomes to tossing a coin

$n_2 = 4$

outcomes to rolling a 4-sided die

Another way to represent this

$$\frac{\overbrace{H/T}^{\text{means H or T}}}{2} \qquad \frac{\overbrace{1-4}^{\text{mean 1,2,3,4}}}{4}$$

$2 \cdot 4 = 8$ possibilities

Ex: In California, a license plate consists of one number (0,1,2,3,...,9) followed by three upper-case letters, followed by 3 numbers.
The only exclusion is that the letters I, O, and Q are not used in spot 2 and spot 4.

Examples are 5AQZ117
0BBC222

How many possible license plates are there?

#	letter not I, O, or Q	letter	letter not I, O, or Q	#	#	#
$n_1 = 10$	$n_2 = 23$	$n_3 = 26$	$n_4 = 23$	$n_5 = 10$	$n_6 = 10$	$n_7 = 10$

total possible license plates:

$10 \cdot 23 \cdot 26 \cdot 23 \cdot 10 \cdot 10 \cdot 10 = \boxed{137,540,000 \text{ possible plates}}$

Birthday Paradox

Suppose there are N people in a classroom. What are the odds that there are at least two people with the same birthday (not year, just day, like two people with Feb 7 birthdays)?

Assumptions:

1. We will assume no one has a Feb 29 leap year birthday
2. We will assume that each day is equally likely.

To be continued...

Proposition Let (S, Ω, P) be a probability space. Let E and F be events. Then

① $P(\overline{E}) = 1 - P(E)$

② If $E \subseteq F$, then $P(E) \leq P(F)$.

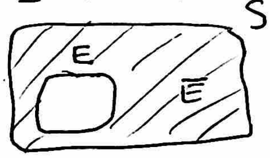
③ $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

④ If $E \cap F = \phi$, then $P(E \cup F) = P(E) + P(F)$

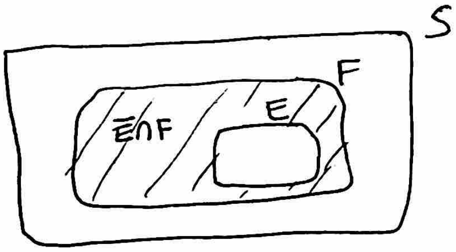
Proof:

① Note that $S = E \cup \overline{E}$. and $E \cap \overline{E} = \phi$.

Thus, $\underset{\boxed{\text{axiom 4}}}{1} = P(S) = P(E \cup \overline{E}) \underset{\boxed{\text{axiom 5}}}{=} P(E) + P(\overline{E})$. Thus, $P(\overline{E}) = 1 - P(E)$.



② Since $E \subseteq F$ we can write $F = E \cup (\overline{E} \cap F)$



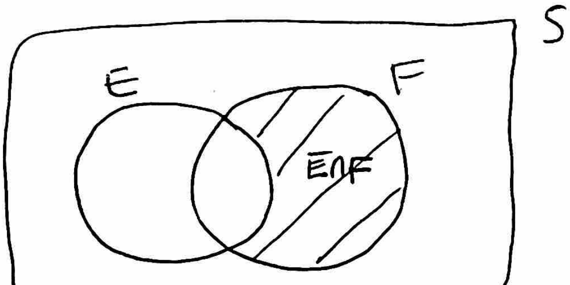
And E and $\overline{E} \cap F$ are disjoint.

Thus, $P(F) = P(E \cup (\overline{E} \cap F))$

$\underset{\boxed{\text{axiom 5}}}{=} P(E) + \underbrace{P(\overline{E} \cap F)}_{\geq 0}$

So, $P(F) \geq P(E)$ [because $P(\overline{E} \cap F) \geq 0$].

③ Note that $E \cup F = E \cup (\overline{E} \cap F)$. And E and $\overline{E} \cap F$ are disjoint. Thus, by axiom 5,

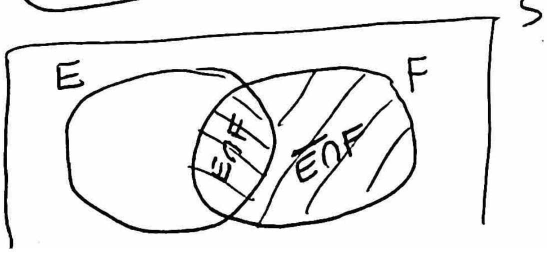


$P(E \cup F) = P(E) + P(\overline{E} \cap F)$.

Furthermore,

$F = (E \cap F) \cup (\overline{E} \cap F)$.

and $E \cap F$ and $\overline{E} \cap F$ are disjoint.



Hence, by axiom 5,

$P(F) = P(E \cap F) + P(\overline{E} \cap F)$.

So, $P(\overline{E} \cap F) = P(F) - P(E \cap F)$.

Thus, $P(E \cup F) = P(E) + P(\overline{E} \cap F)$

$= P(E) + P(F) - P(E \cap F)$.

Math 4740

2/9/22

On Monday we meet at school.

The room has been moved to

SH 162

Different room from syllabus

From last Monday:

Birthday Paradox

Suppose there are N people in a classroom. What are the odds that there are at least two people with the same birthday (not year, just day, like two people with Feb 7 birthdays)?

Assumptions:

1. We will assume no one has a Feb 29 leap year birthday
2. We will assume that each day is equally likely.
3. Assume $N \leq 365$ because if $N > 365$ the probability is 1 or 100%

To analyze this let's think about the sample space size.

Suppose $N=3$.

$$S = \left\{ (\text{date 1}, \text{date 2}, \text{date 3}) \;\middle|\; \begin{array}{l}\text{date } i \\ \text{is a} \\ \text{calendar} \\ \text{day}\end{array} \right\}$$

$$= \Big\{ (\underbrace{\text{Feb 2}}_{\text{student 1}}, \underbrace{\text{April 1}}_{\text{student 2}}, \underbrace{\text{May 3}}_{\text{student 3}}),$$

$$(\underbrace{\text{April 1}}_{\text{student 1}}, \underbrace{\text{Feb 2}}_{\text{student 2}}, \underbrace{\text{May 3}}_{\text{student 3}}),$$

ex of student 1 student 2 having same b-day

$$(\underbrace{\text{May 11}}_{\text{student 1}}, \underbrace{\text{May 11}}_{\text{student 2}}, \underbrace{\text{March 27}}_{\text{student 3}}), \ldots \Big\}$$

lots more

So here when $N=3$.

$$|S| = \underbrace{365}_{\substack{\#\text{ possibilities} \\ \text{for student 1}}} \cdot \underbrace{365}_{\substack{\#\text{ possibilities} \\ \text{for student 2}}} \cdot \underbrace{365}_{\substack{\#\text{ possibilities} \\ \text{for student 3}}}$$

$$= (365)^3$$

For general N, the size of the sample space is $(365)^N$

$$\underset{\text{student 1}}{\underline{\text{365 possibilities}}} \cdot \underset{\text{student 2}}{\underline{\text{365 possibilities}}} \cdots \underset{\text{student } N}{\underline{\text{365 possibilities}}}$$

Let E be the event that there are at least two students in the classroom with the same birthday.

This is too hard so we instead calculate $\bar{E}$ which is the event that there are <u>no</u> students with the same birthday.

$$\underset{\text{Student 1}}{\underline{\text{365 possibilities}}} \cdot \underset{\text{Student 2}}{\underline{\text{364 possibilities}}} \cdot \underset{\text{Student 3}}{\underline{\text{363 possibilities}}} \cdots \underset{\text{Student } N}{\underline{365-(N-1) \text{ possibilities}}}$$

Student 2: Can't be same day as student 1

Student 3: Can't be same day as student 1 or student 2

Student N: Can't be same day as student 1 through student $N-1$

$$|\bar{E}| = \boxed{365} \cdot \boxed{365-1} \cdot \boxed{365-2} \cdots \boxed{365-(N-1)}$$

Thus,

$$P(E) = 1 - P(\overline{E})$$

$$= 1 - \frac{|\overline{E}|}{|S|}$$

since days are equally likely

$$= \boxed{1 - \frac{(365)(364)(363)\cdots(365-(N-1))}{(365)^N}}$$

<u>N=3</u>

$$P(E) = 1 - \frac{365 \cdot 364 \cdot 363}{(365)^3}$$

$$\approx 0.82\,\%$$

N	P(E)
1	0%
2	0.274%
3	0.82%
⋮	⋮
10	11.69%
⋮	⋮
20	41.14%
⋮	
25	56.87%
⋮	
30	70.63%

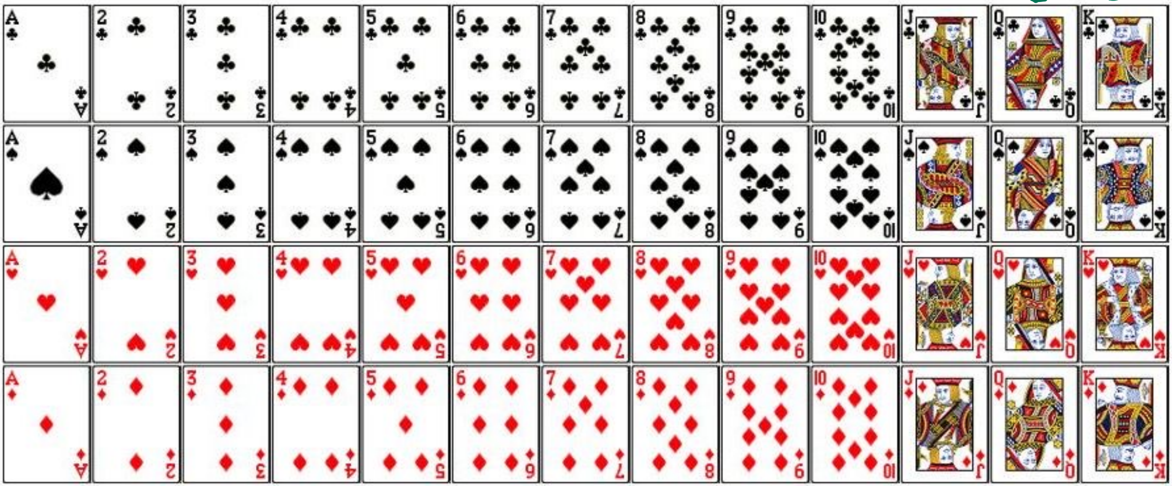
N	P(E)
⋮	⋮
40	89.12%
⋮	
50	97.04%
⋮	⋮

I'll put this full table in the notes online ace

A Deck of Cards – 52 Cards

Jack Pg 7

Queen King



Four Suits:

- Clubs (Black)
- Spades (Black)
- Hearts (Red)
- Diamonds (Red)

13 cards in each suit

This means that there are four of every card – two of each colour.

Picture Cards:
There are three picture cards in each suit – The Jack, the Queen and the King.

Permutations

Suppose you have n objects. A <u>permutation</u> of those n objects is an ordered list of the n objects.

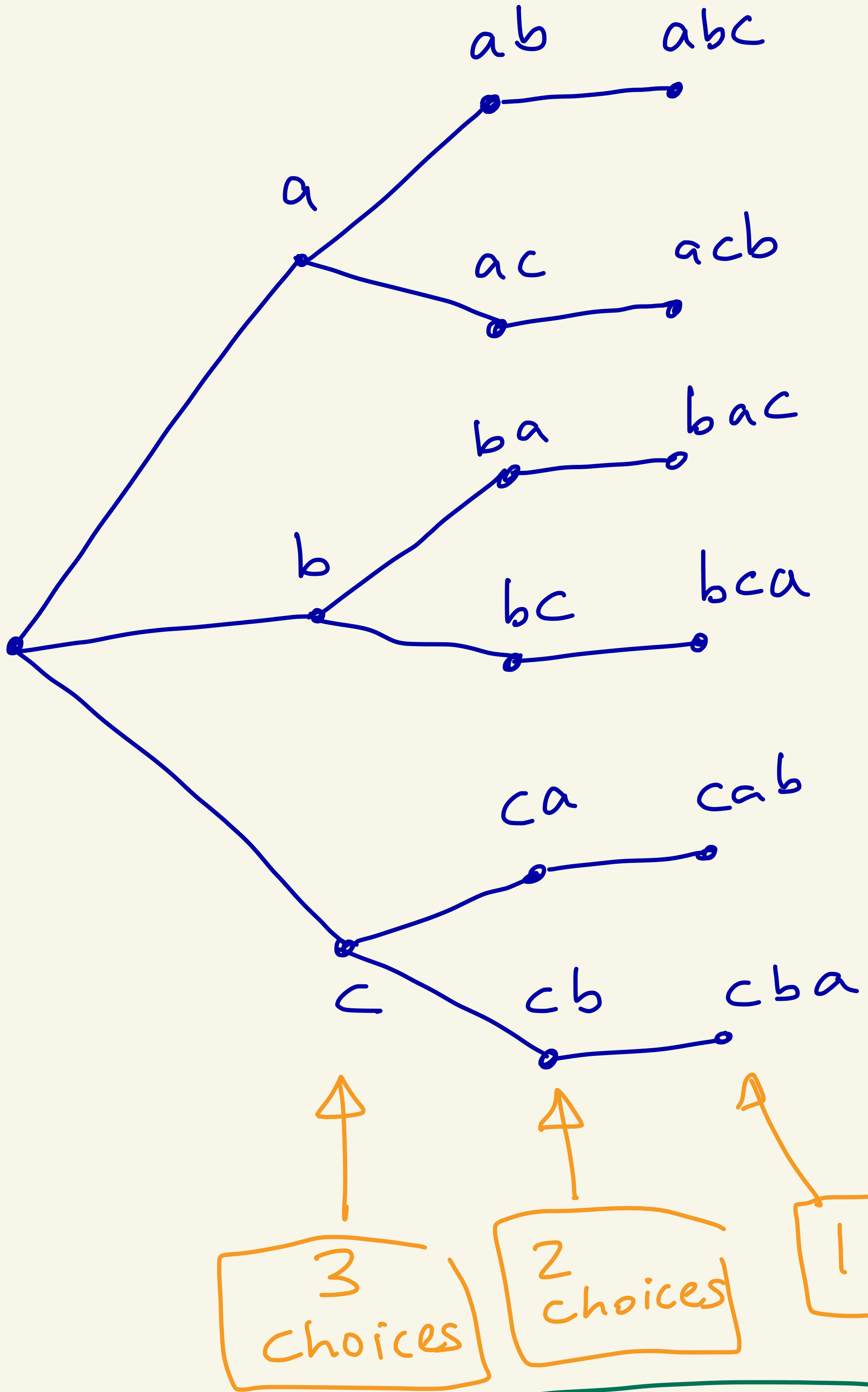
<u>Ex:</u> What are all the permutations of $\underbrace{a, b, c}_{n=3 \text{ objects}}$

<u>permutations</u>

$$\underbrace{\begin{array}{l} a\,b\,c \\ a\,c\,b \\ b\,a\,c \\ b\,c\,a \\ c\,a\,b \\ c\,b\,a \end{array}}_{\text{less writin}} \quad \longleftarrow \quad \underbrace{\begin{array}{l} (a,b,c) \\ (a,c,b) \\ (b,a,c) \\ (b,c,a) \\ (c,a,b) \\ (c,b,a) \end{array}}_{\text{too much writing}}$$

<u>could also do:</u>

6 permutations of a, b, c.



$$\underline{3 \text{ choices}} \cdot \underline{2 \text{ choices}} \cdot \underline{1 \text{ choice}} = 3! = 6$$

There are $n!$ permutations of n objects.

$$\underset{\text{choices}}{\underline{n}} \cdot \underset{\text{choices}}{\underline{n-1}} \cdot \underset{\text{choices}}{\underline{n-2}} \cdots \underset{\text{choices}}{\underline{2}} \cdot \underset{\text{choice}}{\underline{1}}$$

can't repeat 1st choice

can't repeat first 2 choices

etc

$$n! = n(n-1)(n-2)\cdots(2)\cdot(1)$$

Ex: In how many ways can 5 people be seated in a row?

5 people
Andrew
Brian
Clara
Donald
Egor

example seatings:

$\underset{\text{seat 1}}{\underline{A}}$ $\underset{\text{seat 2}}{\underline{C}}$ $\underset{\text{seat 3}}{\underline{D}}$ $\underset{\text{seat 4}}{\underline{E}}$ $\underset{\text{seat 5}}{\underline{B}}$

$\underset{\text{seat 1}}{\underline{E}}$ $\underset{\text{seat 2}}{\underline{A}}$ $\underset{\text{seat 3}}{\underline{C}}$ $\underset{\text{seat 4}}{\underline{D}}$ $\underset{\text{seat 5}}{\underline{B}}$

Answer:

$$\text{possibilities} = \frac{5 \text{ choices}}{\text{seat } 1} \cdot \frac{4 \text{ choices}}{\text{seat } 2} \cdot \frac{3 \text{ choices}}{\text{seat } 3} \cdot \frac{2 \text{ choices}}{\text{seat } 4} \cdot \frac{1 \text{ choice}}{\text{seat } 5}$$

$$= 5! = \boxed{120}$$

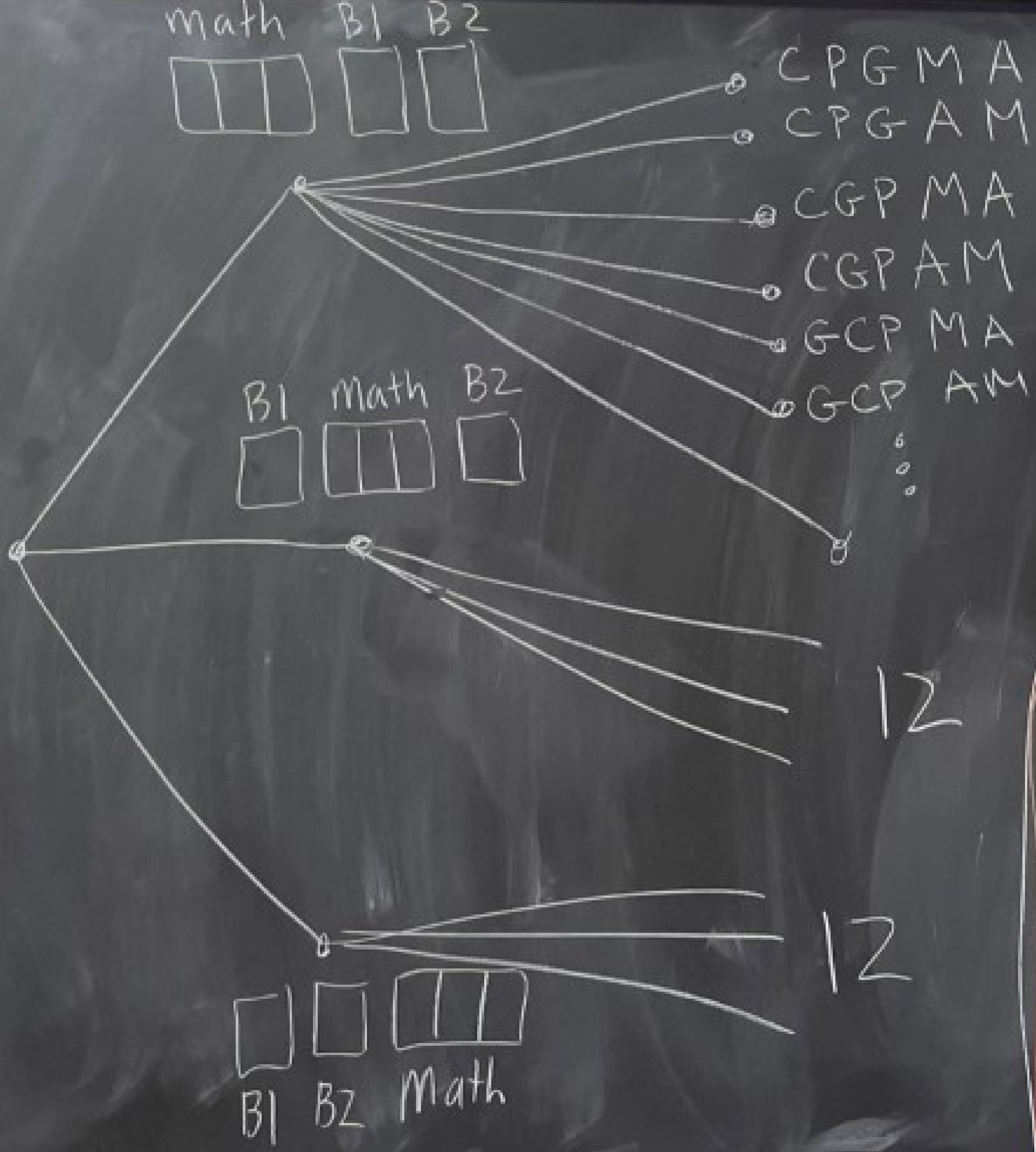
HANDOUT

17

N	Probability that at least two people in a classroom with N people have the same birthday
1	0 %
2	0.274 %
3	0.82 %
4	1.64 %
5	2.71 %
6	4.05 %
7	5.62 %
8	7.43 %
9	9.46 %
10	11.69 %
11	14.11 %
12	16.7 %
13	19.44 %
14	22.31 %
15	25.29 %
16	28.36 %
17	31.5 %
18	34.69 %
19	37.91 %
20	41.14 %
21	44.37 %
22	47.57 %
23	50.73 %
24	53.83 %
25	56.87 %
26	59.82 %
27	62.69 %
28	65.45 %
29	68.1 %
30	70.63 %
31	73.05 %
32	75.33 %
33	77.5 %
34	79.53 %
35	81.44 %
36	83.22 %
37	84.87 %
38	86.41 %
39	87.82 %
40	89.12 %
41	90.32 %
42	91.4 %
43	92.39 %
44	93.29 %
45	94.1 %
46	94.83 %
47	95.48 %
48	96.06 %
49	96.58 %
50	97.04 %

Ex: Suppose we have 3 math books and 2 biology books. How many ways can we put the books on a shelf so that the math books are next to each other?

Math	Bio
Calculus	Marine
Probability	Anatomy
Geometry	



Math B1 B2

$3 \cdot 2 \cdot 1 \cdot 2 \cdot 1$

$(3!)(2!) = 12$

Answer

$3 \cdot (3!)(2!)$

36 ways

Step 1

Pick one of

Math B1 B2

B1 math B2

B1 B2 math

Step 2: Fill in the books

$\underbrace{3!}_{\text{Fill in math}} \cdot \underbrace{2!}_{\text{Fill in Bio.}}$ ways

Suppose we have n objects where

n_1 of them are alike (ie the same or indistinguishable)

n_2 of them are alike,

$\ldots$, n_r are alike.

Then there are $$\frac{n!}{n_1!\, n_2! \cdots n_r!}$$

permutations of the n objects where $n = n_1 + n_2 + \cdots + n_r$

Ex: How many permutations of the letters a, a, b, c are there?

aabc
aacb
abac
acab
abca
acba

baac
caab
baca
caba
bcaa
cbaa

12 ways

Formula method

$n = 4$

$n_1 = 2 \leftarrow \boxed{a\text{'s}}$

$n_2 = 1 \leftarrow \boxed{b\text{'s}}$

$n_3 = 1 \leftarrow \boxed{c\text{'s}}$

$$\frac{n!}{n_1!\,n_2!\,n_3!} = \frac{4!}{2!\,1!\,1!} = \frac{24}{2} = \boxed{12}$$

Combinations

I'll put idea of derivation of formula on website under today's notes

Consider a set of size n.
The number of subsets of size r where $0 \le r \le n$ is

$$\underbrace{\binom{n}{r}}_{\text{read: "n choose r"}} = \frac{n!}{(n-r)!\,r!}$$

This number is the same as the number of ways that r objects can be selected from n objects where order doesn't matter

Ex: Suppose a dealer has the following cards:

$\boxed{A^{\heartsuit}}$ $\boxed{Q^{\diamondsuit}}$ $\boxed{A^{\spadesuit}}$ $\boxed{3^{\clubsuit}}$

Suppose the dealer deals you 2 cards from the 4 that they have. In how many ways can this happen where order doesn't matter.

So for example

$\boxed{A^{\heartsuit}}$ $\boxed{Q^{\diamondsuit}}$

is the same as

$\boxed{Q^{\diamondsuit}}$ $\boxed{A^{\heartsuit}}$

A♥ Q♦ ⟵ {A♥, Q♦}

A♥ A♠ ⟵ {A♥, A♠}

A♥ 3♣ ⟵ {A♥, 3♣}

A♠ Q♦ ⟵ {A♠, Q♦}

A♠ 3♣ ⟵ {A♠, 3♣}

Q♦ 3♣ ⟵ {Q♦, 3♣}

6 ways

$$\underbrace{\binom{4}{2}}_{\substack{\text{choose}\\ \text{2 of the}\\ \text{4 cards}}} = \frac{4!}{(4-2)!\,2!} = \frac{4!}{2!\,2!} = \frac{24}{4} = \boxed{6}$$

"4 choose 2"

Ex: How many 5 card hands are there using a 52-card deck?

Example hand: 10♡ J♡ Q♡ K♡ A♡

same as

J♡ Q♡ 10♡ A♡ K♡

order doesn't matter

$$\binom{52}{5} = \frac{52!}{(52-5)!\,5!} = \frac{52!}{47!\,5!} = \frac{52\cdot 51\cdot 50\cdot 49\cdot 48\cdot \cancel{(47!)}}{\cancel{47!}\,5!} = \frac{52\cdot 51\cdot \cancel{50}^{10}\cdot 49\cdot \cancel{48}^{\cancel{12}\,2}}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}}$$

$$= 52\cdot 51\cdot 10\cdot 49\cdot 2 = 2{,}598{,}960$$

$\boxed{\text{2,598,960 possible hands}}$

Combinations

Consider a set of size n. The number of subsets of size r, where $0 \le r \le n$, is

$$\binom{n}{r} = \frac{n!}{(n-r)!\, r!}$$

read "n choose r"

This number is the same as the number of ways that r objects can be selected from n objects where the order of selection is irrelevant.

<u>proof:</u> Given n objects, there are $n(n-1)\cdots(n-(r-1))$ ways to select r ~~items~~ objects from the n where order is relevant.

Each group of r items will be counted $r!$ times in this count. Hence the number of ways r objects can be selected from n objects where the order of selection is irrelevant is

$$\frac{n(n-1)\cdots(n-(r-1))}{r!} = \frac{n!}{(r-1)!\, r!}$$

CA SuperLotto Plus

A ticket consists of

- 5 "lucky" numbers chosen from 1–47
- 1 "mega" number chosen from 1–27

- No repeat numbers
- Order of lucky numbers doesn't matter.

On the ticket they put it in numerical order.

Example ticket

3 5 7 14 42 — lucky #s

8 — mega #

If you wanted a set theoretic way to make the sample space, you could do

$$S = \{ \underbrace{(\{3,5,7,14,42\},8)}_{\text{our special magical example ticket}}, \underbrace{\ldots}_{\text{all possible tickets}} \}$$

How many possible tickets are there?

$$\underbrace{\binom{47}{5}}_{\substack{\#\text{ of possible}\\ \text{lucky }\#\\ \text{selections}}} \cdot \underbrace{\binom{27}{1}}_{\substack{\#\text{ of}\\ \text{possible}\\ \text{mega}\\ \#\text{s}}} = \frac{47!}{(47-5)!\,5!} \cdot 27 =$$

$$\boxed{\binom{n}{1} = n}$$

$$= \frac{47 \cdot 46 \cdot 45 \cdot 44 \cdot 43 \cdot \cancel{(42!)}}{\cancel{42!} \cdot 5!} \cdot 27$$

$$= \frac{47 \cdot \cancel{46}^{23} \cdot \cancel{45}^{\cancel{9}^{3}} \cdot \cancel{44}^{11} \cdot 43}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} \cdot 27$$

$$= 47 \cdot 23 \cdot 3 \cdot 11 \cdot 43 \cdot 27$$

$$= \boxed{41{,}416{,}353 \text{ possible tickets}}$$

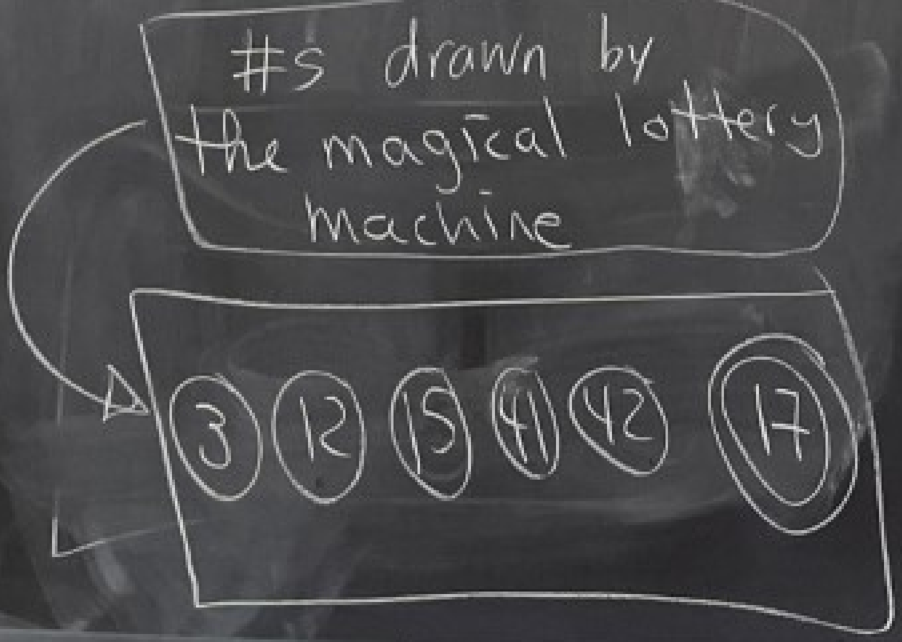
Sample space size is 41,416,353.

Probability of getting all lucky #s and mega number right is

$$\frac{1}{41{,}416{,}353} \approx 0.00000002414\ldots$$

or $0.000002414\,\%$

What are the odds of getting exactly 3 of the 5 lucky numbers and not the mega number?



How many tickets will get exactly 3 of 5 lucky #s and not mega?

$$\boxed{42 = 47 - 5}$$

$$\underbrace{\binom{5}{3}}_{\text{choose 3 of the 5 winning lucky \#s}} \cdot \underbrace{\binom{42}{2}}_{\text{choose 2 of the non-winning lucky \#'s}} \cdot \underbrace{26}_{\text{choose a non-winning mega \#}} =$$

ex: 3, 15, 42 | 18, 24

3, 12, 41 | 7, 11

⋮ | ⋮

1
2
3
4
⋮ (not 17)

$$= \frac{5!}{2!\,3!} \cdot \frac{42!}{40!\,2!} \cdot 26$$

$$= \frac{5 \cdot \cancel{4} \cdot \cancel{3!}}{\cancel{2} \cdot \cancel{3!}} \cdot \frac{42 \cdot 41 \cdot \cancel{40!}}{\cancel{40!}\,\cancel{2}} \cdot 26$$

$$= 5 \cdot 42 \cdot 41 \cdot 26$$

$$= \boxed{223{,}860 \text{ possible tickets}}$$

So the probability of getting exactly 3 of 5 lucky #s and not the mega # is

$$\frac{223{,}860}{41{,}416{,}353} \approx 0.0054051\ldots$$

On the lottery website it says the odds are approximately

$$\frac{1}{185} \approx 0.0054054\ldots$$

<u>Ex:</u> Suppose five 6-sided dice are rolled. What is the probability that exactly two of the dice have 6's showing?

An example is

$$\begin{array}{ccccc} \boxed{6} & \boxed{6} & \boxed{2} & \boxed{1} & \boxed{3} \\ \text{die 1} & \text{die 2} & \text{die 3} & \text{die 4} & \text{die 5} \end{array}$$

Sample space size is

$$\frac{6 \text{ choices}}{\text{die 1}} \quad \frac{6 \text{ choices}}{\text{die 2}} \quad \frac{6 \text{ choices}}{\text{die 3}} \quad \frac{6 \text{ choices}}{\text{die 4}} \quad \frac{6 \text{ choices}}{\text{die 5}}$$

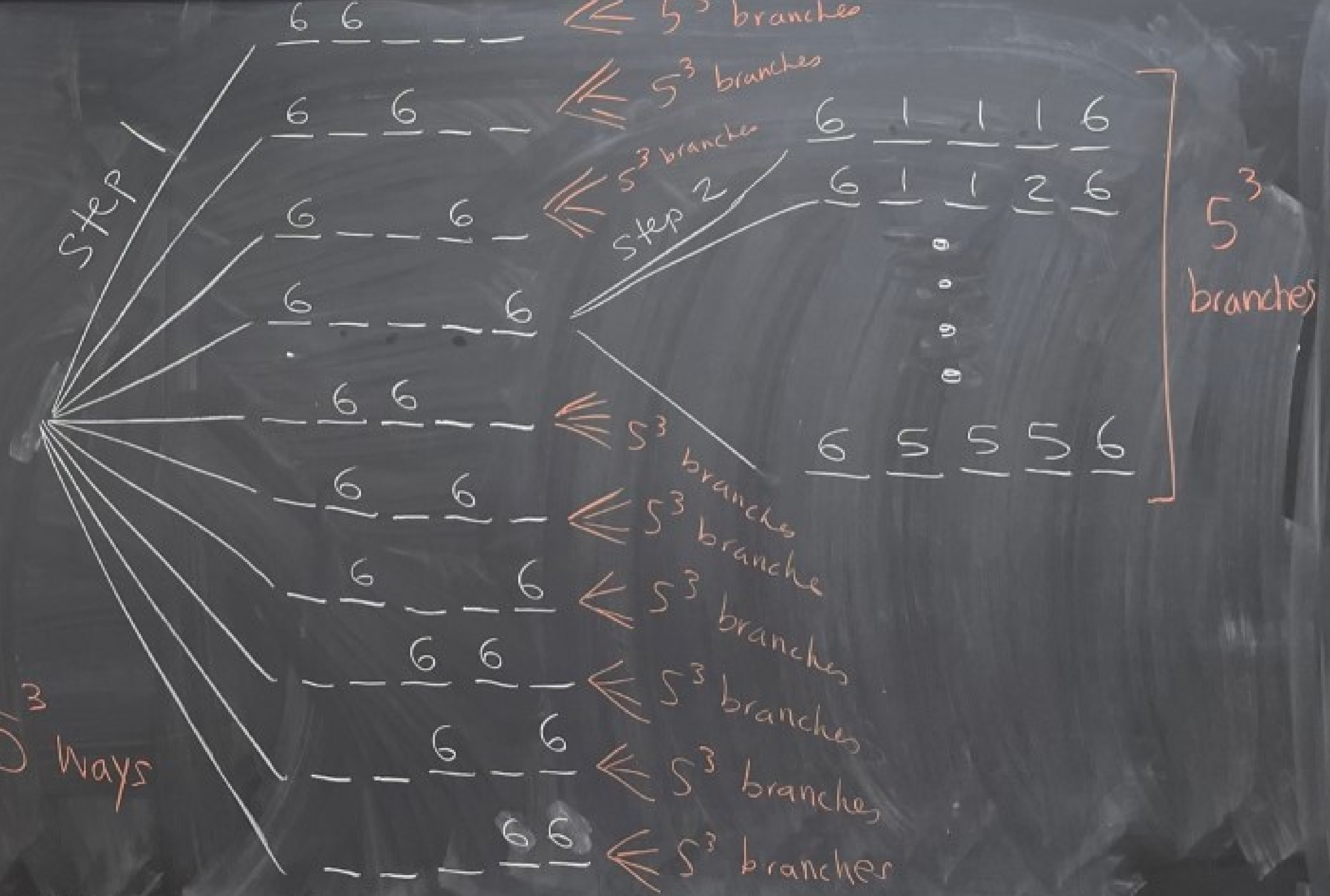
$$= 6^5 = \boxed{7,776}$$

How many of these rolls have exactly two 6's showing?

Step 1: Choose the two dice the 6's go in.
There are $\binom{5}{2} = 10$ ways to do this.

Step 2: Fill in the remaining three dice with #s 1–5.

$$\left.\underline{6}\ \ \underset{\text{choices}}{\underline{5}}\ \ \underset{\text{choices}}{\underline{5}}\ \ \underset{\text{choices}}{\underline{5}}\ \ \underline{6}\right\}\ 5^3 \text{ ways}$$



There are

$$\binom{5}{2} \cdot 5^3 = 10 \cdot 5^3 = 1250$$

ways to roll exactly two 6's.

$$\text{Probability} = \frac{1250}{7776} \approx 0.16075\ldots \approx 16\%$$

pizza rec for week

8142 W 3rd St

Oste

"Pinsa" pizza

Roman style pizza

POKER cheatsheet

selectabet.net

In poker, certain combinations of cards, or hands, outrank other hands, based on the frequency with which these combinations appear.The player with the best poker hand at the showdown wins the pot.

ROYAL FLUSH

A straight from a ten to an ace and all five cards of the same suit. In poker suit does not matter and pots are split between equally strong hands.

STRAIGHT FLUSH

Any straight with all five cards of the same suit.

FOUR OF A KIND

Any four cards of the same rank. If two players share the same Four of a Kind, the fifth card will decide who wins the pot, the bigger card the better.

FULL HOUSE

Any three cards of the same rank together with any two cards of the same rank. Our example shows "Aces full of Kings" and it is a bigger full house than "Kings full of Aces."

FLUSH

Any five cards of the same suit which are not consecutive. The highest card of the five makes out the rank of the flush. Our example shows an Ace-high flush.

STRAIGHT

Any five consecutive cards of different suits. The ace count as either a high or a low card. Our example shows a Five-high straight, which is the lowest possible straight.

THREE OF A KIND

Any three cards of the same rank. Our example shows three of a kind in Aces with a King and a Queen as side cards, which is the best possible three of a kind.

TWO PAIR

Any two cards of the same rank together with another two cards of the same rank. Our example shows the best possible two-pair, Aces and Kings. The highest pair of the two make out the rank of the two-pair.

ONE PAIR

Any two cards of the same rank. Our example shows the best possible one-pair hand.

HIGH CARD

Any hand that does not make up any of the above mentioned hands. Our example shows the best possible High-card hand.

Ex:

J♠ A♠ J♦ J♥ 2♥ ← three-of-a-kind

3♣ 7♥ 4♦ 6♠ 5♥ ← straight
3, 4, 5, 6, 7

3♣ 7♥ 4♦ 6♠ J♦ ← high card (Jack)

2♥ 3♣ 4♦ 2♦ J♣

(one pair)

Ex: Suppose you are dealt five cards from a standard 52-card deck. What's the probability that you get a Royal flush?

Sample space size is $\binom{52}{5} = 2,598,960$
(all possible 5-card hands)

Count how many royal flushes there are.

There are 4 royal flushes.

Probability of getting a royal flush is

$$\frac{4}{2,598,960} = \frac{1}{649,740}$$

$$\approx 0.000001539\ldots$$

Ex: What's the probability of getting one pair and nothing better?

You are dealt 5 cards from a standard 52-card deck

A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K ← $\boxed{\text{rank}}$

♣, ♠, ♡, ◇ ← $\boxed{\text{suit}}$

Sample space size

$$\binom{52}{5} = 2{,}598{,}960$$

Count/enumerate # of one pairs

Step 1: Pick a rank for the pair

A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K

Combos in step 1: $\binom{13}{1} = 13$

← $\boxed{2}\ \boxed{2}\ \boxed{}\ \boxed{}\ \boxed{}$

Step 2: Pick 2 suits for the pair

♣ ◇ ♡ ♠

Combos in step 2

$\binom{4}{2} = 6$

← $\boxed{2^{\heartsuit}}\ \boxed{2^{\diamondsuit}}\ \boxed{}\ \boxed{}\ \boxed{}$

Step 3: Pick the other 3 ranks. They can't be the same rank as step 1, and you can't pick any duplicates

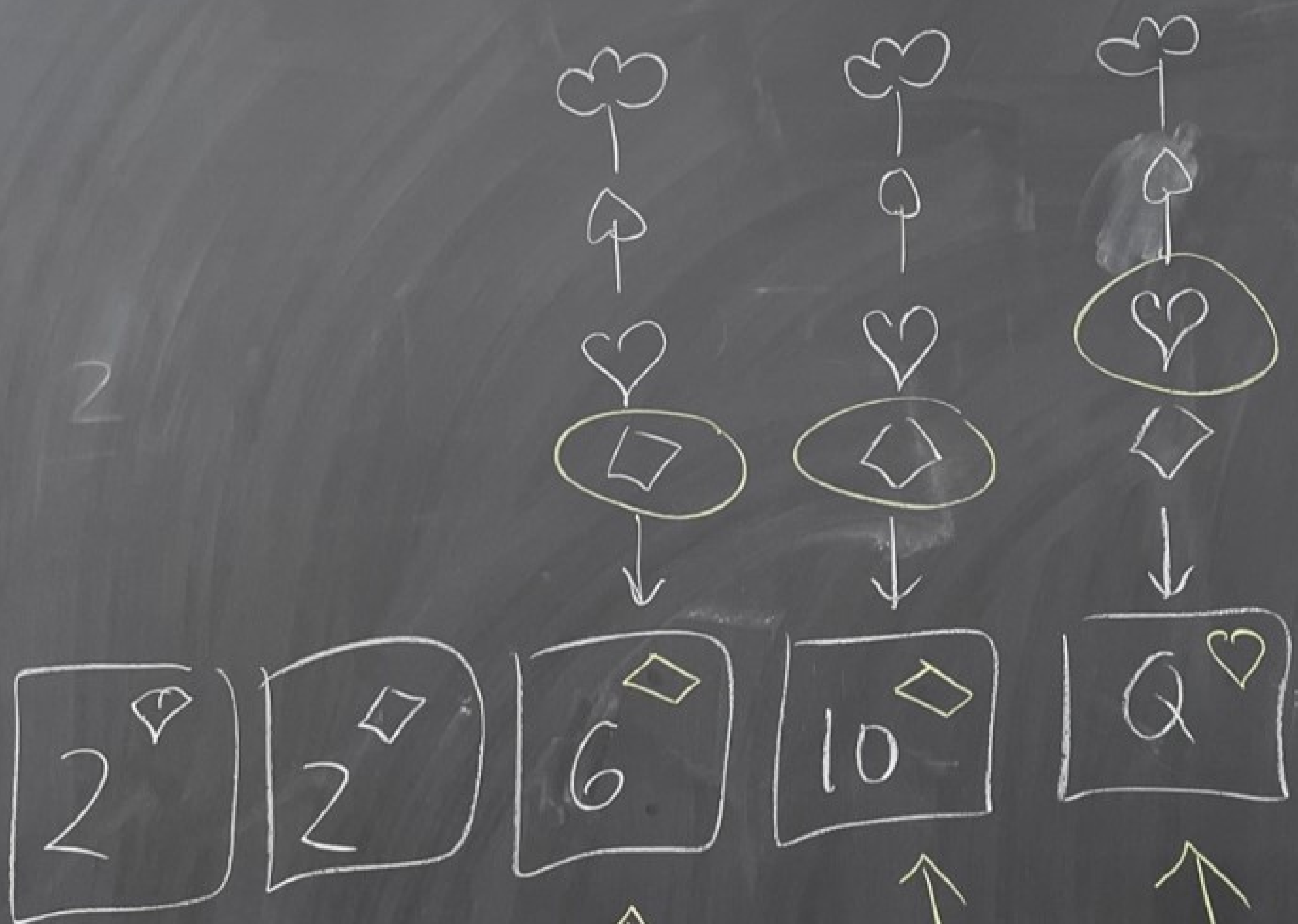
A, ~~2~~, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K

$$\binom{12}{3} = \frac{12!}{9!\,3!} = \frac{12\cdot 11\cdot 10\cdot \cancel{9!}}{\cancel{9!}\cdot 6}$$

$$= \boxed{220 \text{ ways}}$$

← $2^{\heartsuit}$ $2^{\diamondsuit}$ 6 10 Q

Step 4 - Fill in remaing 3 suits



$$\underset{\text{choices}}{4 = \binom{4}{1}} \cdot \underset{\text{choices}}{4} \cdot \underset{\text{choices}}{4} = 4^3 = 64$$

# of possible one pair hands is

$$13 \cdot 6 \cdot 220 \cdot 4^3$$

$$= 1{,}098{,}240$$

probability of getting one pair and nothing better is

$$\frac{1{,}098{,}240}{2{,}598{,}960} \approx 0.422569\ldots$$

$$\approx \boxed{42\%}$$

HW 2 - extra

(1) Suppose you are dealt 2 cards from a standard 52-card deck.

(a) What's the probability they are both aces?

Sample space size
(all possible 2-card hands)

$$\binom{52}{2} = \frac{52!}{50!\,2!}$$

$$= \frac{52 \cdot 51 \cdot \cancel{50!}}{\cancel{50!} \cdot 2}$$

$$= 26 \cdot 51$$

$$= \boxed{1326 \text{ possible 2-card hands}}$$

$$\begin{array}{r} 51 \\ \times\ 26 \\ \hline 306 \\ 1020 \\ \hline 1326 \end{array}$$

How many 2-card hands are both aces?

$\underbrace{\boxed{A♥}\ \boxed{A♦}\ \boxed{A♠}\ \boxed{A♣}}_{\text{pick 2 of these}}$

$$\binom{4}{2} = 6$$

$\boxed{\text{OR}}$

$\boxed{A♥}\ \boxed{A♦}$

$\boxed{A♥}\ \boxed{A♠}$

$\boxed{A♥}\ \boxed{A♣}$

$\boxed{A♦}\ \boxed{A♠}$

$\boxed{A♦}\ \boxed{A♣}$

$\boxed{A♣}\ \boxed{A♠}$

6 possibilities

Answer

$$\frac{6}{1326} = \frac{1}{221} \approx 0.00452\ldots \approx 0.45\%$$

(b) What is the probability both cards have the same face vace

face value is rank

count all 2-card hands with same face value on both cards

13 possibilities

Step 1 – Pick the rank/face value

A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K

$\boxed{7}$ $\boxed{7}$

Step 2 - Pick 2 suits

♠, ♣, ♥, ♦

7♦ 7♥

$\binom{4}{2} = 6$ possibilities

total hands $= 13 \cdot 6 = 78$

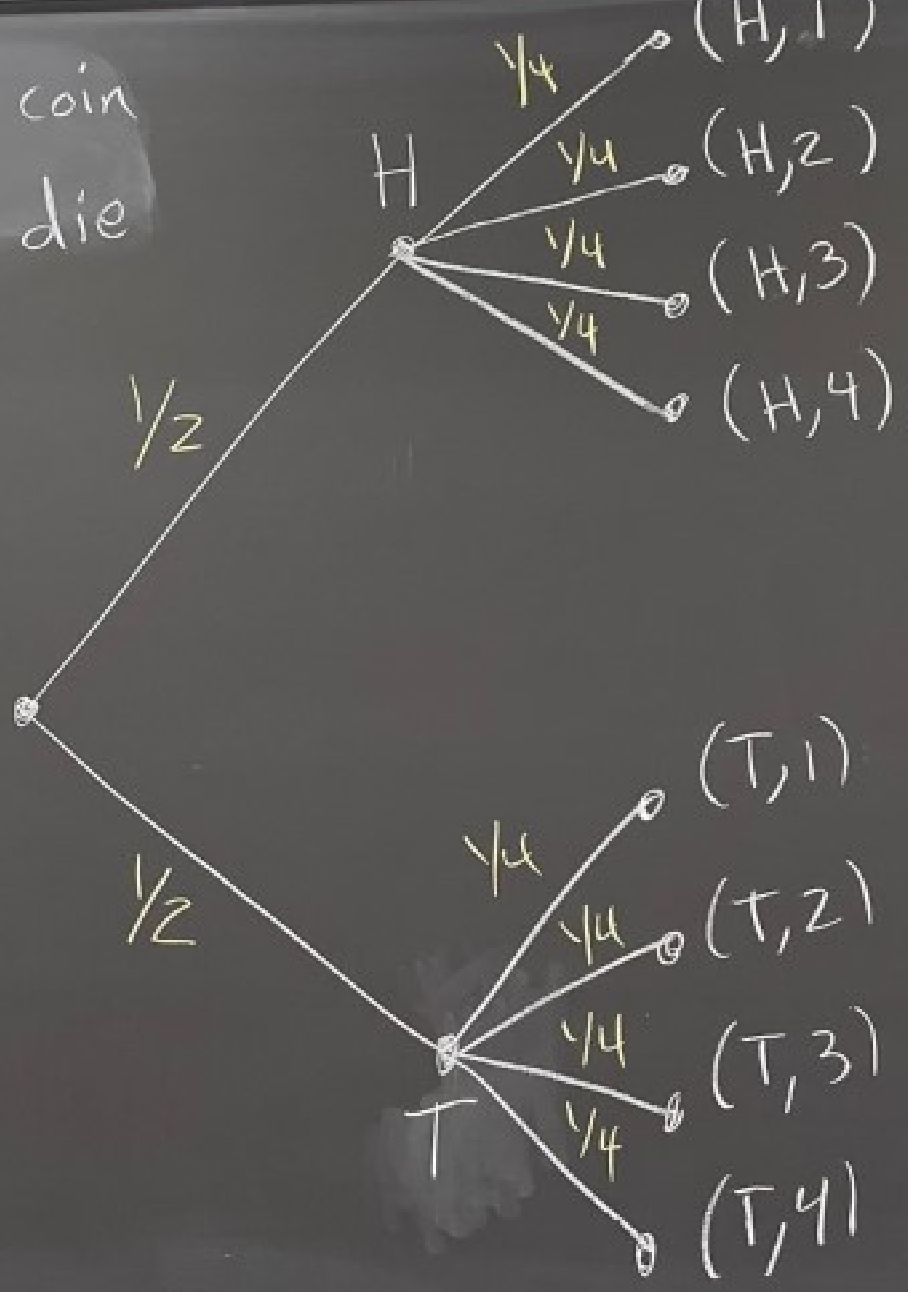
probability $= \dfrac{78}{1{,}326} = \dfrac{1}{17} \approx 0.0588\ldots$

$\approx 5.88\%$

Still in HW 2

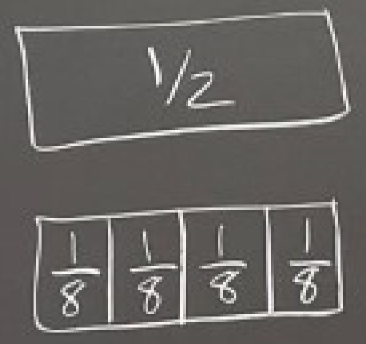
How do you make a probability function when you do two experiments in a row where the outcome of the first experiment does not influence the outcome of the second experiment?

<u>Ex:</u> Suppose you flip a fair coin and then roll a fair 4-sided die. Let's model this with a tree



What's the probability you got H on the coin and 2 on the die?

$$\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{8}$$



$P(\{(H,2)\})$

$$=\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{8}$$

How to do this in general

Suppose we want to do two experiments one after the other and the outcome of each experiment doesn't influence the outcome of the other.

Let (S_1, Ω_1, P_1) and (S_2, Ω_2, P_2) be probability spaces corresponding to the first and second experiments.

Define the space (S, Ω, P) where

$S = S_1 \times S_2$ and

Ω is the smallest σ-algebra containg all subsets of S of the form $E_1 \times E_2$ where $E_1 \in \Omega_1$ and $E_2 \in \Omega_2$

and P is defined by

$$P(\{(w_1, w_2)\}) = P_1(\{w_1\}) \cdot P_2(\{w_2\})$$

where $w_1 \in S_1$ and $w_2 \in S_2$.

If S is finite then if E is an event in Ω_1 and F is an event in Ω_2 then define

$$P(E \times F) = \sum_{(e,f) \in E \times F} P(\{(e,f)\}) = \sum_{(e,f) \in E \times F} P_1(\{e\}) \cdot P_2(\{f\})$$

$$= \sum_{e \in E} \sum_{f \in F} P_1(\{e\}) \cdot P_2(\{f\}) = \sum_{e \in E} P_1(\{e\}) \cdot \sum_{f \in F} P_2(\{f\})$$

$$= P_1(E) \cdot P_2(F)$$

Thus, $P(S) = P(S_1 \times S_2)$

$$= P_1(S_1) \cdot P_2(S_2) = 1 \cdot 1 = 1$$

This construction gives a probability space.

Ex: Suppose you have a 4-sided weighted die labeled 1, 2, 3, 4.

The probabilities are

# on die	1	2	3	4
Probability	1/8	1/4	1/2	1/8

Let's first roll the die and then flip a fair coin.

roll die space

$S_1 = \{1, 2, 3, 4\}$

Ω_1 is set of all subsets of S_1

$P_1(\{1\}) = 1/8$

$P_1(\{2\}) = 1/4$

$P_1(\{3\}) = 1/2$

$P_1(\{4\}) = 1/8$

coin space

$S_2 = \{H, T\}$

Ω_2 is set of all subsets of S_2

$P_2(\{H\}) = 1/2$

$P_2(\{T\}) = 1/2$

roll die then flip coin space

$$S = S_1 \times S_2 = \{(1,H), (2,H), (3,H), (4,H), (1,T), (2,T), (3,T), (4,T)\}$$

Ω is set of all subsets of S

$$P(\{(1,H)\}) = P_1(\{1\}) \cdot P_2(\{H\}) = \frac{1}{8} \cdot \frac{1}{2} = \frac{1}{16}$$

$$P(\{(2,H)\}) = P_1(\{2\}) \cdot P_2(\{H\}) = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$$

$$P(\{(3,H)\}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

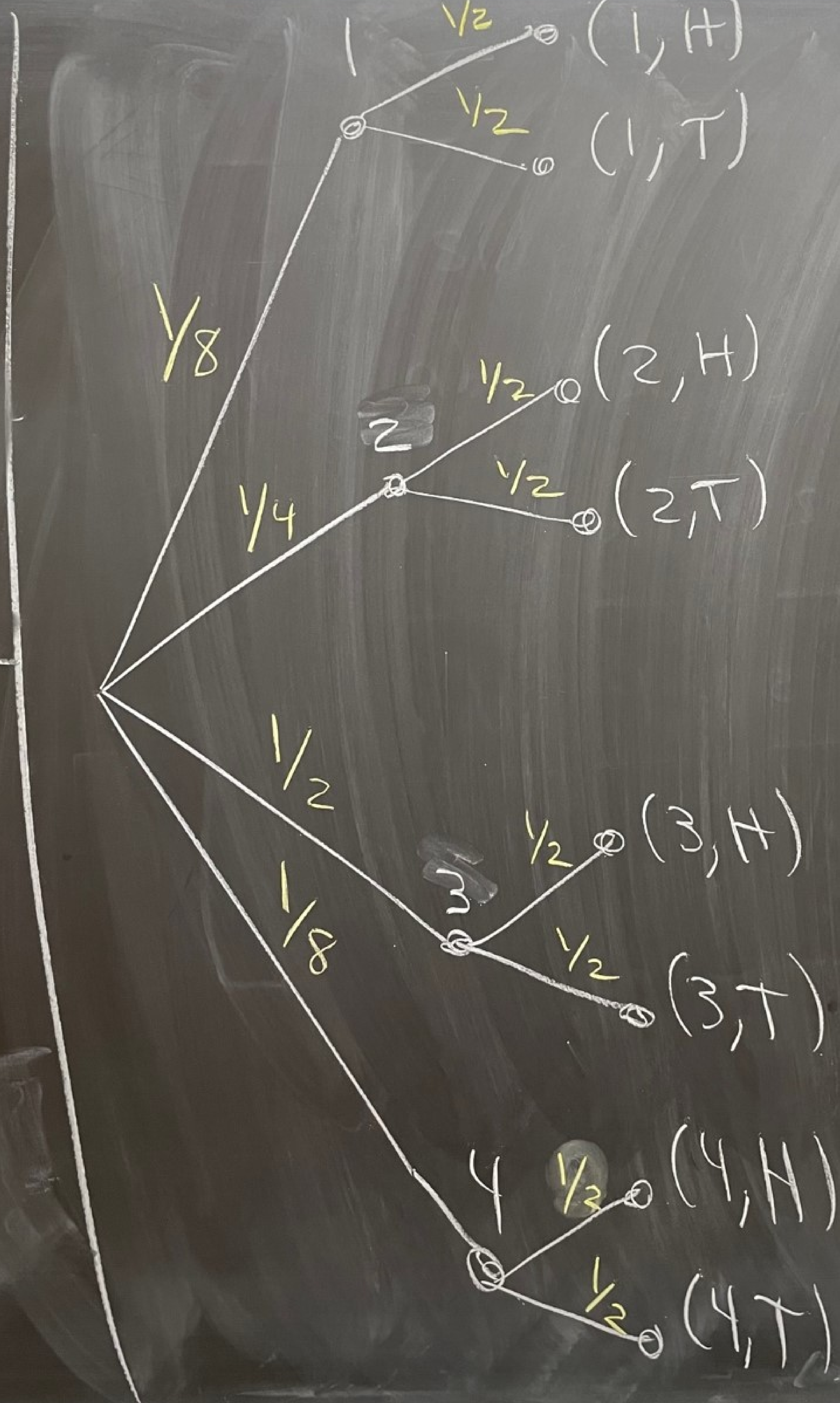
$$P(\{(4,H)\}) = \frac{1}{8} \cdot \frac{1}{2} = \frac{1}{16}$$

$$P(\{(1,T)\}) = \frac{1}{8} \cdot \frac{1}{2} = \frac{1}{16}$$

$$P(\{(2,T)\}) = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$$

$$P(\{(3,T)\}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$P(\{(4,T)\}) = \frac{1}{8} \cdot \frac{1}{2} = \frac{1}{16}$$


1
1/2
(1, H)
1/2
(1, T)
1/8
2
1/2
(2, H)
1/2
(2, T)
1/4
1/2
1/8
3
1/2
(3, H)
1/2
(3, T)
4
1/2
(4, H)
1/2
(4, T)

HW 2

(14) Suppose that five numbers are selected at random from the numbers (no repeats)

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20

What is the probability that the smallest number selected is larger than 6?

Ex:

#s picked	smallest #	
7, 2, 11, 5, 9	2	$2 \not> 6$
10, 17, 7, 12, 11	7	$7 > 6$

$$\text{Sample space size} = \binom{20}{5} = \frac{20!}{15!\,5!} = \frac{20\cdot19\cdot18\cdot17\cdot16\cdot\cancel{15!}}{\cancel{15!}\;5\cdot4\cdot3\cdot2\cdot1} = \frac{1{,}860{,}480}{120}$$

$$\boxed{= 15{,}504}$$

We want # of selections where lowest # is > 6

(pick 5 #s from 7, 8, 9, 10, 11, ..., 20)

$$= \binom{14}{5} = \frac{14!}{9!\,5!} = \frac{14\cdot13\cdot12\cdot11\cdot10\cdot\cancel{9!}}{\cancel{9!}\;5!}$$

$$= \frac{14\cdot13\cdot12\cdot11\cdot10}{5!}$$

$$= 2{,}002$$

$$\text{Answer} = \frac{2{,}002}{15{,}504} \approx 0.1291$$

HW 2

10 You roll four 8-sided dice.

(a) Whats the probability you get exactly two 3's?

sample space size $= 8 \cdot 8 \cdot 8 \cdot 8 = 8^4$

$\underset{\text{die 1}}{\underline{\quad}}\ \underset{\text{die 2}}{\underline{3}}\ \underset{\text{die 3}}{\underline{3}}\ \underset{\text{die 4}}{\underline{\quad}}$ ← Step 1

pick 2 spots from the 4 spots where 3's go $= \binom{4}{2} = 6$

3	3	_	_
3	_	3	_
3	_	_	3
_	3	3	_
_	3	_	3
_	_	3	3

$\underline{8}\ \underline{3}\ \underline{3}\ \underline{1}$ ← Step 2

7 choices (die 1) 7 choices (die 4)

Fill in the remaining 2 spots with #s that aren't 3 $= 7 \cdot 7 = 49$

ways to get exactly two 3's $= 6 \cdot 49 = 294$

Answer $= \dfrac{294}{8^4}$

$\approx 0.071\ldots$

Hw

10

(a

HW 3 Topic - Conditional Probability

Ex: Suppose we roll two 6-sided dice, a green die and red die. Suppose the green die stops rolling and lands on a 3, but the red die keeps rolling. What's the probability that the sum of the dice is 8?

beginning sample space: S

(1,1)
(1,2)
⋮
(3,1)
(3,2)
(3,3)
(3,4)
(3,5)
(3,6)
⋮
(5,6)
(6,6)

36 possibilities

(green, red)

once green die is 3

sample space shrinks

new sample space

S'

(3,1)
(3,2)
(3,3)
(3,4)
(3,5)
(3,6)

Now there are only 6 possibilities and only 1 of them (3,5) gives that the sum of the dice is 8.

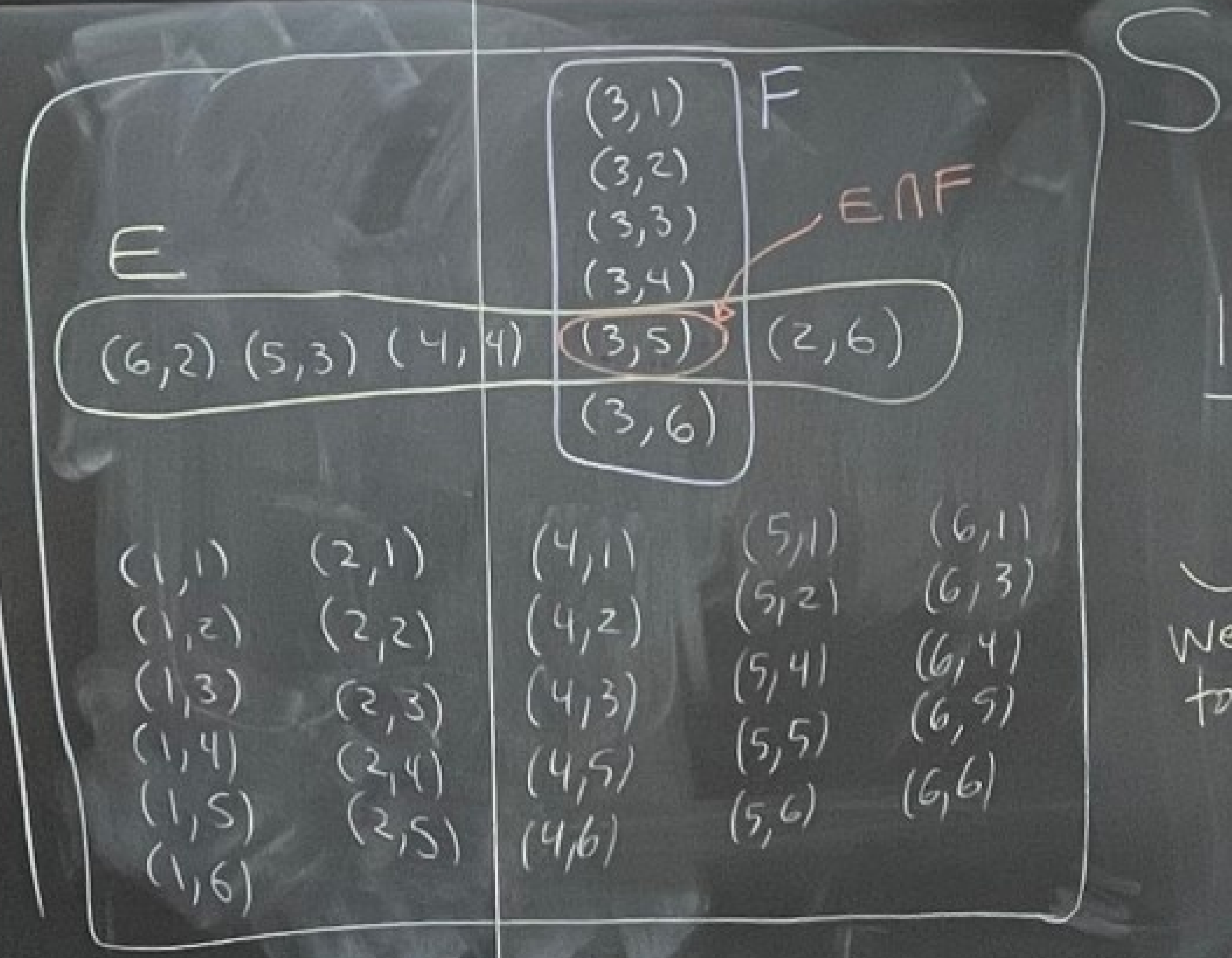
So, the answer is $\frac{1}{6}$.

Let's make a formula for this without having to shrink the sample space S and also a method that generalizes to any probability space.

Let $E =$ the event in S where the sum is 8.

Let $F = S' =$ the event where the green die is 3

We want to know the "conditional probability" of the event E occuring "given" that F has already occured.



Probability in S

$$\underbrace{\frac{|E\cap F|}{|F|}}_{\text{we did this to get } 1/6} = \frac{|E\cap F|/|S|}{|F|/|S|} = \frac{P(E\cap F)}{P(F)}$$

Can only do this step because all outcomes have equal probability

this is $\dfrac{1/36}{6/36} = \dfrac{1}{6}$

Def: Let (S, Ω, P) be a probability space. Let E and F be events.

Suppose $P(F) > 0$.

We define the conditional probability that E occurs given that F occured as

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

(These probabilities are all calculated in S)

HW 3

3. Suppose you roll two 6-sided dice. You can't see the outcome of the roll, but someone else can.

They tell you that the sum is divisible by 5.

What's the probability that both of the dice have landed on 5's?

Let E be the event that both dice are 5's.

Let F be the event that the sum of the dice is divisible by 5.

We want $P(E|F)$.

$$E=\{(5,5)\}$$

$$F=\{(2,3),(4,1),(3,2),(1,4),(4,6),(6,4),(5,5)\}$$

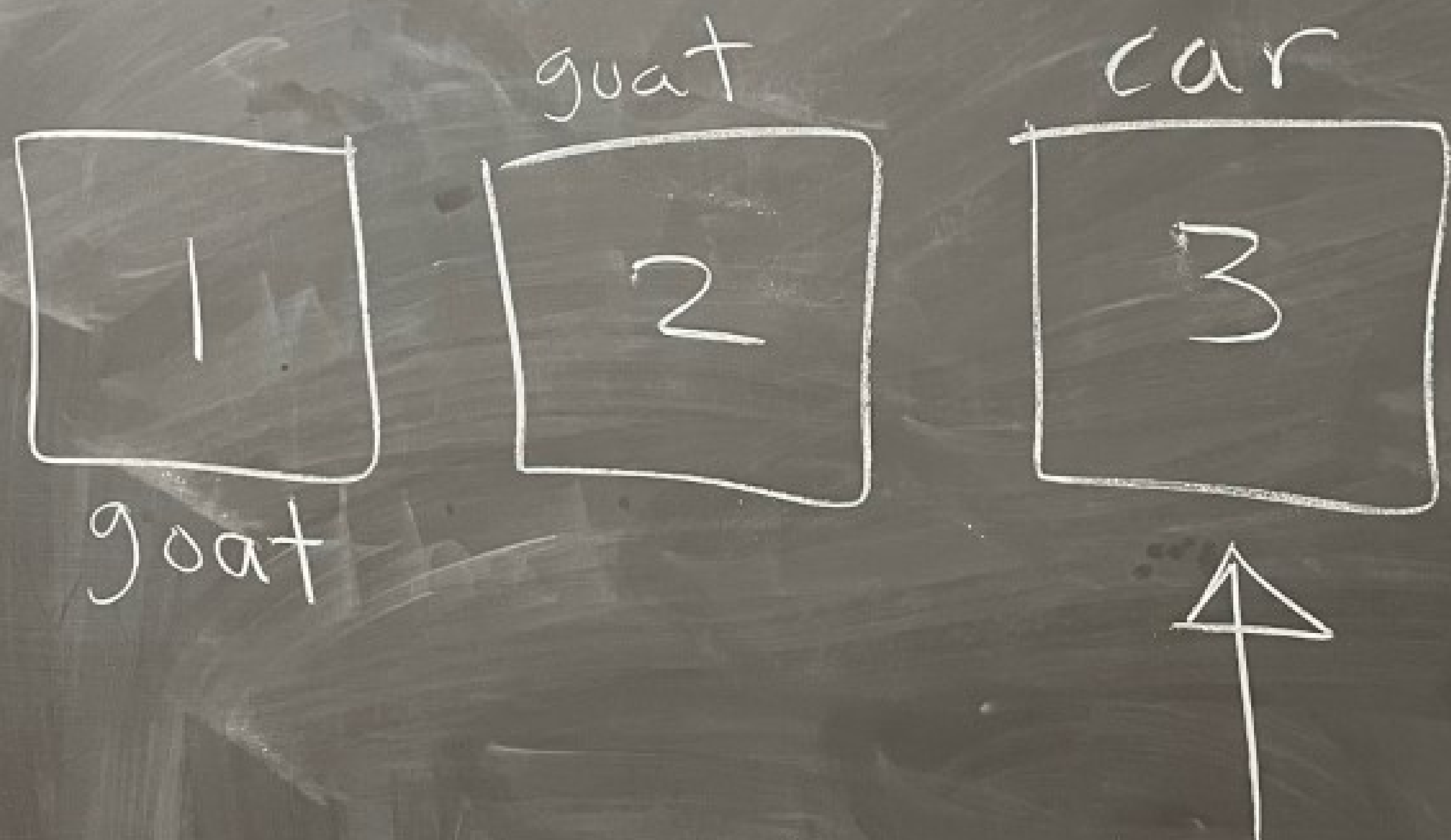
$$E\cap F=\{(5,5)\}$$

$$P(E|F)$$

$$=\frac{P(E\cap F)}{P(F)}=\frac{1/36}{7/36}=\frac{1}{7}$$

$$\approx 0.1428$$

Monty Hall Problem



Keep?

Switch?

movie 21 has a scene about this

HW 2

(16) You are dealt 5 cards from a standard 52-card deck. You know three of the cards, they are

2♣ 3♣ 4♣

You don't know the other two.

2♣ 3♣ 4♣ ? ?

(a) What's the probability the other two cards are clubs?

Clubs left (10 left)

A♣ 5♣ 6♣ 7♣ 8♣ 9♣ 10♣ J♣ Q♣ K♣

How many ways can we put 2 clubs in ?'s.

$$\binom{10}{2} = \frac{10!}{8!\,2!} = \frac{10 \cdot 9}{2} = 45$$

How many total ways to put any two cards in ?'s

52 total cards

$- \boxed{2\clubsuit} \ \boxed{3\clubsuit} \ \boxed{4\clubsuit}$

49 cards left to fill in ?'s

$$\binom{49}{2} = \frac{49!}{47!\,2!} = \frac{49 \cdot 48}{2} = 49 \cdot 24 = 1176$$

Sample space size

Answer

$$\frac{45}{1176} \approx 0.0383\ldots$$

$$\approx 3.8\%$$

Test 1

next Weds

HW 1

HW 2

Monty Hall

Switch or stay?

Win = car
lose = goat

Strategy 1 – Switch

Suppose you pick door 1 and then you switch after Monty reveals a goat.

Strategy 2 – Stay

Suppose you pick door 1 and stay on door 1 after Monty reveals a goat

door 1	door 2	door 3	switch	stay
car	goat	goat	lose	win
goat	car	goat	win	lose
goat	goat	car	win	lose

Switch wins $2/3$

Stay wins $1/3$

Recall:

$$\underbrace{P(E|F)}_{\boxed{\text{probability of } E \text{ given } F}} = \frac{P(E \cap F)}{P(F)}$$

(Assume $P(F) > 0$)

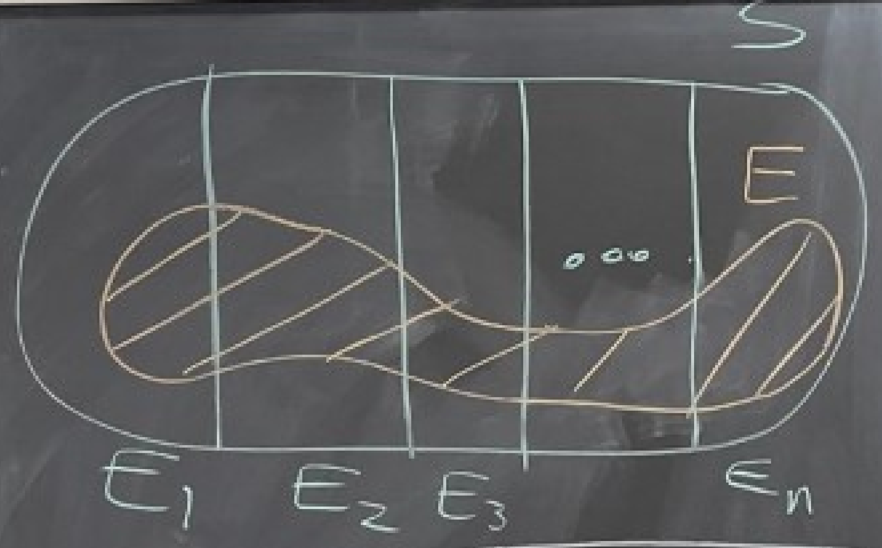
Theorem: Let (S, Ω, P) be a probability space.

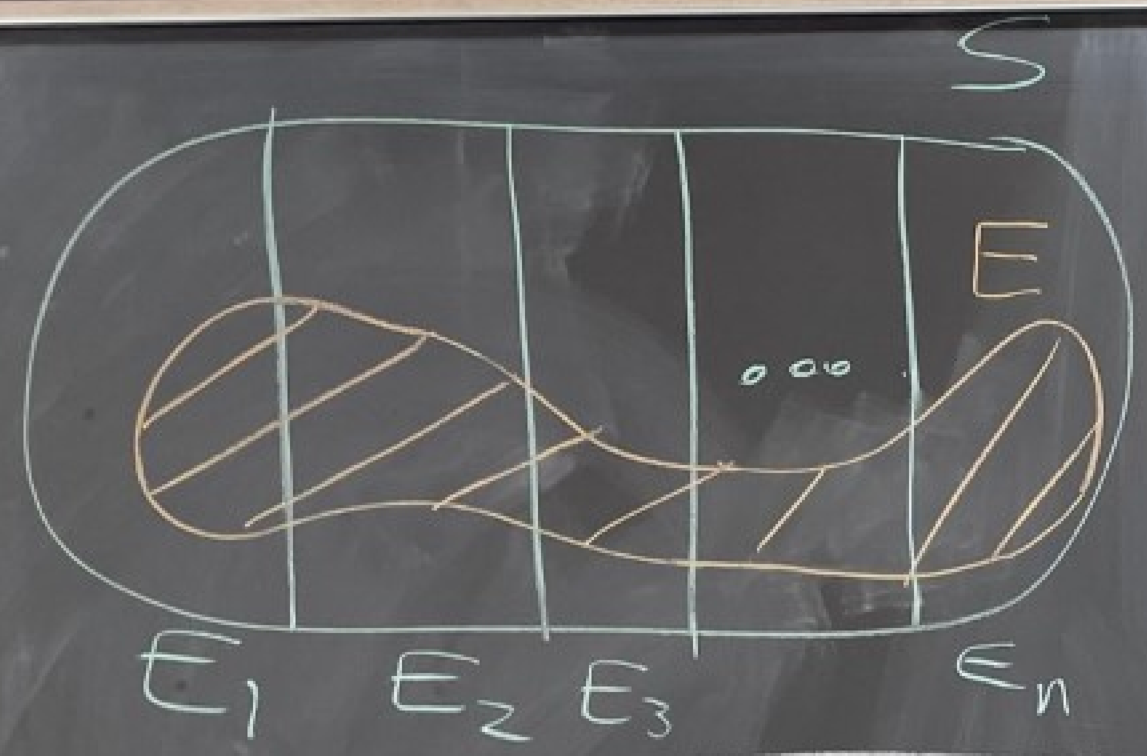
① Let A and B be events with $P(A)>0$.
Then $P(A\cap B) = P(A)\cdot P(B\,|\,A)$

② Let $A_1, A_2, \ldots, A_n$ be events with $P(A_1\cap A_2\cap\cdots\cap A_n)>0$
then

$$P(A_1\cap A_2\cap\cdots\cap A_n) = P(A_1)\cdot P(A_2|A_1)\cdot P(A_3|A_1\cap A_2)\cdot P(A_4|A_1\cap A_2\cap A_3)\cdots P(A_n|A_1\cap A_2\cap\cdots\cap A_{n-1})$$

③ (Law of total probability) Suppose $S = E_1\cup E_2\cup\cdots\cup E_n$ where each E_i is non-empty and





$$\ldots_1 \cap A_2 \cap A_3) \cdots P(A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1})$$

$\ldots \cup E_n$ where each E_i is non-empty and

$E_i \cap E_j = \phi$ when $i \neq j$ (that is, the E_k's are disjoint from each other).

Suppose also $P(E_i) \neq 0$ for all i.

Then if E is any event, we have

$$P(E) = P(E|E_1)\cdot P(E_1) + P(E|E_2)\cdot P(E_2) + P(E|E_3)\cdot P(E_3) + \cdots + P(E|E_n)\cdot P(E_n)$$

Ex: Suppose there are three boxes

In box 1 are two 4-sided dice.

In box 2 are two 6-sided dice.

In box 3 are two 8-sided dice.

Suppose you randomly pick a box (each box is equally likely to be chosen), then you take the dice out of that box and roll them. What is the probability that the sum of the dice is 8?

box 1 chosen	box 2 chosen	box 3 chosen
Two 4-sided dice	Two 6-sided dice	Two 8-sided dice
$4^2 = 16$ combos	$6^2 = 36$ combos	$8^2 = 64$ combos
(4,4) ← gives sum 8	(2,6), (6,2) (5,3), (3,5) (4,4)	(1,7), (7,1) (2,6), (6,2) (3,5), (5,3) (4,4)
$\frac{1}{16}$ get sum 8	5 ways to get sum 8 Prob get sum 8 is $\frac{5}{36}$	probability get sum 8 is $\frac{7}{64}$

Need law of total probability.

$$P(\text{sum is } 8) = P(\text{sum is } 8 \mid \text{box 1 chosen}) \cdot P(\text{box 1 is chosen})$$
$$+ P(\text{sum is } 8 \mid \text{box 2 is chosen}) \cdot P(\text{box 2 chosen})$$
$$+ P(\text{sum is } 8 \mid \text{box 3 is chosen}) \cdot P(\text{box 3 is chosen})$$
$$= \frac{1}{16} \cdot \frac{1}{3} + \frac{5}{36} \cdot \frac{1}{3} + \frac{7}{64} \cdot \frac{1}{3}$$
$$= \frac{11{,}456}{110{,}592} \approx 0.1036 \approx 10.36\%$$

$$P(E) = P(E \mid E_1)P(E_1) + P(E \mid E_2)P(E_2) + P(E \mid E_3)P(E_3)$$

Monty Hall

Let's redo the switch strategy using the law of total probability.

pick door 1 and switch when offered

$$
\begin{aligned}
P(\text{win}) = {} & \underbrace{P\left(\text{win} \,\middle|\, \begin{matrix}\text{car behind} \\ \text{door 1}\end{matrix}\right)}_{0} \cdot \underbrace{P\left(\begin{matrix}\text{car behind} \\ \text{door 1}\end{matrix}\right)}_{1/3} \\
& + \underbrace{P\left(\text{win} \,\middle|\, \begin{matrix}\text{car behind} \\ \text{door 2}\end{matrix}\right)}_{1} \cdot \underbrace{P\left(\begin{matrix}\text{car behind} \\ \text{door 2}\end{matrix}\right)}_{1/3} \\
& + \underbrace{P\left(\text{win} \,\middle|\, \begin{matrix}\text{car behind} \\ \text{door 3}\end{matrix}\right)}_{1} \cdot \underbrace{P\left(\begin{matrix}\text{car behind} \\ \text{door 3}\end{matrix}\right)}_{1/3}
\end{aligned}
$$

$$= (0)\left(\tfrac{1}{3}\right) + (1)\left(\tfrac{1}{3}\right) + (1)\left(\tfrac{1}{3}\right) = \boxed{\tfrac{2}{3}}$$

Some properties

Let (S, Ω, P) be a probability space.

① Let A and B be events with $P(A)>0$. Then

$$P(A\cap B) = P(A)\,P(B|A)$$

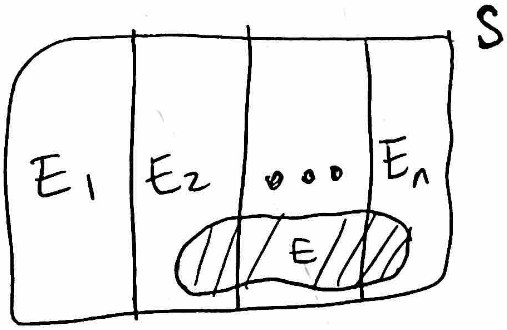
② Let $A_1, \dots, A_n$ be events with $P(A_1\cap A_2\cap\dots\cap A_n)>0$

then

$$P(A_1\cap A_2\cap\dots\cap A_n) = P(A_1)\cdot P(A_2|A_1)\cdot P(A_3|A_1\cap A_2)\cdot P(A_4|A_1\cap A_2\cap A_3)\cdots P(A_n|A_1\cap A_2\cap\dots\cap A_{n-1})$$

(Law of total probability)

③ Suppose that $S = E_1\cup E_2\cup\dots\cup E_n$ and $E_i\cap E_j=\emptyset$ where each E_i is non-empty and if $i\neq j$ (ie each of the E_i and E_j are disjoint). Suppose further that $P(E_i)\neq 0$ for all i.



Then if E is any event then

$$P(E) = P(E|E_1)\cdot P(E_1) + P(E|E_2)\cdot P(E_2) + \dots + P(E|E_n)\cdot P(E_n)$$

proof: ① From def of $P(B|A) = \dfrac{P(A\cap B)}{P(A)}$.

② Here it is for 3 sets A_1, A_2, and A_3: $P(A_1\cap A_2\cap A_3) = P((A_1\cap A_2)\cap A_3)$

$$\overset{①}{=} P(A_1\cap A_2)\cdot P(A_3|A_1\cap A_2) \overset{①}{=} P(A_1)\cdot P(A_2|A_1)\cdot P(A_3|A_1\cap A_2).$$

③ Note $P(E) = P((E\cap E_1)\cup(E\cap E_2)\cup\dots\cup(E\cap E_n)) \xrightarrow{\text{axiom ⑥}} \sum_{i=1}^{n} P(E\cap E_i)$

$$= \sum_{i=1}^{n} P(E|E_i)\cdot P(E_i) \quad ▨$$

HW 2

Order doesn't matter

Tony, Bob, Larry

Larry, Bob, Tony

Same committees

9 From a group of 10 women and 8 men a committee consisting of 5 women and 4 men must be formed.

(a) How many possible committees?

(b) How many different committees are possible if 2 of the men refuse to serve together?

(a

(b)

8

To
M
C

(a) $\binom{10}{5}\binom{8}{4}$

(b)

8 men
Tony
Mike
Gary
Tom
Jerry
Jack
Bill
Harry

refuse to work together

Method 1

Count committees where they do work together and subtract from total

$$\underbrace{\binom{10}{5}}_{\text{choose 5 women}} \cdot \underbrace{\binom{6}{2}}_{\text{need 4 men with Tom \& Jerry included}}$$

$$\underline{\text{Tom}}\ \ \underline{\text{Jerry}}\ \ \underbrace{\underline{\ ?\ }\ \ \underline{\ ?\ }}_{\text{6 men can go here}}$$

Answer

$$\underbrace{\binom{10}{5}\binom{8}{4}}_{\text{total}} - \underbrace{\binom{10}{5}\binom{6}{2}}_{\text{committees Tom \& Jerry are both on}}$$

$$= 13{,}860$$

Method 2

Directly count
Break into disjoint cases
and add.

committees both Tom & Jerry
aren't on is

$$\underbrace{\binom{10}{5}}_{\text{women}}\underbrace{\binom{6}{4}}_{\substack{\text{pick 4 men}\\\text{not Tom}\\\text{not Jerry}}}$$

Tony
Mike
Gary
~~Tom~~
~~Jerry~~
Jack
Bill
Harry

committees Tom is on
but not Jerry

$$\underbrace{\binom{10}{5}}_{\text{women}}\cdot\underbrace{\binom{6}{3}}_{\substack{\text{Tom + }\\\text{pick 3 more}\\\text{that aren't}\\\text{Tom or Jerry}}}$$

Tom ? ? ?

Tony
Mike
Gary
Tom ✓
~~Jerry~~
Jack
Bill
Harry

committees Jerry is
on but not Tom.

Same calculation

$$\binom{10}{5}\cdot\binom{6}{3}$$

Answer

$$\binom{10}{5}\binom{6}{4}+\binom{10}{5}\binom{6}{3}+\binom{10}{5}\binom{6}{3}$$

$$= 13,860$$

HW 2 – Part 2

(15) (c) You are dealt a 5-card poker hand. What is the probability you get two pair?

Ex of two pair → 2♣ 2♥ A♦ A♣ 10♠

Sampl

Count

(1) Pick
tw

A, 2

Sample space size = $\binom{52}{5}$

Count # of two-pair hands

① Pick face values of the two pair.

$\boxed{2}\ \boxed{2}\ \boxed{A}\ \boxed{A}\ \boxed{}$

$$\binom{13}{2}$$

A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K ← 13 face values

pick 2

② Pick the remaining face value

$\boxed{2}\ \boxed{2}\ \boxed{A}\ \boxed{A}\ \boxed{10}$

$$\binom{11}{1}$$

~~A~~, ~~2~~, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K ← Pick 1 of these

③ Pick 2 suits from

♠, ♣, ♥, ♦

for the first pair.

$$\binom{4}{2}$$

2♥ 2♦ A A 10

④ Pick 2 suits from

♠, ♣, ♥, ♦

for the second pair

$$\binom{4}{2}.$$

2♥ 2♦ A♥ A♦ 10

⑤ Pick suit for last card

$$\binom{4}{1}$$

2♥ 2♦ A♥ A♦ 10♥

for

d

A♦ 10♥

Answer

$$\boxed{\binom{n}{1} = n \quad \Bigg| \quad \binom{n}{2} = \frac{n!}{(n-2)!\,2!} = \frac{n(n-1)[(n-2)!]}{(n-2)!\,2} = \frac{n(n-1)}{2}}$$

$$\frac{\left(\text{total \# of two pair hands}\right)}{\left(\text{total \# of poker hands}\right)} = \frac{\binom{13}{2}\binom{11}{1}\binom{4}{2}\binom{4}{2}\binom{4}{1}}{\binom{52}{5}}$$

$$= \frac{\left(\frac{13\cdot 12}{2}\right)(11)(6)(6)(4)}{2{,}598{,}960} = \frac{123{,}552}{2{,}598{,}960}$$

$$\approx 0.0475\ldots$$

HW 2 - Part 1

13 Coin tossed 20 times.

(a) Probability at least 2 heads occur.

At least 2 means
≥ 2

Instead do the complement.
Which is exactly 0 heads
or exactly 1 head.

Exactly 0 heads

T T T T ... T T

1 way
all tails

Exactly 1 head

$$
\left.
\begin{array}{cccccc}
\underline{H} & \underline{T} & \underline{T} & \cdots & \underline{T} \\
\underline{T} & \underline{H} & \underline{T} & \cdots & \underline{T} \\
\underline{T} & \underline{T} & \underline{H} & \cdots & \underline{T} \\
 & \vdots & & \vdots & \\
\underline{T} & \underline{T} & \underline{T} & \cdots & \underline{H}
\end{array}
\right\}
\begin{array}{l} 20 \\ \text{ways} \end{array}
$$

OR

Pick where the head goes in the 20 spots

$$\binom{20}{1} = 20$$

then fill the rest with tails

# of ways to get exactly 0 heads or exactly 1 head

is

$$1 + 20 = 21$$

Sample space size is 2^{20}.

Probability of ≥ 2 heads occuring is

$$1 - \underbrace{\frac{21}{2^{20}}}_{< 2 \text{ heads}}$$

shes

2

HW 2 - Part 1

(16)(b) You're dealt 5 cards from 52-card deck. You know three of the cards are

2♣, 3♣, 4♣

You don't know other two.

Probability straight but not straight flush.

$\boxed{2♣}$ $\boxed{3♣}$ $\boxed{4♣}$ $\boxed{?}$ $\boxed{?}$

Sample space is filling in the ?'s with cards that aren't $2^{\clubsuit}, 3^{\clubsuit}, 4^{\clubsuit}$.

$$\binom{52-3}{2} = \binom{49}{2}$$

straights, not straight flushes

$\boxed{A^{?}}\ \boxed{2^{\clubsuit}}\ \boxed{3^{\clubsuit}}\ \boxed{4^{\clubsuit}}\ \boxed{5^{?}}$

$\boxed{2^{\clubsuit}}\ \boxed{3^{\clubsuit}}\ \boxed{4^{\clubsuit}}\ \boxed{5^{?}}\ \boxed{6^{?}}$

count straights + straight flushes

$$(4\cdot 4 + 4\cdot 4) - 2$$

Answer

$$\frac{4\cdot 4 + 4\cdot 4 - 2}{\binom{49}{2}}$$

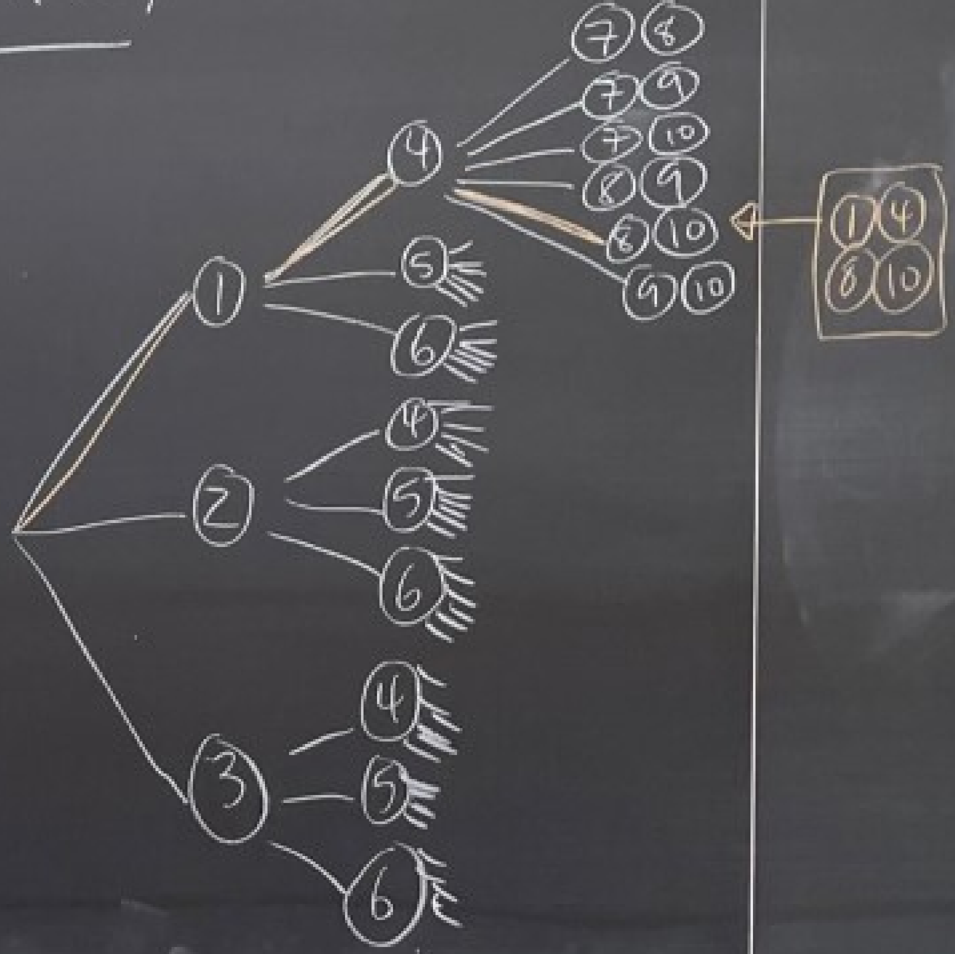
Test problem 5

bag with
- black balls: 1 2 3
- red balls: 4 5 6
- orange balls: 7 8 9 10

(a) Pick 4 balls at once from bag. Want probability picked 1 black, 1 red, 2 orange

ex: 2, 4, 8, 9

$$\frac{\binom{3}{1}\binom{3}{1}\binom{4}{2}}{\binom{10}{4}}$$



(b

7 8
7 9
7 10
8 9
8 10
9 10

1 4
8 10

(b) Pick 2 balls from bag

Want probability both are even

$$\frac{\binom{5}{2}}{\binom{10}{2}} = \frac{10}{45}$$

pick 2 evens from 5 even numbers

10 possibilities

2 4	4 10
2 6	6 8
2 8	6 10
2 10	8 10
4 6	
4 8	

HW 3 continued...

Recall $P(E|F) = \dfrac{P(E\cap F)}{P(F)}$

Sometimes $P(E|F)$ is not equal to $P(E)$, sometimes it is. Suppose $P(E|F)=P(E)$

Then, $\dfrac{P(E\cap F)}{P(F)} = P(E)$

which gives $P(E\cap F)=P(E)P(F)$

Def: We say that two events E and F are <u>independent</u> if

$$P(E\cap F) = P(E)P(F)$$

Otherwise we say E and F are <u>dependent</u>

<u>Note:</u> $\boxed{\text{E and F are independent}}$

is equivalent to

$$\boxed{P(E \cap F) = P(E)\,P(F)}$$

is equivalent to

$$\boxed{\frac{P(E \cap F)}{P(E)} = P(F) \quad \text{and} \quad \frac{P(E \cap F)}{P(F)} = P(E)}$$

is equivalent to

$$\boxed{P(F|E) = P(F) \quad \text{and} \quad P(E|F) = P(E)}$$

Ex: Suppose you roll two 6-sided dice, one green and one red.
Let E be the event that the green die is 1.
Let F be the event that the red die is 3.

Are E and F independent?

$$S = \left\{ (g, r) \;\middle|\; \begin{matrix} g = 1,2,3,4,5,6 \\ r = 1,2,3,4,5,6 \end{matrix} \right\}$$

$$E = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)\}$$

$$F = \{(1,3), (2,3), (3,3), (4,3), (5,3), (6,3)\}$$

$$E \cap F = \{(1,3)\}$$

$$P(E \cap F) = \frac{1}{36}$$

$$P(E) \cdot P(F) = \underbrace{\frac{6}{36}}_{1/6} \cdot \underbrace{\frac{6}{36}}_{1/6} = \frac{1}{36}$$

$P(E \cap F) = P(E)\,P(F)$

E and F are independent

Ex: Suppose you roll two 6-sided dice, one green and one red.

Let E be the event that the sum of the dice is 6.

Let F be the event that the red die equals 4.

$$S = \left\{ (g, r) \,\middle|\, \begin{array}{l} g = 1,2,3,4,5,6 \\ r = 1,2,3,4,5,6 \end{array} \right\}$$

$$E = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$$

$$F = \{(1,4), (2,4), (3,4), (4,4), (5,4), (6,4)\}$$

$$E \cap F = \{(2,4)\}$$

$$P(E \cap F) = \frac{1}{36} \approx 0.0278\ldots$$

$$P(E)P(F) = \frac{5}{36} \cdot \frac{6}{36} = \frac{30}{36^2} \approx 0.0231\ldots$$

$P(E \cap F) \neq P(E)P(F)$

E and F are dependent

General definition of independence ($n \geq 2$)

The events $E_1, E_2, \ldots, E_n$ are said to be independent if for every $2 \leq k \leq n$ we have that

$$P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}) = P(E_{i_1}) P(E_{i_2}) \cdots P(E_{i_k})$$

whenever $1 \leq i_1 < i_2 < \ldots < i_k \leq n$

$E_{i_k})$

Ex: E_1, E_2, E_3 are independent if the following equations are true:

$$\boxed{\begin{aligned} P(E_1 \cap E_2) &= P(E_1)P(E_2) \\ P(E_1 \cap E_3) &= P(E_1)P(E_3) \\ P(E_2 \cap E_3) &= P(E_2)P(E_3) \\ P(E_1 \cap E_2 \cap E_3) &= P(E_1)P(E_2)P(E_3) \end{aligned}}$$

need all 4 equations to be true for E_1, E_2, E_3 to be independent

Theorem: Let S be a sample space of a repeatable experiment. Let A and B be events where $A \cap B = \phi$ [this called mutually exclusive or disjoint events].

Suppose further that each time we repeat S, the experiment is independent of the previous times we did the experiment S.

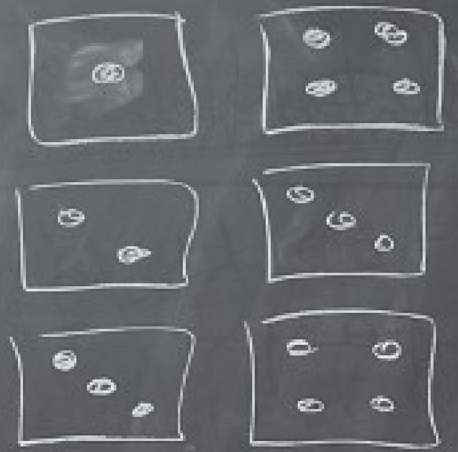
Suppose we repeat S until either A or B occurs. Then the probability that A occurs before B is given by the formula $$\frac{P(A)}{P(A)+P(B)}$$

<u>Ex:</u> Suppose we roll two 6-sided dice over and over. Let A be the event that the sum of the dice is 4. Let B be the event that the sum of the dice is 7. We keep rolling the dice until either A or B happens, then we stop.

roll 1 →

roll 2 →

roll 3 →



← (sum is 5)

← (sum is 5)

← (sum is 7) STOP

Ex of sum is 7 before sum is 4

The probability that A occurs before B

is $$\frac{P(A)}{P(A)+P(B)} = \frac{3/36}{3/36+6/36} = \frac{3}{9} = \frac{1}{3}$$

$A = \{(1,3),(2,2),(3,1)\}$

$B = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

The probability that B occurs before A is

$$\frac{P(B)}{P(B)+P(A)} = \frac{6/36}{6/36 + 3/36} = \frac{6}{9} = \frac{2}{3}$$

Theorem:

Let S be a sample space of a repeatable experiment. Let A and B be mutually exclusive events in S [ie $A \cap B = \phi$]. Suppose further that each time we repeat S the experiment is independent of the previous experiments. Suppose we repeat S until either A or B occurs. Then the probability that A occurs before B is $\frac{P(A)}{P(A)+P(B)}$.

Ex: Suppose we roll two 6-sided dice over and over. Let A be the event that the sum of the dice is 4. Let B be the event that the sum of the dice is 7. The probability that A occurs before B is

$$\frac{P(A)}{P(A)+P(B)} = \frac{3/36}{3/36 + 6/36} = \frac{3}{9} = 1/3$$

The probability that B occurs before A is

$$\frac{P(B)}{P(B)+P(A)} = \frac{6/36}{6/36+3/36} = \frac{6}{9} = 2/3$$

proof of theorem:

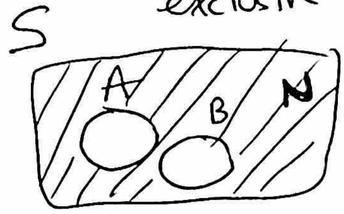
Let E be the event that A occurs before B.
Let A_1, B_1, N_1 be the events that A occurs on the first experiment, B occurs on the first experiment, or neither occurs on the first experiment.

Then

$$P(E) = P(E|A_1)P(A_1) + P(E|B_1)P(B_1) + P(E|N_1)P(N_1)$$

$$= 1 \cdot P(A) + 0 \cdot P(B) + P(E|N_1) \cdot \underbrace{[1 - P(A) - P(B)]}_{\text{because mutually exclusive}}$$

$$= P(A) + P(E)[1 - P(A) - P(B)]$$



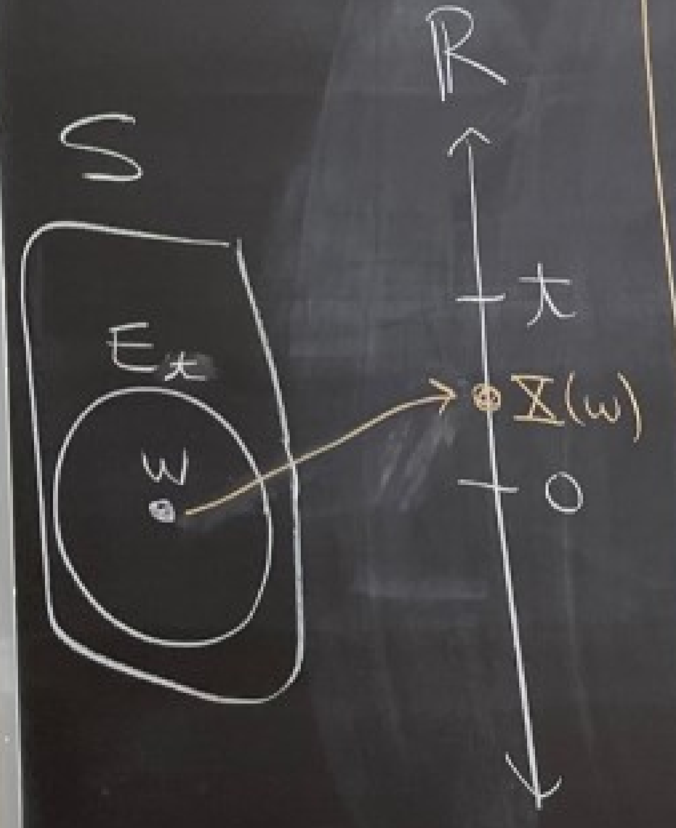
$P(E|N_1) = P(E)$
since the outcomes of successive experiments are all independent of each other, when the second experiment begins, the whole procedure probabilistically starts over again. Therefore, if in the 1st experiment neither A nor B occurs, the probability of E before doing the 1st experiment and after doing the 1st experiment is the same

Thus,

$$P(E) - P(E)[1 - P(A) - P(B)] = P(A)$$

So,

$$P(E) = \frac{P(A)}{P(A) + P(B)} \quad \blacksquare$$



Random Variables

Def: Let (S, Ω, P) be a probability space

A random variable is a function $\underbrace{X : S \longrightarrow \mathbb{R}}$

means X is a function
input = elements of S
output = elements of real numbers $\mathbb{R}$

such that for all real numbers t we have that $E_t = \{w \mid w \in S \text{ and } X(w) \le t\}$ is an element of Ω

R

unction
of S
rs of real
bers $\mathbb{R}$

The condition on E_t means we can measure the probability of E_t for all t.

The condition on E_t will always be true when S is finite and Ω consists of all subsets of S.

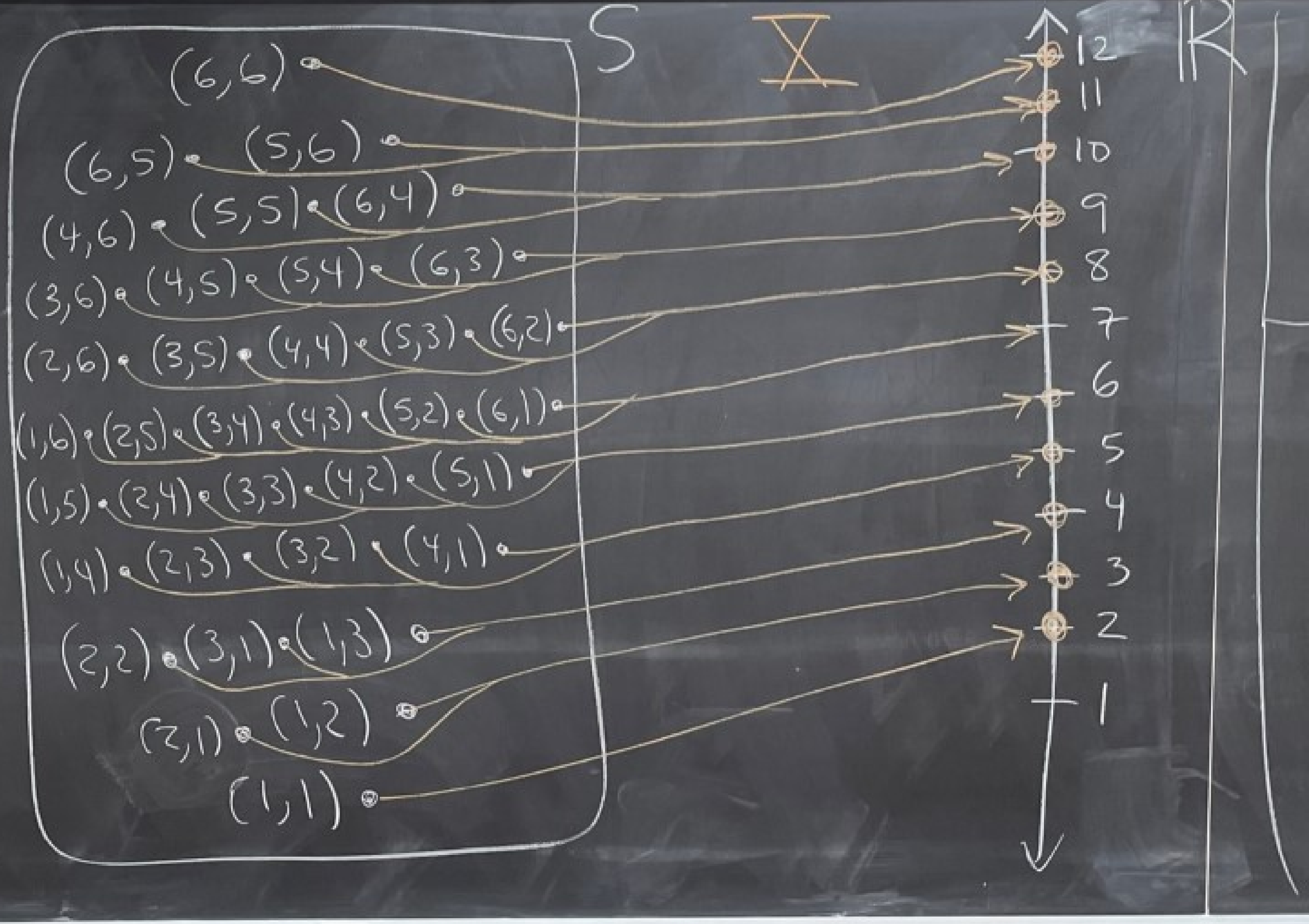
Def: Let X be a random variable on a probability space (S, Ω, P). We say that X is discrete if the range of X can be enumerated as a list of values $x_1, x_2, x_3, \ldots$

Math 3450: Ie, the range of X is finite or countably infinite

Ex: Let (S, Ω, P) be the probability space corresponding to rolling two 6-sided dice. Let X be the sum of the dice.

$$S = \{(1,1),(1,2),\ldots,(6,6)\}$$

$$X((2,3)) = 2+3 = 5$$



The range of X can be written
as a list $2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$.
So, X is a discrete random variable.

$t = 3.2$

$$E_t = \{ w \mid w \in S \text{ and } X(w) \leq 3.2 \}$$
$$= \{ (2,1), (1,2), (1,1) \}$$

$\leftarrow$ The def of random variable says we can always measure the probability of E_t

Def: Let X be a random variable on a probability space (S, Ω, P).

- We write $P(X = i)$ instead of
$$P(\{w \mid w \in S \text{ and } X(w) = i\})$$

- We write $P(X \leq i)$ instead of
$$P(\{w \mid w \in S \text{ and } X(w) \leq i\})$$

- Similarly for $P(X < i)$, etc

(4,
(3,6
(2,6
(1,6)
(1,5)
(1,4)
(2,

Def continued...

- The <u>probability function</u> of X is

$$p(i) = P(X = i)$$

So, $p : \mathbb{R} \to \mathbb{R}$

- The <u>cumulative distribution function</u> of X is

$$F(i) = P(X \leq i)$$

So, $F : \mathbb{R} \to \mathbb{R}$

Ex: Consider the probability space (S, Ω, P) corresponding to rolling two 6-sided die.

Let X be the sum of the dice.

Let's draw the probability function p and the cumulative distribution F.

$\mathbb{R}$

$$p(2) = P(X=2) = P(\{(1,1)\}) = \frac{1}{36}$$

$$p(3) = P(X=3) = P(\{(2,1),(1,2)\}) = \frac{2}{36}$$

$$p(4) = P(X=4) = P(\{(2,2),(1,3),(3,1)\}) = \frac{3}{36}$$

$$p(5) = P(X=5) = 4/36$$

$$p(6) = 5/36$$

$$p(7) = 6/36$$

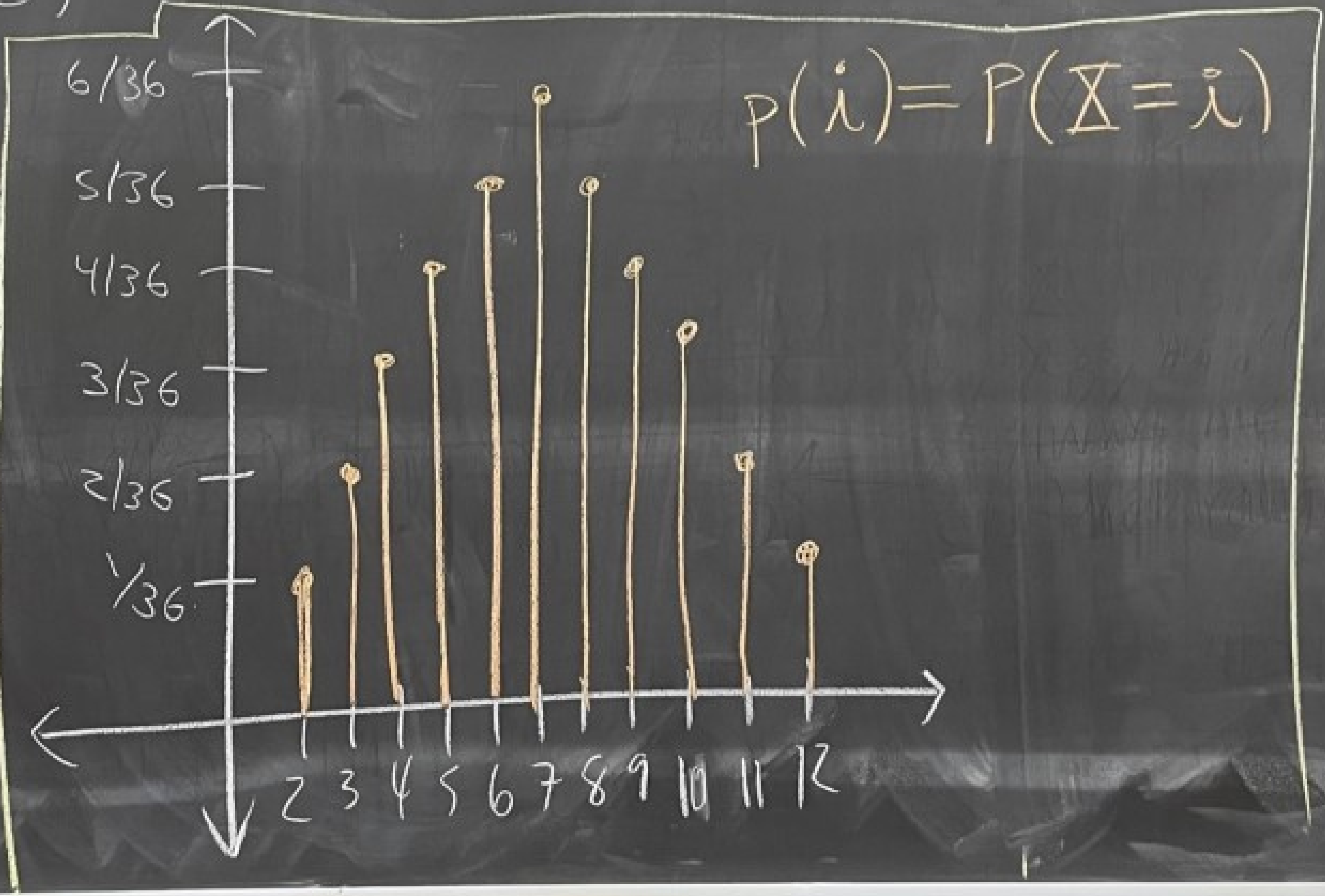
$$p(8) = 5/36$$

$$p(9) = 4/36$$

$$p(10) = 3/36$$

$$p(11) = 2/36$$

$$p(12) = 1/36$$



$$F(2) = P(X \le 2) = P(\{(1,1)\}) = \frac{1}{36}$$

$$F(3) = P(X \le 3) = P(X=2) + P(X=3) = \frac{1}{36} + \frac{2}{36} = \frac{3}{36}$$

$$F(3.5) = P(X \le 3.5) = P(X \le 3) = \frac{3}{36}$$

$$F(4) = P(X \le 4) = P(X=2) + P(X=3) + P(X=4) =$$

$$= \frac{1}{36} + \frac{2}{36} + \frac{3}{36} = \frac{6}{36}$$

and so on...

$$F(2) = P(X \le 2) = P(\{(1,1)\}) = \frac{1}{36}$$

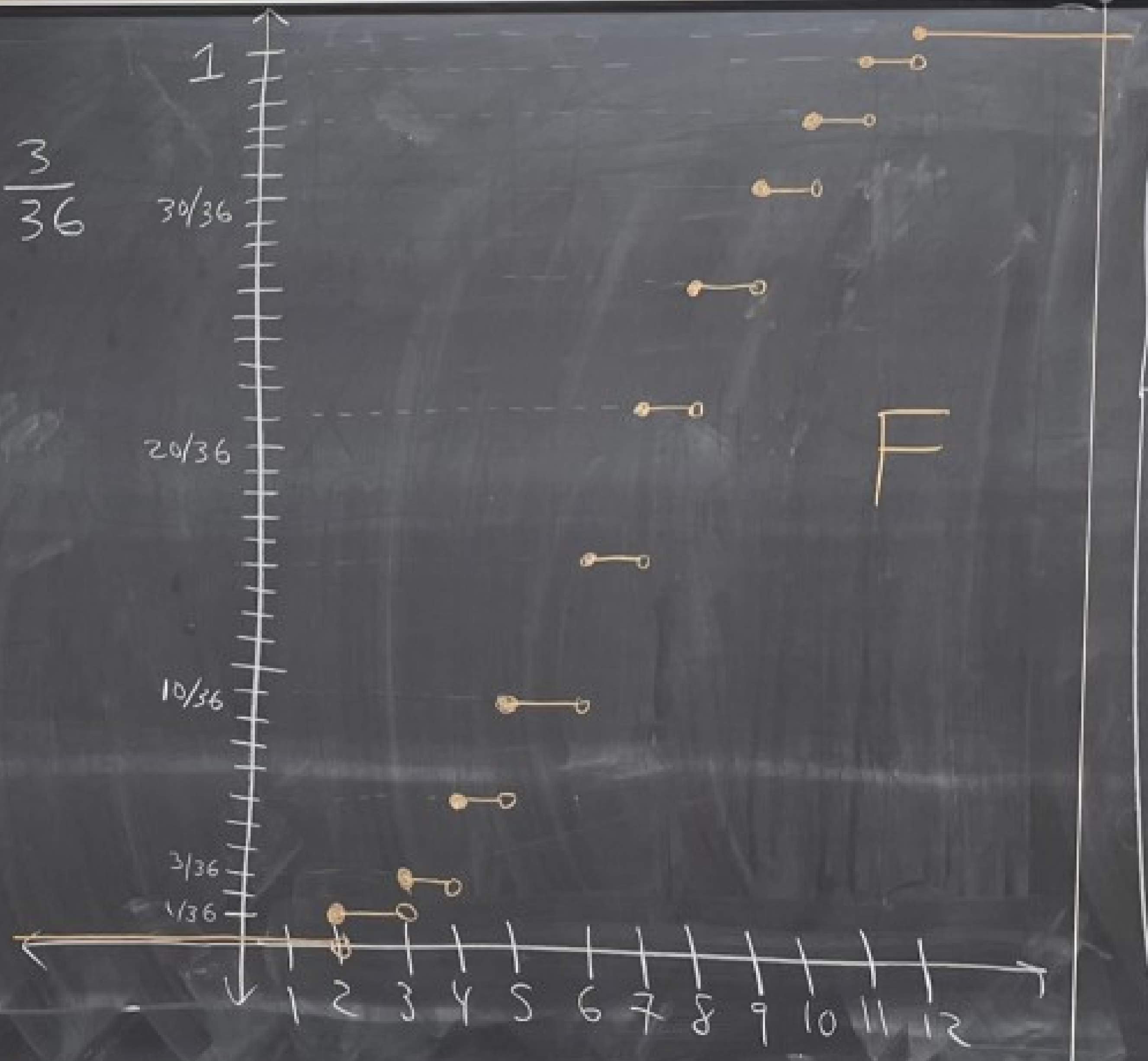
$$F(3) = P(X \le 3) = P(X=2) + P(X=3) = \frac{1}{36} + \frac{2}{36} = \frac{3}{36}$$

$$F(3.5) = P(X \le 3.5) = P(X \le 3) = \frac{3}{36}$$

$$F(4) = P(X \le 4) = P(X=2) + P(X=3) + P(X=4) =$$

$$= \frac{1}{36} + \frac{2}{36} + \frac{3}{36} = \frac{6}{36}$$

and so on...



I rewrote some of the HW 3 solutions

Krebs will teach this class on Wednesday

He will start HW 5

HW 4 continued...

Def: Let X be a discrete random variable on a probability space (S, Ω, P). The *expected value* of X is

$$E[X] = \sum_{\underbrace{w \in S}_{\text{sum over elements } w \text{ of } S}} X(w) \cdot P(\{w\})$$

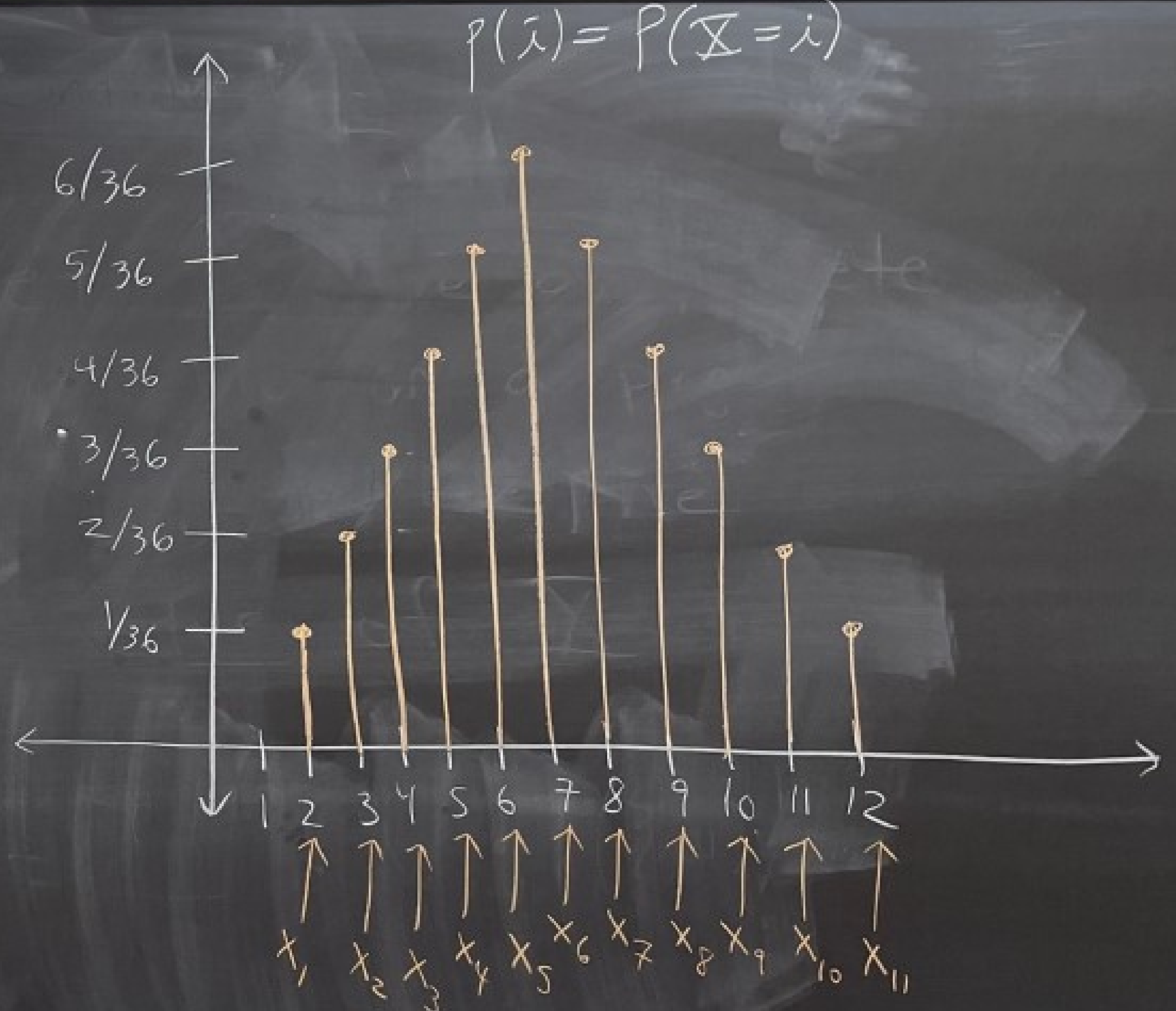
If $x_1, x_2, x_3, \ldots$ are all the outputs of X [ie the range of X] then you can rewrite

$$E[X] = \sum_i x_i \cdot P(X = x_i)$$

$$p(x_i) = P(X = x_i)$$

$$= \sum_i x_i \cdot p(x_i)$$

Ex: Consider the experiment of rolling two 6-sided dice. Let X be the sum of the dice. Let p be the probability function for X, ie $p(i) = P(X = i)$



One way to calculate $E[X]$

$$E[X] = \sum x_i \cdot P(X = x_i)$$

$$= \sum x_i \cdot p(x_i)$$

$$= (2)\left(\frac{1}{36}\right) + (3)\left(\frac{2}{36}\right) + (4)\left(\frac{3}{36}\right) + (5)\left(\frac{4}{36}\right) + (6)\left(\frac{5}{36}\right)$$

$$+ (7)\left(\frac{6}{36}\right) + (8)\left(\frac{5}{36}\right) + (9)\left(\frac{4}{36}\right) + (10)\left(\frac{3}{36}\right)$$

$$+ (11)\left(\frac{2}{36}\right) + (12)\left(\frac{1}{36}\right)$$

$$= 7$$

This is a weighted average
You weight each x_i by $p(x_i)$

Another way to calculate $E[X]$

$$E[X] = \sum_{\omega \in S} X(\omega)\cdot P(\{\omega\}) = \overbrace{X(1,1)}^{2}\cdot\overbrace{P(\{(1,1)\})}^{1/36} + \overbrace{X(1,2)}^{3}\cdot\overbrace{P(\{(1,2)\})}^{1/36} + \overbrace{X(2,1)}^{3}\cdot\overbrace{P(\{(2,1)\})}^{1/36}$$

$$+ \overbrace{X(1,3)}^{4}\cdot\overbrace{P(\{(1,3)\})}^{1/36} + \overbrace{X(2,2)}^{4}\cdot\overbrace{P(\{(2,2)\})}^{1/36} + \overbrace{X(3,1)}^{4}\cdot\overbrace{P(\{(3,1)\})}^{1/36}$$

$$+ \overbrace{X(1,4)}^{5}\cdot\overbrace{P(\{(1,4)\})}^{1/36} + \cdots + \overbrace{X(6,6)}^{12}\cdot\overbrace{P(\{(6,6)\})}^{1/36}$$

Same sum as before once you re-group the terms

$$= (2)\left(\tfrac{1}{36}\right) + (3)\left(\tfrac{2}{36}\right) + (4)\left(\tfrac{3}{36}\right) + \cdots + (12)\left(\tfrac{1}{36}\right)$$

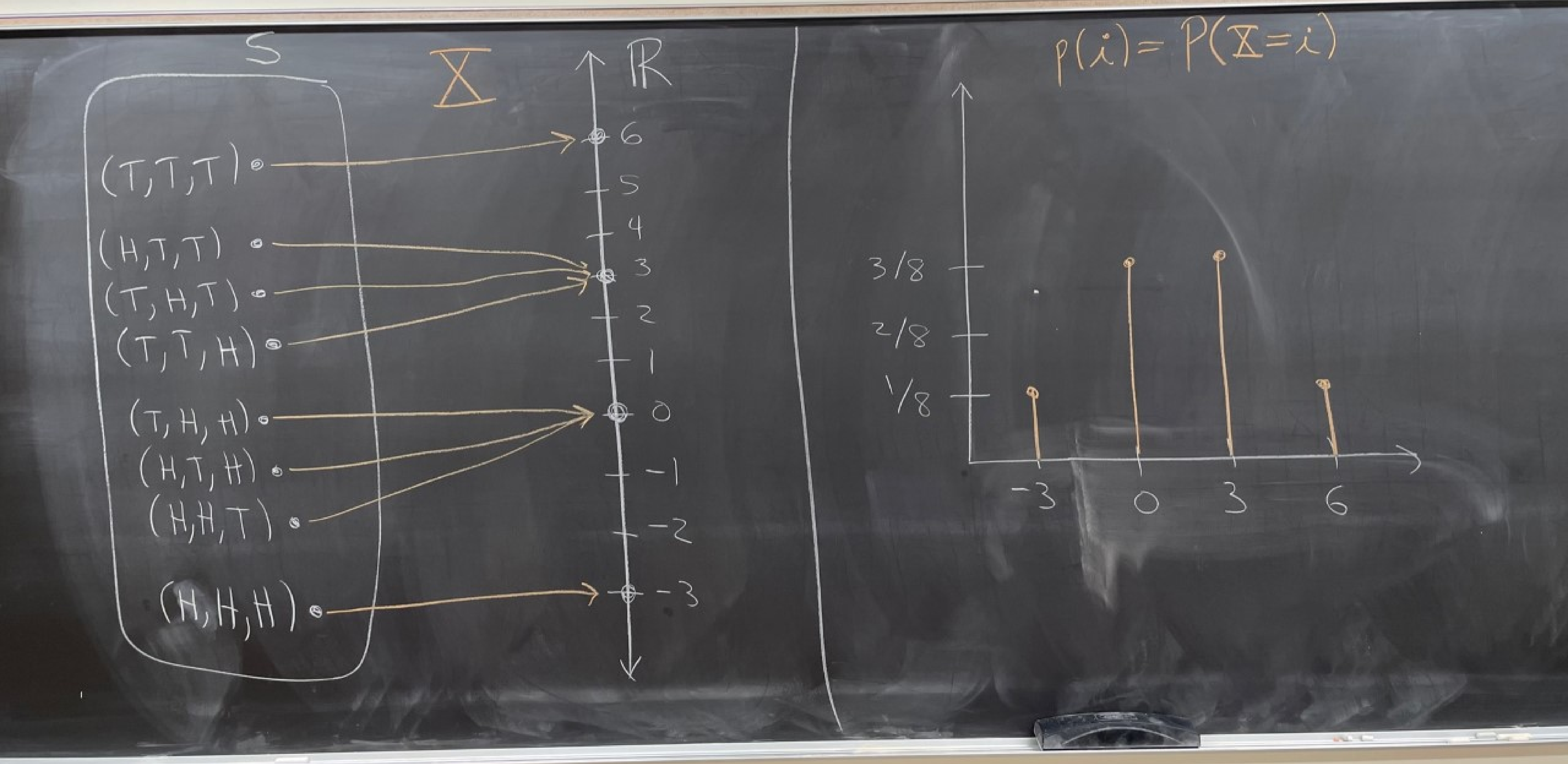
$$= 7$$

Ex: Suppose you flip a coin 3 times.
For every head you lose \$1
For every tail you win \$2.
Let X be the total amout won or lost.
Draw X and the probability function $p(i) = P(X = i)$.
Calculate $E[X]$.

$$(H, H, T)$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ -\$1 & -\$1 & +\$2 \end{array}$$

$$X(H,H,T) = -1-1+2$$
$$= 0$$


S
X
R
(T,T,T)
(H,T,T)
(T,H,T)
(T,T,H)
(T,H,H)
(H,T,H)
(H,H,T)
(H,H,H)
6
5
4
3
2
1
0
-1
-2
-3
p(i) = P(X = i)
3/8
2/8
1/8
-3
0
3
6

$$E[X] = (-3)\left(\frac{1}{8}\right) + (0)\left(\frac{3}{8}\right) + (3)\left(\frac{3}{8}\right) + (6)\left(\frac{1}{8}\right)$$

$$= \frac{-3+0+9+6}{8}$$

$$= \frac{12}{8} = 1.5$$

You can interpret this as on average if you played the game a lot then you'd average about \$1.50 win per game.

If you played the game 1,000,000 times you would have won probably around

$$(1{,}000{,}000)(\$1.50)$$

$$= \$1{,}500{,}000$$

$\boxed{\text{Odds}}$

Let E be an event

We define

$$\text{odds for } E = \frac{P(E)}{P(\overline{E})} = \frac{P(E)}{1-P(E)}$$

$$\text{odds against } E = \frac{P(\overline{E})}{P(E)} = \frac{1-P(E)}{P(E)}$$

Casino uses this

<u>Ex:</u> Suppose you roll a 4-sided dice. Let E be the event that you roll a 1.

So, $P(E) = \frac{1}{4}$.

$$\text{odds for } E = \frac{P(E)}{1-P(E)} = \frac{1/4}{3/4} = \frac{1}{3}$$

written 1 : 3
read "1 to 3"

$$\text{odds against } E = \frac{1-P(E)}{P(E)} = \frac{3/4}{1/4} = \frac{3}{1}$$

written 3 : 1
read "3 to 1"

How to convert back

$$\boxed{\begin{array}{c}\text{odds for } E\\ a:b\end{array}} \longrightarrow \boxed{P(E) = \frac{a}{a+b}}$$

$$\boxed{\begin{array}{c}\text{odds against } E\\ c:d\end{array}} \longrightarrow \boxed{P(E) = \frac{d}{c+d}}$$

Ex:

Suppose the odds for E are $3:4$

Then, $P(E) = \frac{3}{3+4} = \frac{3}{7}$

Ex:

Suppose the odds against E are $10:1$

Then, $P(E) = \frac{1}{10+1} = \frac{1}{11}$

Binomial Random Variables

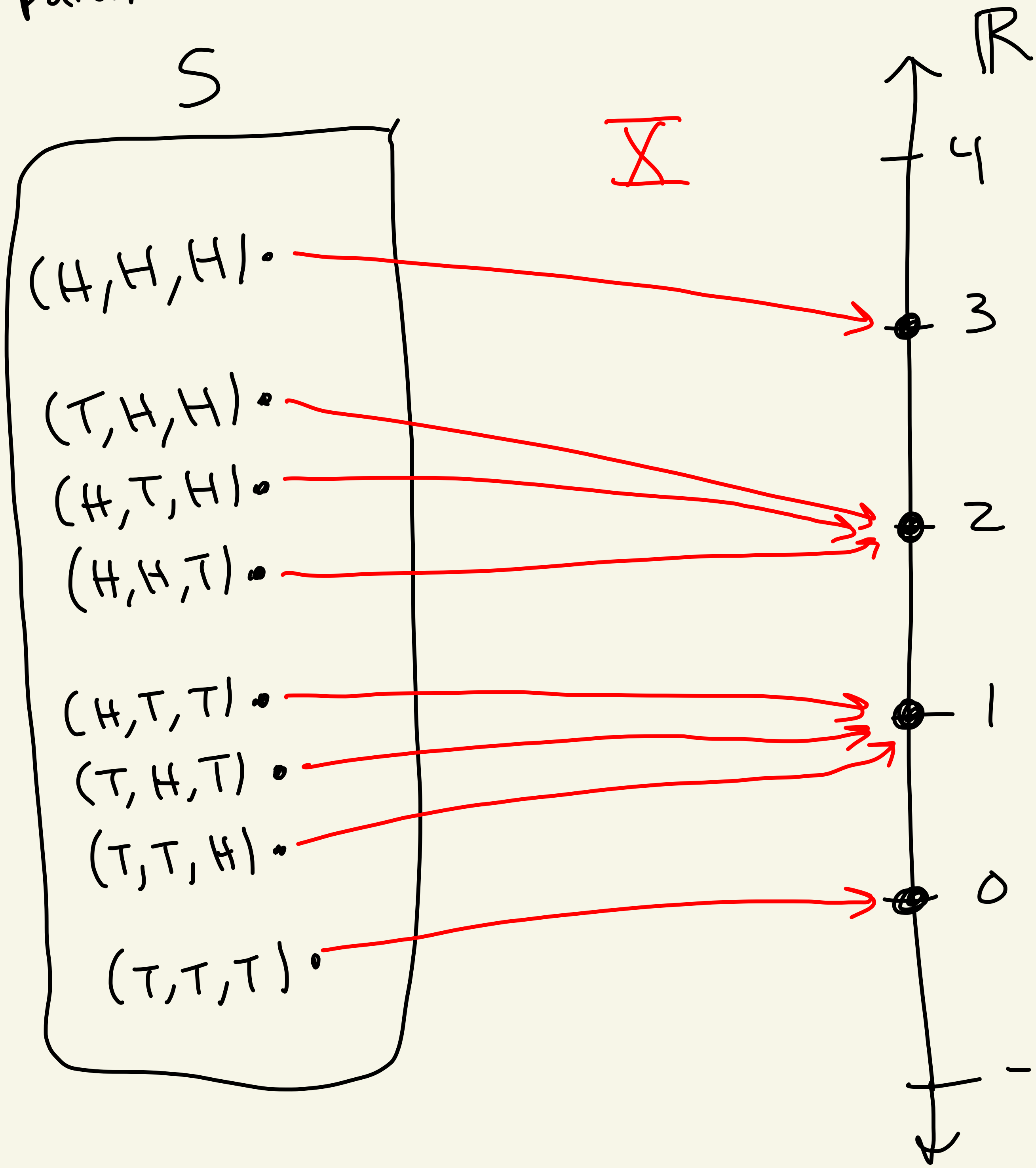
A Bernoulli trial is an experiment with two possible outcomes: success and failure. Suppose success occurs with probability p and failure with probability $1-p$.

Ex: Flipping a coin.
Say heads is success and tails is failure.
Here $p = \frac{1}{2}$ and $1-p = \frac{1}{2}$.

Now suppose that n Bernoulli trials, each with probability of success p, are performed independently. Let X be the number of successes. Then X is called a binomial random variable with parameters n and p.

[redacted]

Ex: Suppose you flip a coin 3 times in a row. Let $X : S \to \mathbb{R}$ be the random variable that counts the number of heads that occur. Here heads is "success" with probability $p = \frac{1}{2}$ and tails is "failure" with probability $1-p = \frac{1}{2}$. X is a binomial random variable with parameters $p = \frac{1}{2}$ and $n = 3$.



Thm: Let X be a binomial random variable with parameters n and p. Let p be the probability function of X, that is $p(i) = P(X=i)$. Then

$$p(i) = \begin{cases} \binom{n}{i} p^i (1-p)^{n-i}, & \text{if } i = 0,1,2,\ldots,n \\ 0, & \text{otherwise} \end{cases}$$

Proof: Let i be fixed as one of the numbers $0,1,2,\ldots,$ or n. Let's calculate $p(i) = P(X=i)$.

How many ways can i successes and $n-i$ failures occur in n trials?

$$\underbrace{\_\ \underline{s}\ \underline{s}\ \_\ \_\ \cdots\ \underline{s}\ \_}_{\binom{n}{i} \text{ ways to fill in } i \text{ successes.}}$$

In $\binom{n}{i}$ ways!

Given such a sequence of successes and failures the probability of such a sequence is $p^i (1-p)^{n-i}$ because of independence.

Ex: $(i=3)$
H T H H
$\uparrow \uparrow \uparrow \uparrow$
$p \ \ 1-p \ \ p \ \ p$
$p^3 (1-p)^{4-3}$

Therefore, $p(i) = \binom{n}{i} p^i (1-p)^{n-i}$.

If i is not one of the numbers $0,1,\ldots,n$ then $p(i) = 0$ since we can't have for example $\frac{1}{2}$ successes or $n+1$ successes in n trials. $\blacksquare$

65 Ex: Suppose we flip a coin 100 times. What is the probability of exactly 48 heads occuring?

$$p(48) = \binom{100}{48}\left(\frac{1}{2}\right)^{48}\left(\frac{1}{2}\right)^{100-48}$$

$$= \frac{\binom{100}{48}}{2^{100}} = \frac{93{,}206{,}558{,}875{,}049{,}876{,}949{,}581{,}681{,}100}{1{,}267{,}650{,}600{,}228{,}229{,}401{,}496{,}703{,}205{,}376}$$

$$\approx 0.073527\ldots$$

$$\approx 7.35\%$$

Ex: Suppose we flip a coin 20 times. What is the probability of getting between 10 and 12 heads? Let X be the number of heads that occurs.

$$p(10 \leq X \leq 12) = P(X=10) + P(X=11) + P(X=12)$$

$$= \binom{20}{10}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^{20-10} + \binom{20}{11}\left(\frac{1}{2}\right)^{11}\left(\frac{1}{2}\right)^{20-11} + \binom{20}{12}\left(\frac{1}{2}\right)^{12}\left(\frac{1}{2}\right)^{20-12}$$

$$= \frac{\binom{20}{10} + \binom{20}{11} + \binom{20}{12}}{2^{20}}$$

$$= \frac{184{,}756 + 167{,}960 + 125{,}970}{1{,}048{,}576} \approx 0.456511 \approx 45.65\%$$

Ex: Suppose that we roll two 6-sided dice 20 times. We say that a suppose

sum of seven or eleven is a success and any other sum is a failure. Let X be the number of successes.

What is $p(12) = P(X=12)$?

sum of 7

$$p = \frac{6}{36} + \frac{2}{36} = \frac{8}{36} = \frac{2}{9}$$

sum of 11

$$1-p = \frac{7}{9}$$

$$p(12) = \binom{20}{12}\left(\frac{2}{9}\right)^{12}\left(\frac{7}{9}\right)^{8} = \frac{\binom{20}{12}\cdot 2^{12}\cdot 7^{8}}{9^{20}}$$

$$= \frac{(125,970)(4096)(5,764,801)}{12,157,665,459,056,928,801}$$

$$\approx 0.000244659\ldots \approx 0.024\%$$

1

BINOMIAL THM

Let $x, y \in \mathbb{R}$ and $n \geq 0$ be an integer. Then

$$(x+y)^n = \sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^{i} = \sum_{i=0}^{n}\binom{n}{i}x^{i}y^{n-i}$$

Ex: $(a+b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3$

$= a^3 + 3a^2b + 3ab^2 + b^3$

2 DO FIRST

3 Thm: Let X be a binomial random variable with parameters n and p. Then $E[X] = np$.

MOTIVATE THM FIRST WITH THIS

Intuition: If say we toss a coin 100 times we expect that the average number of heads will be $100 \cdot \frac{1}{2} = n \cdot p$.

proof of thm: $E[X] = \sum_{i=0}^{n} i\binom{n}{i}p^{i}(1-p)^{n-i} =$

67

$$= \sum_{i=1}^{n} i \frac{n!}{i!(n-i)!} p^{i}(1-p)^{n-i}$$

$$= \sum_{i=1}^{n} \frac{n!}{(i-1)!(n-i)!} p^{i}(1-p)^{n-i}$$

$$= np \cdot \left[\sum_{i=1}^{n} \frac{(n-1)!}{(i-1)!(n-i)!} p^{i-1}(1-p)^{n-i} \right]$$

$$= np \cdot \left[\sum_{i=1}^{n} \binom{n-1}{i-1} p^{i-1}(1-p)^{n-i} \right]$$

$$= np \left[\sum_{k=0}^{n-1} \binom{n-1}{k} p^{k}(1-p)^{(n-1)-k} \right]$$

Let $k=i-1$

$$= np\left[p+(1-p)\right]^{n-1} = np. \quad \blacksquare$$

BINOMIAL THM

prob. of success in a single trial

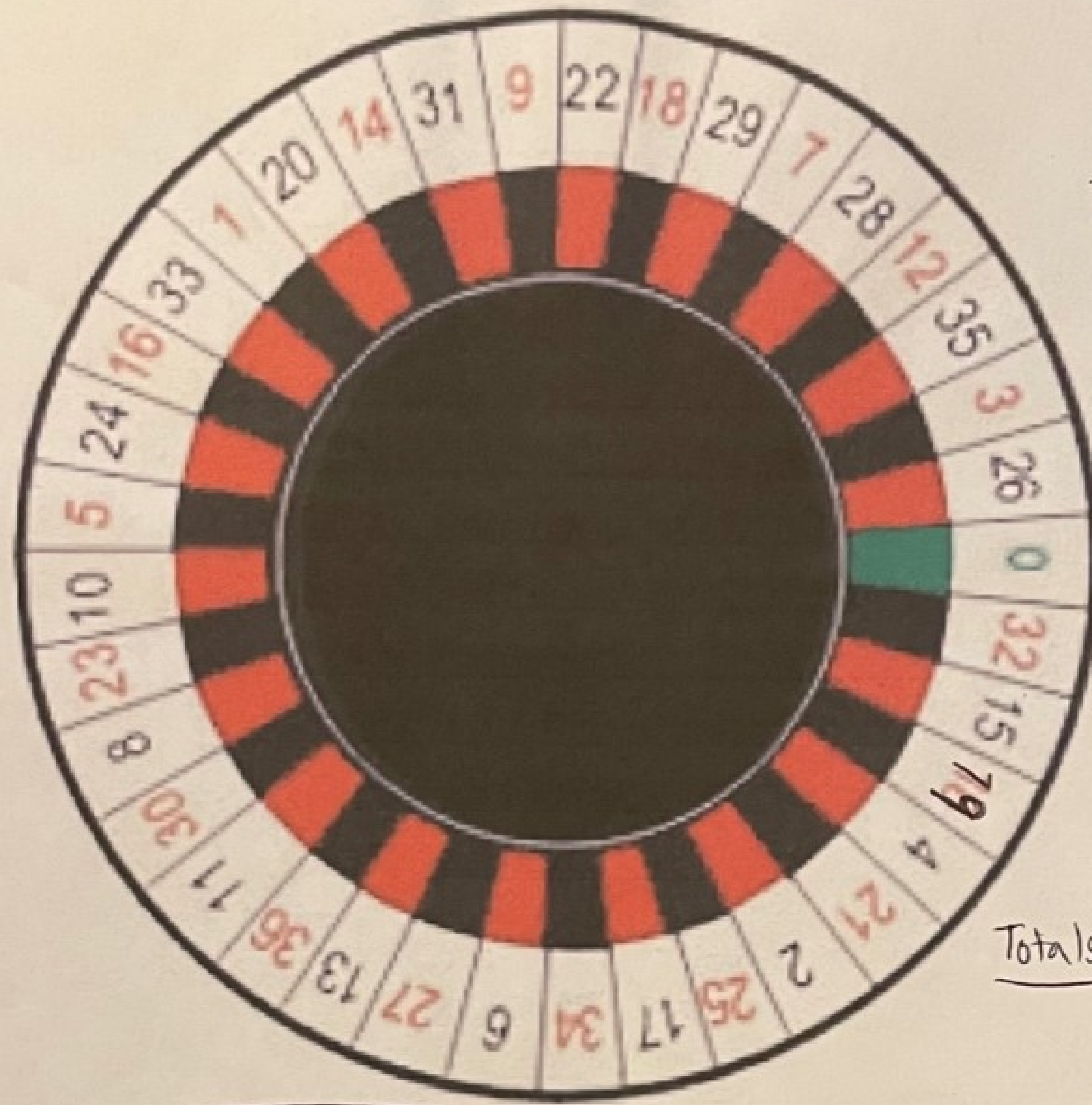
#successes

#trials

#successes

$$\underbrace{\binom{n}{i}}_{\#\text{ ways to get } i \text{ successes}} p^{i} \underbrace{(1-p)^{n-i}}_{\text{same but with failures}}$$

EUROPEAN



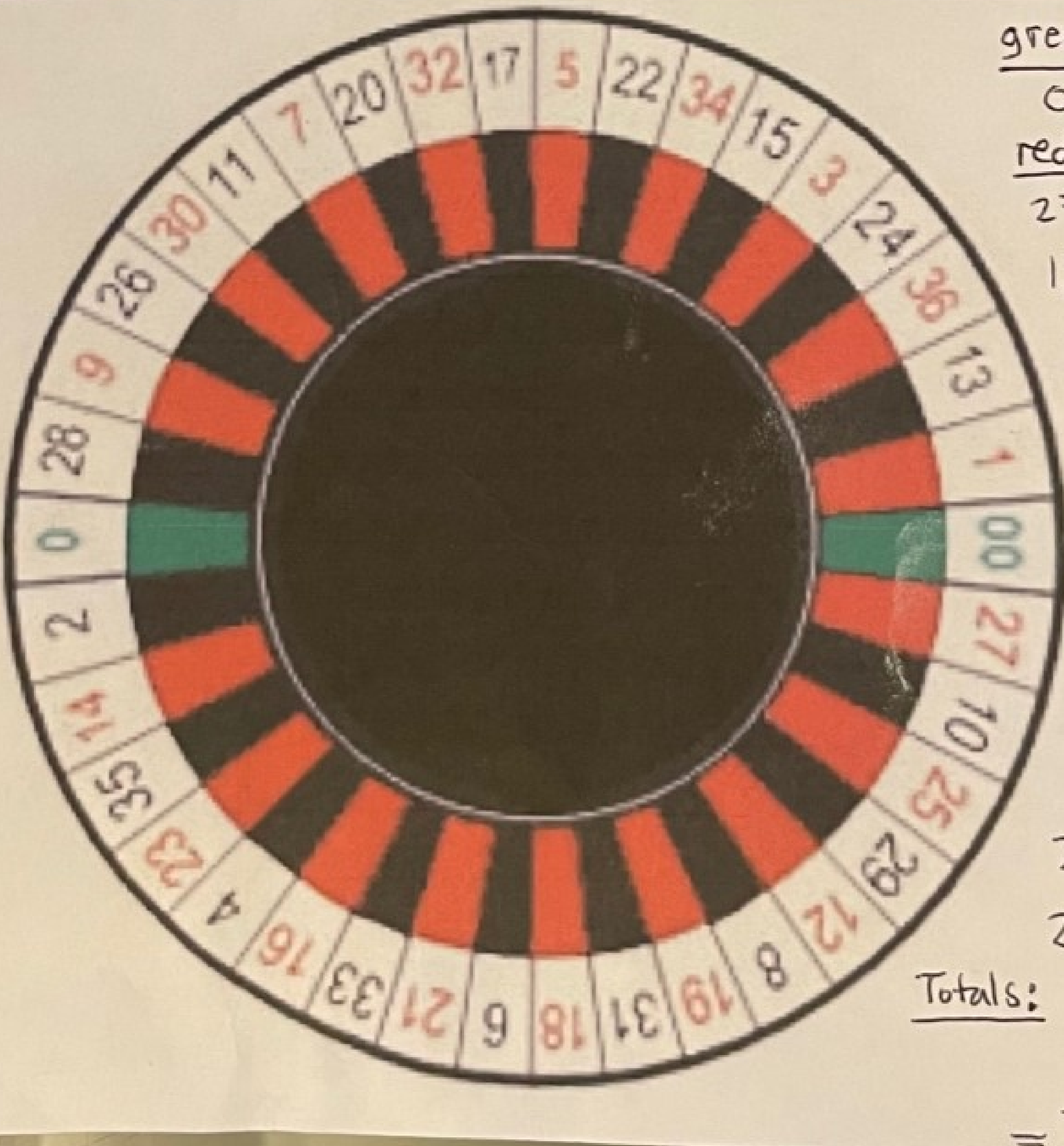
green
0

red
32, 19, 21, 25,
34, 27, 36, 30,
23, 5, 16, 1,
14, 9, 18, 7
12, 3

black
15, 4, 2, 17
6, 13, 11, 8
10, 24, 33, 20
31, 22, 29, 28
35, 26

Totals: 1 green
18 red
18 black
= 37 total

AMERICAN



green
0, 00

red
27, 25, 12, 19
18, 21, 16, 23
14, 9, 30, 7
32, 5, 34, 3
36, 1

black
10, 29, 8, 31
6, 33, 4, 35
2, 28, 26, 11
20, 17, 22, 15
24, 13

Totals: 2 green
18 red
18 black
= 38 total

American version / Handout

Type of Bets And ~~Winning Chances~~ Casino payouts

Inside bets

Bet Name	Ex.	Numbers to bet on	Payout	~~Chances~~	True odds
Straight up	A	30	35:1	~~38:1~~	37:1
Split Bet	B	11 or 14	17:1	~~38:2~~	36:2
Street Bet	C	19, 20, 21	11:1	~~38:3~~	35:3
Corner	D	25, 26, 28, 29	8:1	~~38:4~~	34:4
Five Numbers	E	0, 00, 1, 2, 3	6:1	~~38:5~~	
Line Bet	F	4, 5, 6, 7, 8, 9	5:1	~~38:6~~	32:6

Outside Bets

Bet Name	Ex.	Numbers to bet on	Payout	~~Chances~~	True odds
Column	G	Set of column numbers	2:1	~~38:12~~	26:12
Dozen	H	25 through 36	2:1	~~38:12~~	26:12
Red or Black	I	Red numbers	1:1	~~38:18~~	20:18
Even or Odd	J	Odd numbers	1:1	~~38:18~~	20:18
Low or High	K	19 through 36	1:1	~~38:18~~	20:18

Last time Krebs covered HW 5. We will finish HW 4 and then pickup where he left off.

Roulette

Sample space

$$S = \{0, 00, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36\}$$

Each number is equally likely with probability $\frac{1}{38}$

Straight up bet

Suppose we bet \$1 on 7

Let X be the amount won or lost.

$$X(w) = \begin{cases} -1 & \text{if } w \neq 7 \\ 35 & \text{if } w = 7 \end{cases}$$

$$E[X] = (-1)\cdot P(X \neq 7) + (35)\cdot P(X = 7)$$

$$= (-1)\cdot\left(\frac{37}{38}\right) + (35)\left(\frac{1}{38}\right) = \frac{-2}{38} = \frac{-1}{19} = \boxed{-0.0526}$$

$$\boxed{\begin{aligned} P(X = 7) &= \frac{1}{38} \\ P(X \neq 7) &= \frac{37}{38} \end{aligned}}$$

On average you lose 5.26 cents per \$1 bet.

The casio pays 35 : 1 on this Kind of bet.
What are the real odds (ie "odds against")?

Here the event we win is $E = \{7\}$.

$$\text{odds against } E = \frac{P(\overline{E})}{P(E)} = \frac{37/38}{1/38} = \frac{37}{1}$$

odds against is 37 : 1

If the casino paid 37:1 instead of 35:1 then the expected value would change to

$$(-1)\cdot\left(\frac{37}{38}\right)+(37)\cdot\left(\frac{1}{38}\right)=0$$

So if they paid you 37:1 then "on average" in the long run everyone breaks even.

Column bet (2:1 payout)

Suppose we bet \$1 on the first column.

Let $E = \{1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34\}$

E consists of the winning numbers.

Let X be the amount won or lost.

$$X(w) = \begin{cases} 2 & \text{if } w \text{ is in } E \\ -1 & \text{if } w \text{ is not in } E \end{cases}$$

$$E[X] = (2)\cdot P(X=2) + (-1)\cdot P(X=-1)$$

$$= \underbrace{(2)\left(\frac{12}{38}\right)}_{P(E)} + \underbrace{(-1)\left(\frac{26}{38}\right)}_{P(\overline{E})}$$

$$= \frac{-2}{38} = \frac{-1}{19} \approx \boxed{-0.0526}$$

On average you lose 5.26 cents per $1 bet.

What are the true odds, ie the odds against E?

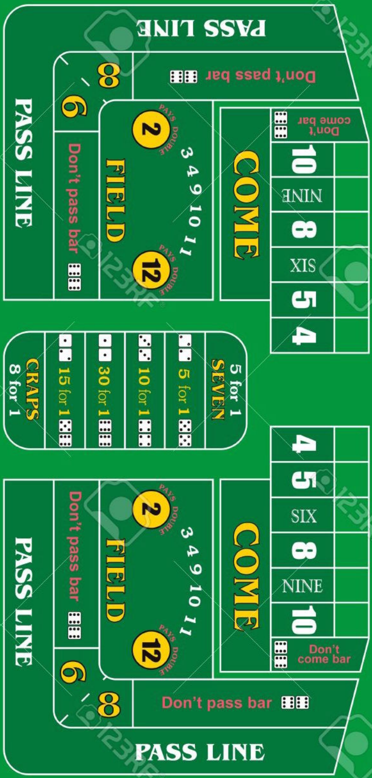
$$\text{Odds against } E = \frac{P(\bar{E})}{P(E)} = \frac{26/38}{12/38} = \frac{26}{12} = \frac{13}{6}$$

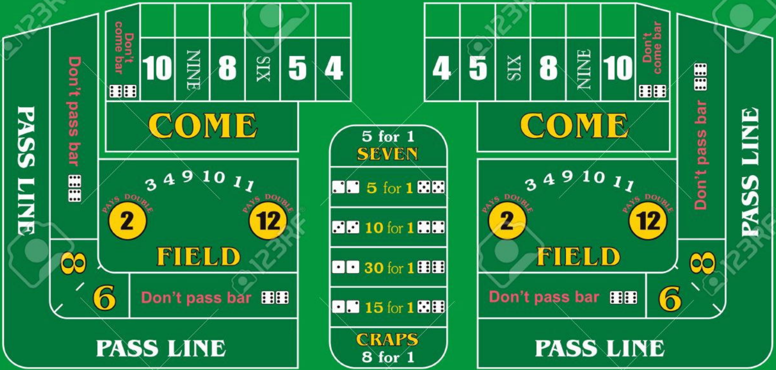
Odds against are $13:6$

If the casino paid $\$\frac{13}{6}$ for a win on a bet of \$1

then the expected valve would be

$$\left(\frac{13}{6}\right)\left(\frac{12}{38}\right) + (-1)\left(\frac{26}{38}\right) = 0$$


PASS LINE
Don't pass bar
Don't come bar
COME
FIELD
PAYS DOUBLE
2
3 4 9 10 11
12
10
NINE
8
SIX
5
4
6
8
CRAPS
8 for 1
15 for 1
30 for 1
10 for 1
5 for 1
SEVEN
5 for 1
PASS LINE
Don't pass bar
FIELD
COME
Don't come bar
PASS LINE
Don't pass bar
PASS LINE

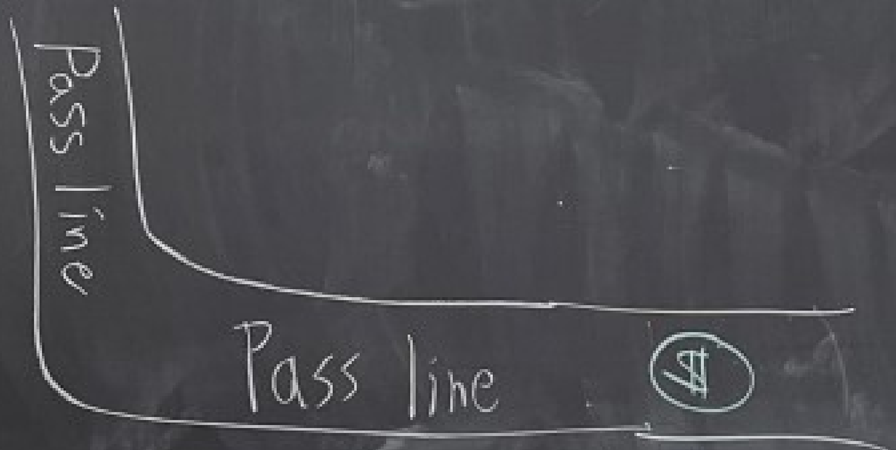


Don't come bar
10
NINE
8
SIX
5
4
COME
Don't pass bar
PASS LINE
3 4 9 10 11
PAYS DOUBLE
2
PAYS DOUBLE
12
FIELD
8
6
Don't pass bar
PASS LINE
5 for 1
SEVEN
5 for 1
10 for 1
30 for 1
15 for 1
CRAPS
8 for 1
4
5
SIX
8
NINE
10
Don't come bar
COME
Don't pass bar
PASS LINE
3 4 9 10 11
PAYS DOUBLE
2
PAYS DOUBLE
12
FIELD
8
Don't pass bar
6
PASS LINE

The main bet in craps is called the pass line bet.

People place their bets on the table and the game starts.

Suppose we put money on the pass line.



Some player is rolling the dice. Two 6-sided dice are rolled.

- The first roll is called the "come out roll."

$\boxed{\text{The sum is the dice is measured on each roll.}}$

ice.

alled

<u>case 1</u>: If a 7 or 11 is rolled then you win the pass line bet.

This roll is called a "natural"

<u>Case 2</u>: If a 2, 3, or 12 is rolled then you lose the pass line bet.

This roll is called "craps"

<u>case 3</u>: Suppose a 4, 5, 6, 8, 9, or 10 is rolled. The number rolled is called the "point". Now the dice are rolled over and over again until either 7 is rolled or the point is rolled again.

If the point comes up first
the pass line bet is won.
If 7 comes up first then
the pass line bet is lost.

Then the game repeats
with new pass line bets
and a new come out roll.

The casino payout is 1:1 on pass line bets ts

pass line bet
$2

Come out roll

7

win $2

pass line bet
$50

Come out roll

2

lose $50

pass line bet
$10

Come out roll	roll 2	roll 3	roll 4	roll 5
6	2	5	4	6

6 is the point

6 is marked on the table

Point happened before 7

all pass line bets are won.

We get $10

pass line bet
$200

Come out roll	roll 2	roll 3	roll 4
4	6	5	7

7 rolled before the point.
We lose $200

the point is 4

Craps

Sum of dice	# of ways to roll
2	1
3	2
4	3
5	4
6	5
7	6
8	5
9	4
10	3
11	2
12	1

probabilities for come out roll

roll	probability	
7 or 11	8/36	Win
2, 3, or 12	4/36	lose
4	3/36	
5	4/36	
6	5/36	
8	5/36	
9	4/36	
10	3/36	

Point is made 24/36

Previous theorem

Let S be a sample space of a repeatable experiment. Let A and B be mutually exclusive events [means $A \cap B = \phi$]

Suppose each time we repeat the experiment S it is independent of the previous times we did S.

Suppose we repeat S over and over until either A or B occurs. The probability that A occurs before B is

$$\frac{P(A)}{P(A)+P(B)}$$

proof is online 3/14

Let's now calculate the probabilities once a point is made

So 8 is the point

Suppose on the come out roll an 8 is rolled.

Let A be the event the sum of the dice is 8.

Let B be the event the sum of the dice is 7.

The probability 8 is rolled before a 7 is rolled is

$$\frac{P(A)}{P(A)+P(B)} = \frac{5/36}{5/36 + 6/36} = \frac{5}{11}$$

The probability 7 is rolled before an 8 is rolled is

$$\frac{P(B)}{P(B)+P(A)} = \frac{6/36}{6/36 + 5/36} = \frac{6}{11}$$

You can do this for all the different point values.

point	Probability of point being rolled before a 7	Probability of 7 being rolled before the point
4	3/9	6/9
5	4/10	6/10
6	5/11	6/11
8	5/11	6/11
9	4/10	6/10
10	3/9	6/9

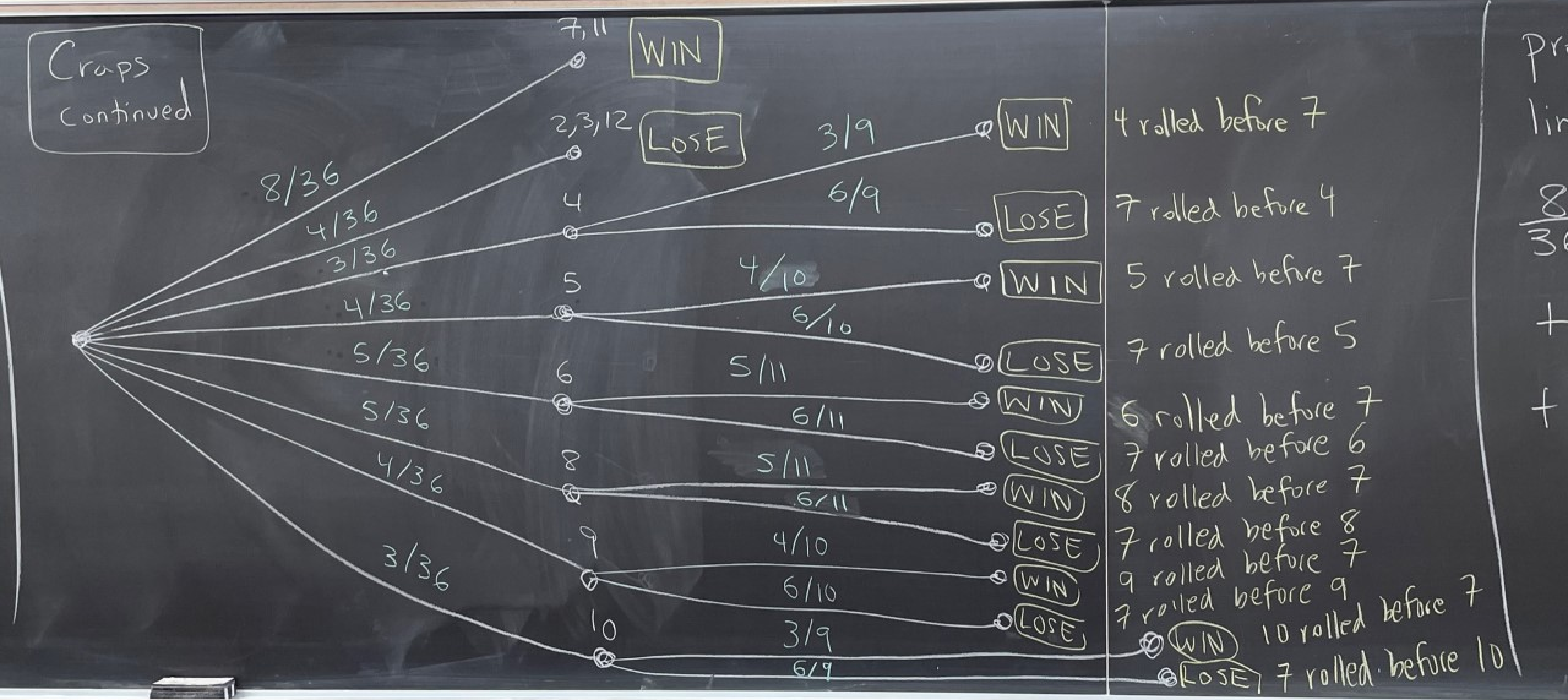
Test 2

Covers

HW 3 / HW 4

Monday

April 25



Craps continued
7, 11
WIN
2, 3, 12
LOSE
8/36
4/36
3/36
4/36
5/36
5/36
4/36
3/36
4
5
6
8
9
10
3/9
6/9
4/10
6/10
5/11
6/11
5/11
6/11
4/10
6/10
3/9
6/9
WIN
LOSE
WIN
LOSE
WIN
LOSE
WIN
LOSE
WIN
LOSE
WIN
LOSE
4 rolled before 7
7 rolled before 4
5 rolled before 7
7 rolled before 5
6 rolled before 7
7 rolled before 6
8 rolled before 7
7 rolled before 8
9 rolled before 7
7 rolled before 9
10 rolled before 7
7 rolled before 10

Probability of winning a pass line bet is

$$\frac{8}{36} + \frac{3}{36}\cdot\frac{3}{9} + \frac{4}{36}\cdot\frac{4}{10} + \frac{5}{36}\cdot\frac{5}{11} + \frac{5}{36}\cdot\frac{5}{11} + \frac{4}{36}\cdot\frac{4}{10} + \frac{3}{36}\cdot\frac{3}{9} = \frac{244}{495} \approx 0.4929\ldots$$

7
5
7
6
7
8
7
7
ed before 7

You can do the same method to get the probability of losing a pass line bet, or just do $1 - p(\text{winning})$

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probability of losing a pass line bet is

$$1 - \frac{244}{495} = \frac{251}{495} \approx 0.5071$$

C

c

Expected value (Pass line bet paid 1:1)

Suppose we bet \$1 on the pass line. Let X be the amount won or lost.

$$E[X] = (\$1)\left(\frac{244}{495}\right) + (-\$1)\left(\frac{251}{495}\right)$$

$$= -\$\frac{7}{495} \approx \boxed{-\$0.01414}$$

The 1:1 payout on the pass line bet is less than the true odds

The true odds are

$$\frac{p(\text{losing})}{p(\text{win})} = \frac{251/495}{244/495} = \frac{251}{244} \approx 1.0415$$

If the casino paid you $\frac{251}{244} : 1$ then expected value would be

$$\left(\$\frac{251}{244}\right)\left(\frac{244}{495}\right) + (-\$1)\left(\frac{251}{495}\right) = \$0$$

The casino does allow an extra "free odds" bet if a point is established. The free odds bets are paid off at their true odds making them "fair" bet. [fair bet means expected valve 0, ie the casino has no edge]

point	true odds
4	2 : 1
5	3 : 2
6	6 : 5
8	6 : 5
9	3 : 2
10	2 : 1

Point is 4

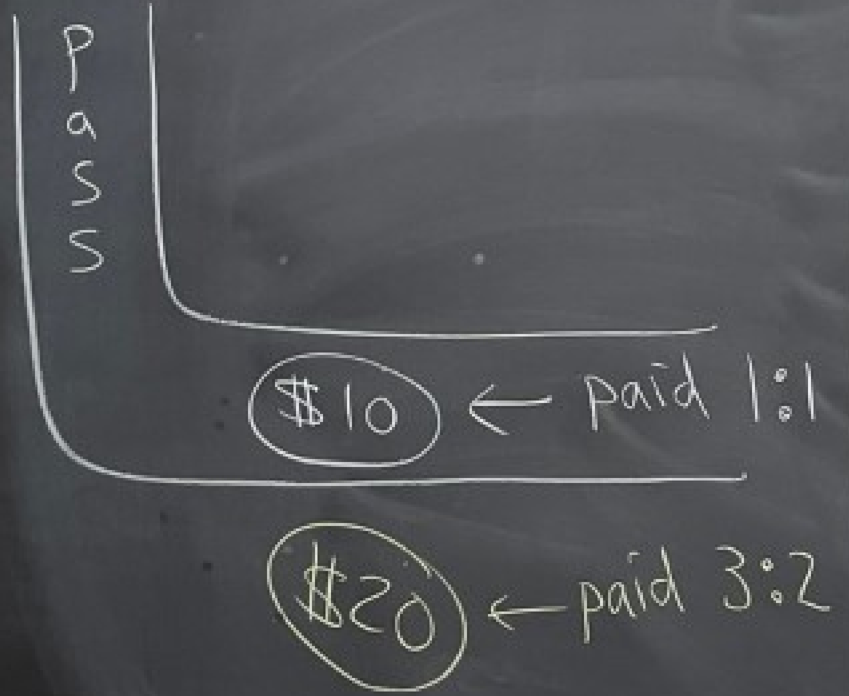
$P(\text{win}) = 3/9$

$P(\text{losing}) = 6/9$

$$\text{odds against} = \frac{P(\text{losing})}{P(\text{win})} = \frac{6/9}{3/9} = \frac{2}{1}$$

Ex: Suppose you bet \$10 on the pass line. The first roll makes the point 5. Now that the points made we can make a "free odds" bet. Let's bet \$20 more dollars as free odds bet.

This second bet is paid at 3:2 instead of 1:1.



Come out roll	roll 2	roll 3	roll 4
5	2	3	5
Point is 5			WIN

If you lose these bets
You lose $30.

odds"

If you win you win

$$\left(\frac{1}{1}\right)(\$10)+\left(\frac{3}{2}\right)(\$20)=\boxed{\$40}$$

Expected Value

Suppose you bet \$10 on the pass line and if a point is made then you bet an additional \$10 as a free odds bet. Let X be the amount won or lost.

$$E[X] = \underbrace{(\$10)\left(\tfrac{8}{36}\right)}_{\text{7 or 11 on come out roll}} + \underbrace{(-\$10)\left(\tfrac{4}{36}\right)}_{\text{2,3, or 12 on come out roll}}$$

$$+ \underbrace{2}_{\boxed{\text{4 or 10 point}}} \cdot \underbrace{(\$30)}\cdot\left(\tfrac{3}{36}\right)\cdot\left(\tfrac{3}{9}\right) + \underbrace{2\cdot(-\$20)\left(\tfrac{3}{36}\right)\left(\tfrac{6}{9}\right)}_{\text{losing point is 4 or 10}} +$$

$\$10 \leftarrow 1:1$
$\$10 \leftarrow 2:1$
$\$30$

WINNING

$$+ \underbrace{2\cdot(\$25)\left(\tfrac{4}{36}\right)\left(\tfrac{4}{10}\right)}_{\text{win 5 or 9 point}} + \underbrace{2\cdot(-\$20)\left(\tfrac{4}{36}\right)\left(\tfrac{6}{10}\right)}_{\text{lose 5 or 9}}$$

$\$10 \leftarrow 1:1$
$\$10 \leftarrow 3:2$
$\$25$

$$+ \underbrace{2\cdot(\$22)\left(\tfrac{5}{36}\right)\left(\tfrac{5}{11}\right)}_{\text{win 6 or 8}} + \underbrace{2\cdot(-\$20)\left(\tfrac{5}{36}\right)\left(\tfrac{6}{11}\right)}_{\text{lose 6 or 8}} = -\$\tfrac{14}{99} \approx -\$0.1414$$

$\$10 \leftarrow 1:1$
$\$10 \leftarrow 6:5$
$\$22$

Last time...

Suppose you bet \$10 on the pass line and if a point is made we bet \$10 more as a free odds bet.

X = amount won/lost

$$E[X] = -\frac{14}{99} \approx -\$0.1414\ldots$$

Let's put this in "per \$1 bet" terms

$$\text{average amount bet} = \underbrace{(\$10)\left(\frac{12}{36}\right)}_{\substack{\text{Come out roll} \\ \text{is } 7,11,2,3,12}} + \underbrace{(\$20)\left(\frac{24}{36}\right)}_{\substack{\text{Come out roll} \\ \text{is } 4,5,6,8,9,10 \\ \text{(a point is made)}}}$$

$$= (\$10)\left(\frac{1}{3}\right) + (\$20)\left(\frac{2}{3}\right)$$

$$= \frac{\$50}{3} \approx \$16.67$$

Expected value per dollar wagered is

$$\approx \frac{-\$0.1414}{\$16.67} \approx \boxed{-\$0.0085}$$

Recall with \$1 bet on pass line the expected value was $\boxed{-\$0.014}$

St. Petersburg Paradox

Goes back to 1700's.

A casino offers a game to a single player.
A fair coin is tossed at each stage.

The pot (amount won) starts at \$2 and doubles everytime a head appears
The first time a tails appears, the game ends and the player wins whatever is in the pot.

How much would you pay to play this game? You don't get back what you paid, just what you win.

Ex:

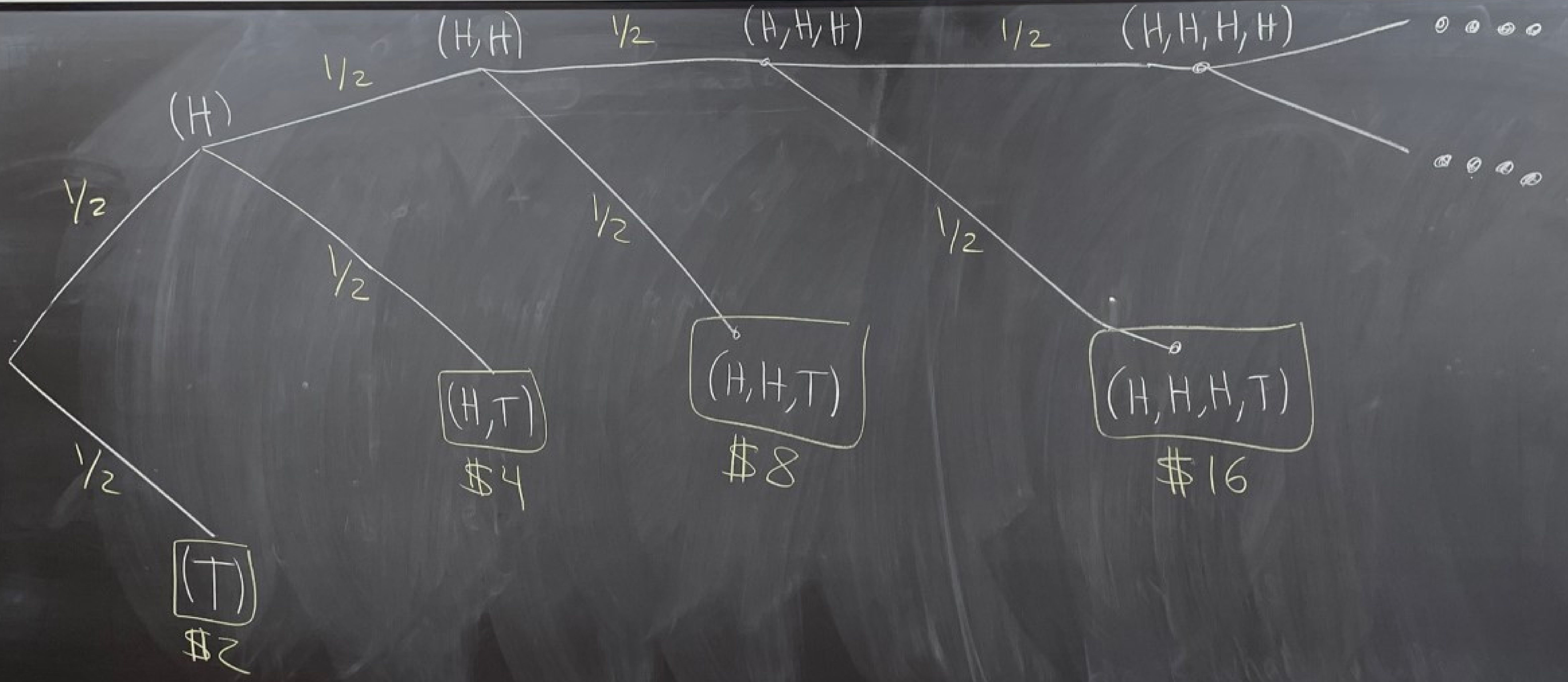
pay $\$5$ to play

pot	flip
$\$2$	H
$\$4$	H
$\$8$	H
$\$16$	T

win $\$16$

paid $\$5$

total winnings = $\$11$



(H)
1/2
(H,H)
1/2
(H,H,H)
1/2
(H,H,H,H)
1/2
1/2
1/2
1/2
1/2
(T)
$2
(H,T)
$4
(H,H,T)
$8
(H,H,H,T)
$16

Let X be the amount won or lost.

$$E[X] = (\$2)\left(\frac{1}{2}\right) + (\$4)\underbrace{\left(\frac{1}{4}\right)}_{\left(\frac{1}{2}\right)^2} + (\$8)\underbrace{\left(\frac{1}{8}\right)}_{\left(\frac{1}{2}\right)^3} + \cdots$$

$$= \$1 + \$1 + \$1 + \cdots$$

$$= \infty$$

To win at least $\$2^{20} = \$1{,}048{,}576$ this would happen with probability

$$\frac{1}{2^{20}} + \frac{1}{2^{21}} + \frac{1}{2^{22}} + \cdots = \frac{1}{2^{20}}\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots\right] = \frac{1}{2^{20}} \cdot \frac{1}{1 - 1/2}$$

$$\boxed{1 + x + x^2 + \cdots = \frac{1}{1-x},\ -1 < x < 1}$$

$$= \frac{1}{2^{19}}$$

$$\approx 0.000001907\ldots$$

$\boxed{\frac{1}{2^n} \text{ gets small fast}}$

This game has infinite expected value. However, you probably wouldn't pay alot to play since you would win $\$2^n$ with probababilility $\frac{1}{2^n}$

Before spring break
Krebs covered the
HW 5 topic
Binomial random
variables

Let's do another HW 5 topic example.
Consider the game where you are
dealt 2 cards from a 52-card
deck. What is the probability
that you get blackjack?

Answer

$$\frac{\binom{16}{1}\binom{4}{1}}{\binom{52}{2}} = \frac{16 \cdot 4}{\frac{52 \cdot 51}{2}} = \frac{64}{1326} = \boxed{\frac{32}{663}} \approx \boxed{0.04826}$$

$10 \leftarrow 4$ 10's $J \leftarrow 4$ J's $Q \leftarrow 4$ Q's $K \leftarrow 4$ K's } 16 total	4 aces

Let's repeat this game 20 times, shuffling the deck each time.

What's the probability we get at least 2 blackjacks?

Let X be the number of blackjacks that we get. X is a Binomial random variable with $n = 20$, $p = \frac{32}{663}$ ← getting a blackjack or "success" on a given round/trial

$1 - p = \frac{631}{663}$ ← not getting blackjack on a round, ie "failure"

Formula for exactly k successes/blackjacks

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} = \binom{20}{k} \cdot \underbrace{\left(\frac{32}{663}\right)}_{\approx 0.048}^{k} \cdot \underbrace{\left(\frac{631}{663}\right)}_{\approx 0.952}^{20-k}$$

$\approx 0.048^k \quad \approx 0.952^{n-k}$

Probability of at least 2 successes is

$$\frac{32}{663}$$

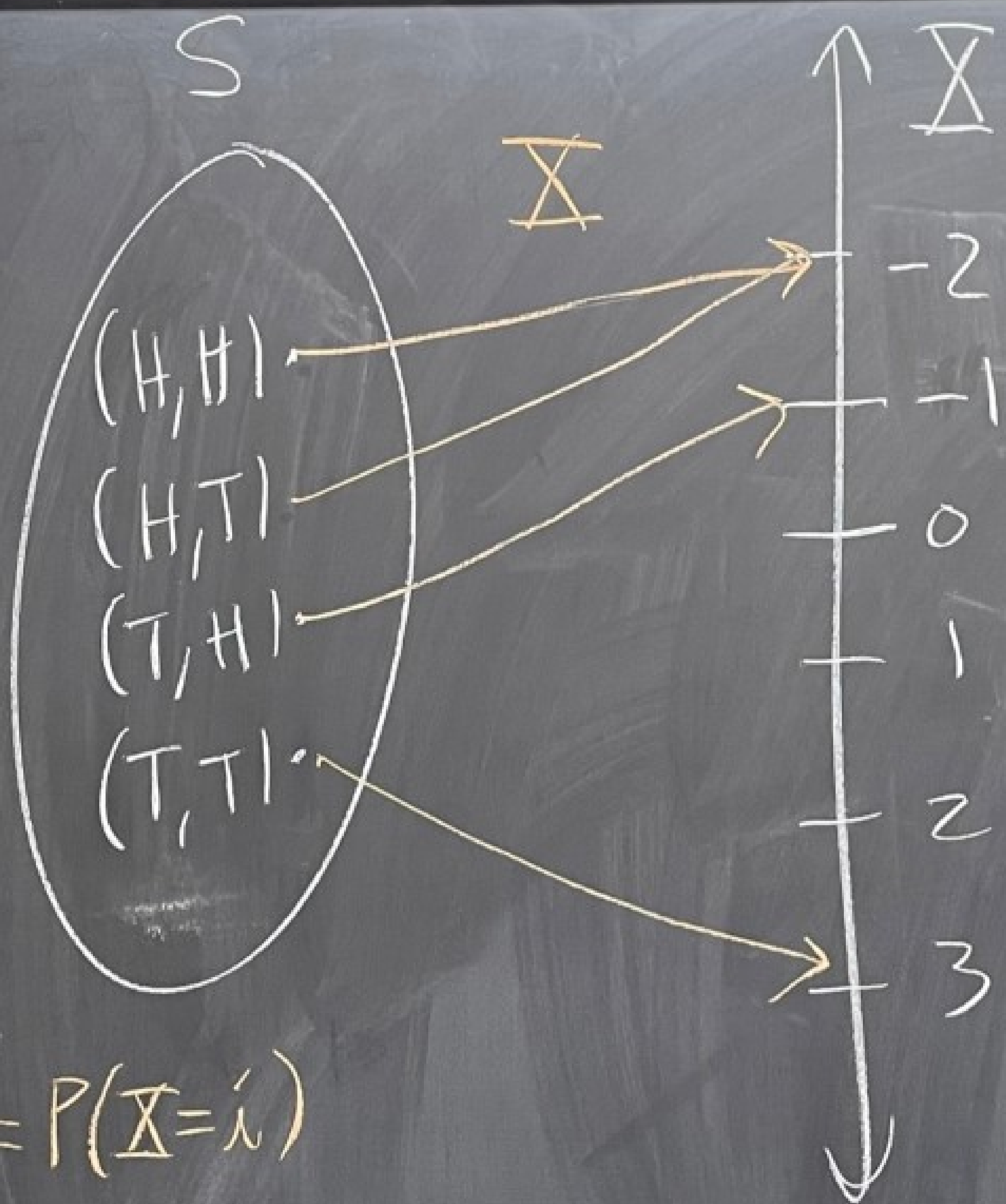
$$P(X=2)+P(X=3)+P(X=4)+\dots+P(X=20)$$

$$= 1 - P(X=0) - P(X=1)$$

$$\approx 1 - \underbrace{\binom{20}{0}}_{1}\underbrace{(0.048)^0}_{1}(0.952)^{20-0} - \underbrace{\binom{20}{1}}_{20}(0.048)^1(0.952)^{20-1}$$

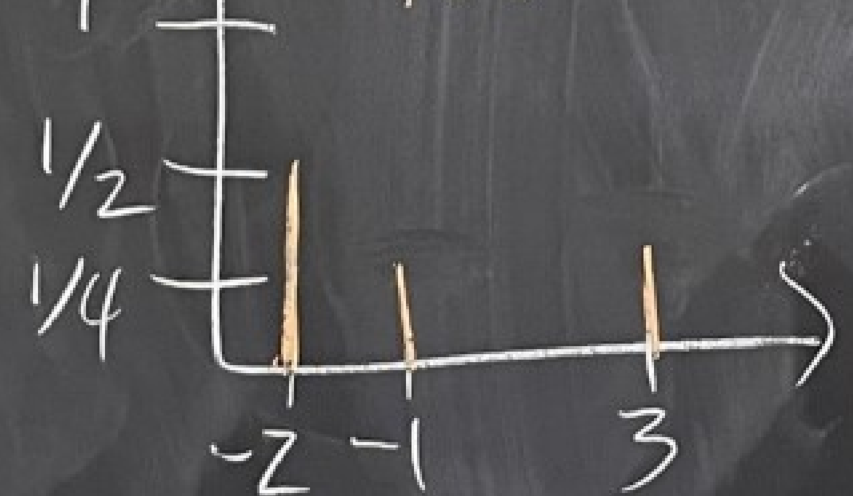
$$\approx 1 - 0.3738 - 20(0.048)(0.3927)$$

$$\approx 1 - 0.3738 - 0.3769 \approx \boxed{0.2493} \approx \boxed{24.93\%}$$



SIDE EXAMPLE TO REMEMBER STUFF

$$P(i) = P(X = i)$$



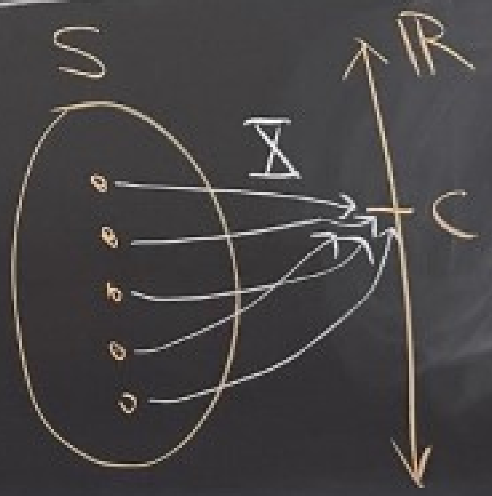
Test 2 is Monday 4/25 next week

Test 2 covers HW 3 / HW 4

Let's review on Weds

I'm going to rewrite HW 4 #7 on Tuesday

I rewrote a lot of the other solutions this past weekend



HW 5/6 Topics continued...

Theorem: Let X be discrete random variable on a probability space (S, Ω, P).
If X is constant, that is there is a constant c where $X(w) = c$ for all w in S, then $E[X] = c$

$= c$

Theorem: Let X be a discrete random variable with values $x_1, x_2, x_3, \ldots$

Let p be the probability function for X.

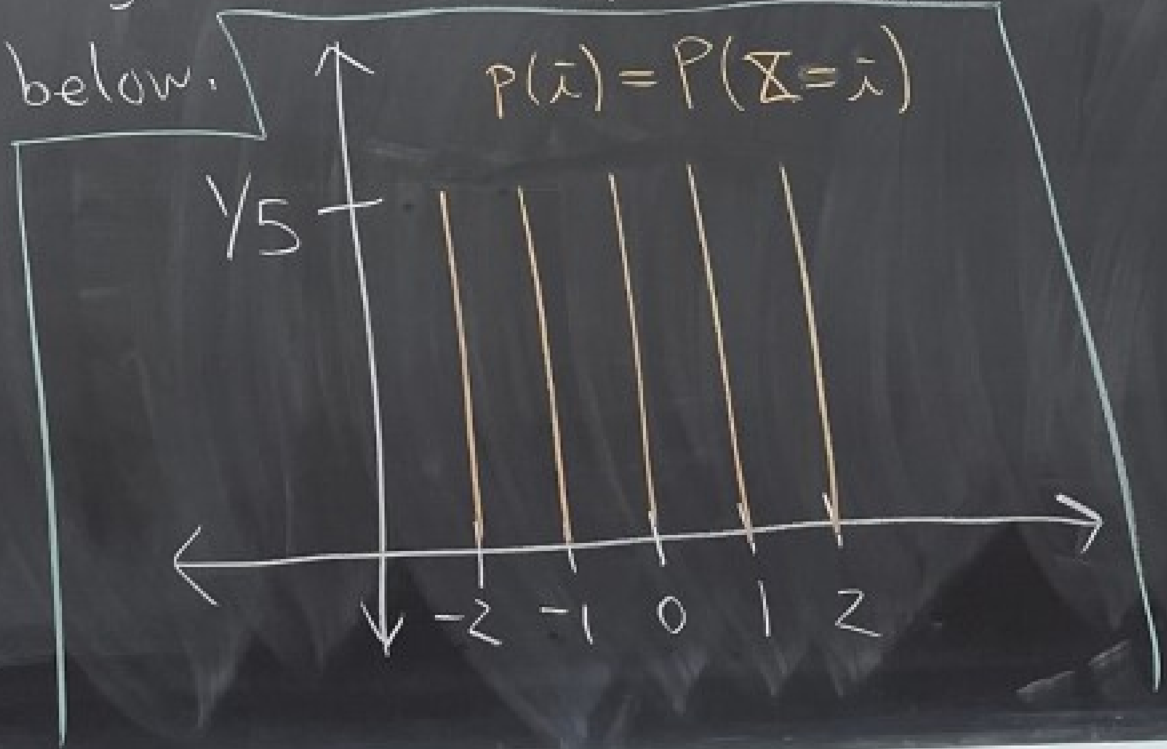
Let $f : \mathbb{R} \to \mathbb{R}$ be any function.

input for f output for f

Then the expected valve of $f(X) = f \circ X$ is

$$E\left[f(X)\right] = \sum_i f(x_i) \cdot p(x_i)$$

Ex: Suppose X is a discrete random variable with values $-2, -1, 0, 1, 2$ and probability function $p(i) = P(X = i)$ drawn below.



$$E[X] = (-2)\left(\tfrac{1}{5}\right) + (-1)\left(\tfrac{1}{5}\right) + (0)\left(\tfrac{1}{5}\right) + (1)\left(\tfrac{1}{5}\right) + (2)\left(\tfrac{1}{5}\right)$$

$$= 0$$

$$E[X^2] = (-2)^2\left(\tfrac{1}{5}\right) + (-1)^2\left(\tfrac{1}{5}\right) + (0)^2\left(\tfrac{1}{5}\right) + (1)^2\left(\tfrac{1}{5}\right) + (2)^2\left(\tfrac{1}{5}\right) = \frac{10}{5} = 2$$

$f(x) = x^2$

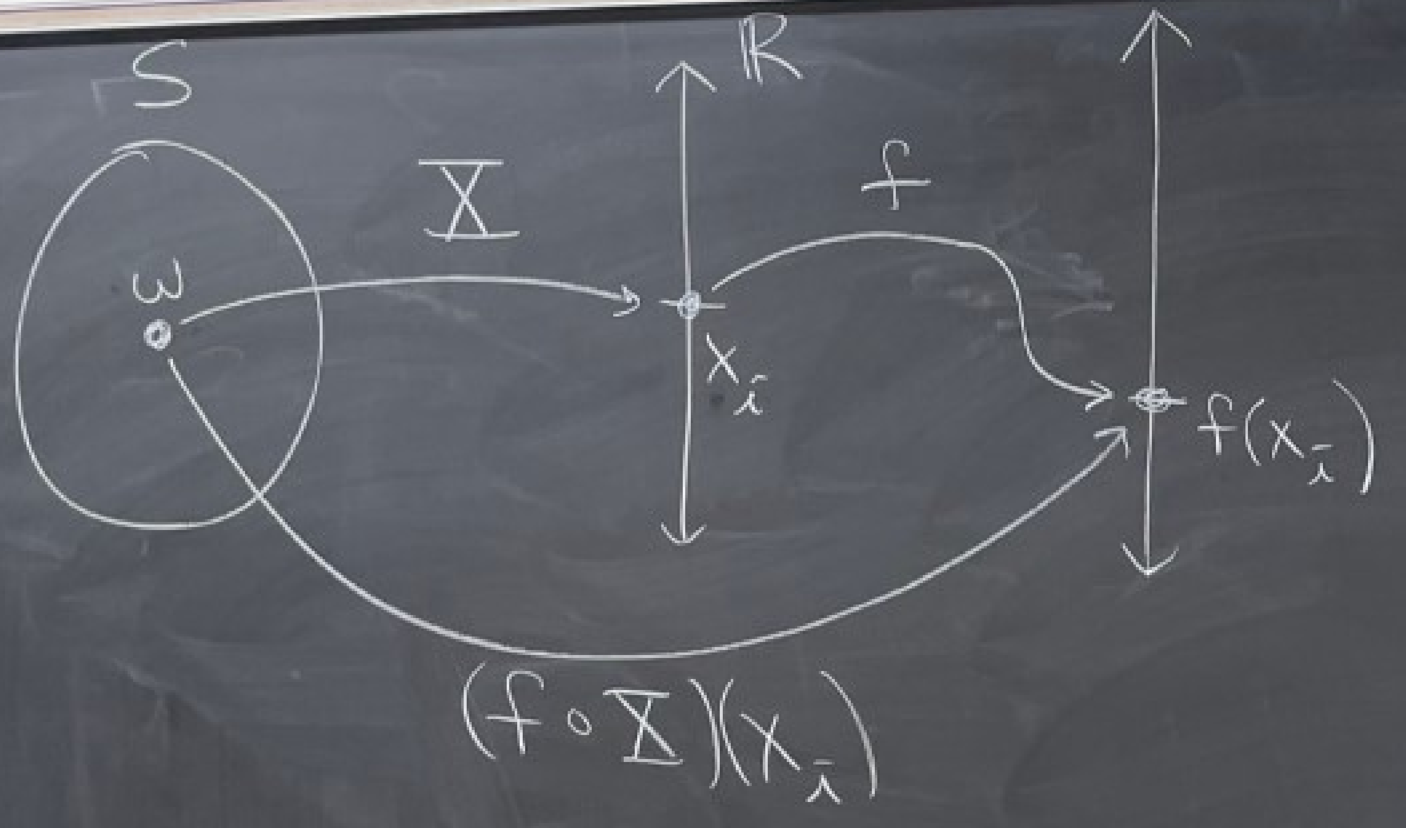
$$E[-X+1] = (-(-2)+1)\left(\frac{1}{5}\right) + (-(-1)+1)\left(\frac{1}{5}\right)$$

$$+ (-(0)+1)\left(\frac{1}{5}\right) + (-(1)+1)\left(\frac{1}{5}\right)$$

$$+ (-(2)+1)\left(\frac{1}{5}\right)$$

$$= \frac{3}{5} + \frac{2}{5} + \frac{1}{5} + 0 - \frac{1}{5} = 1$$

$$\boxed{f(x) = -x+1}$$



<u>Theorem:</u> Let X be a discrete random variable and functions $f_1, f_2, \ldots, f_n$ functions from $\mathbb{R}$ to $\mathbb{R}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are real numbers.

Then,

$$E\left[\alpha_1 f_1(X) + \alpha_2 f_2(X) + \cdots + \alpha_n f_n(X)\right]$$

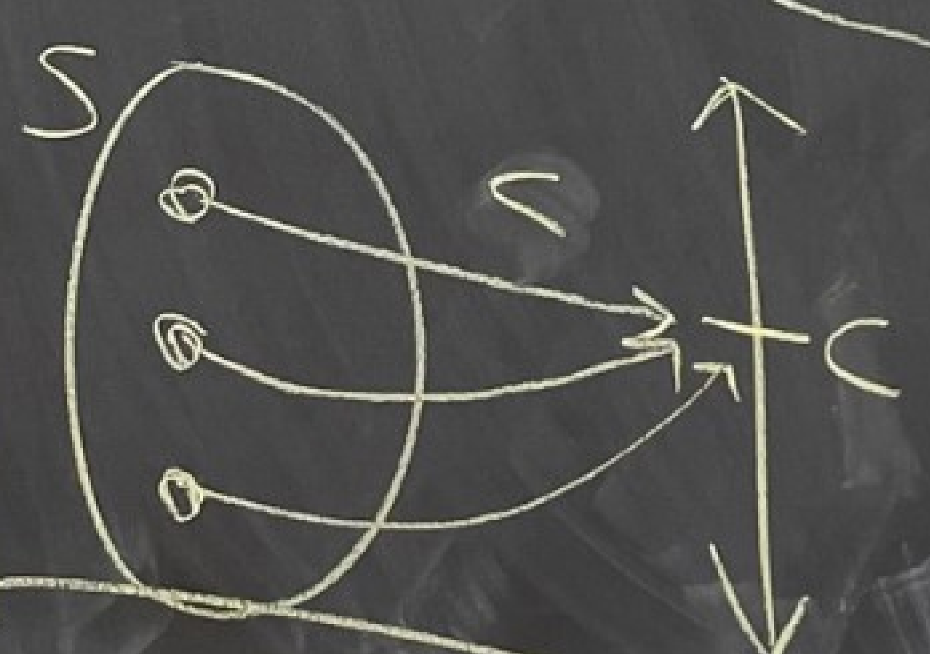
$$= \alpha_1 E\left[f_1(X)\right] + \alpha_2 E\left[f_2(X)\right] + \cdots + \alpha_n E\left[f_n(X)\right]$$

Ex: Suppose X has $E[X] = 0$ like in the last example, then

$$E[-X+1] = -E[X] + E[1]$$

$$= -0 + 1 = 1$$

$E[c] = c$



$\mu = E[X]$ doesn't capture all

$\boxed{\text{mu}}$

the details of X.

Check out these two probability functions with the same μ

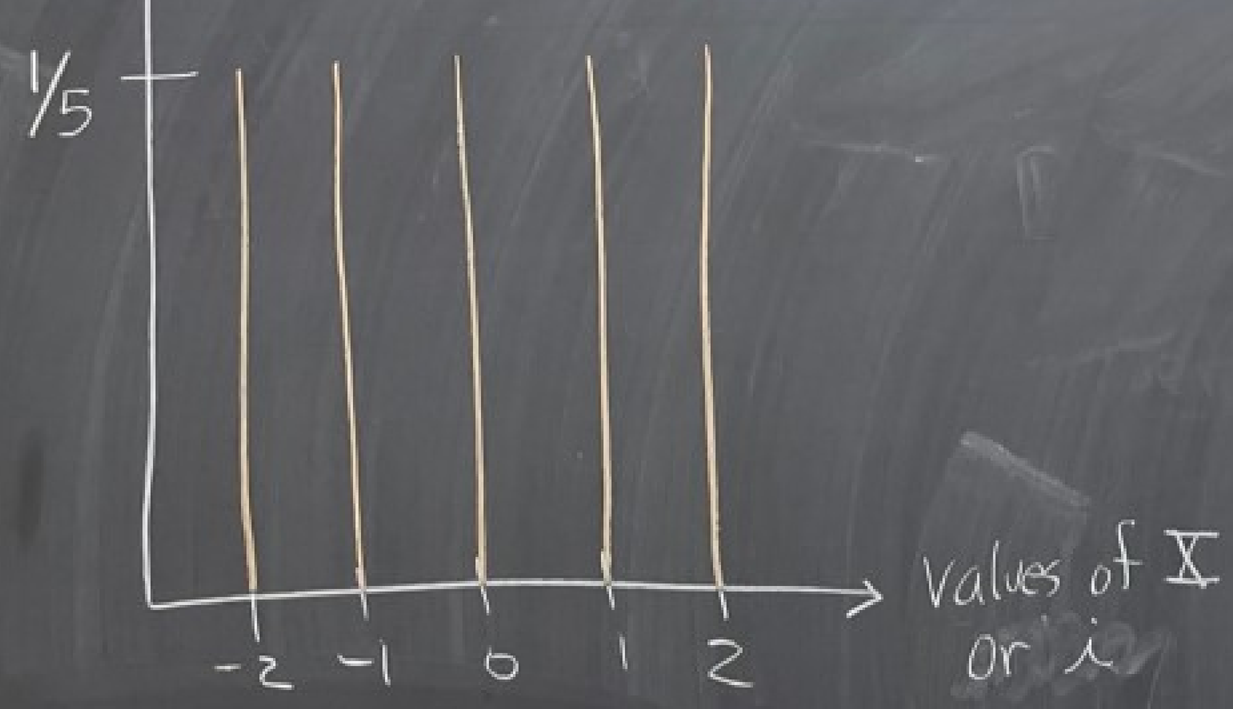
but whose values are "spread out" very differently.

E

W

Ex:

$p(i) = P(X = i)$

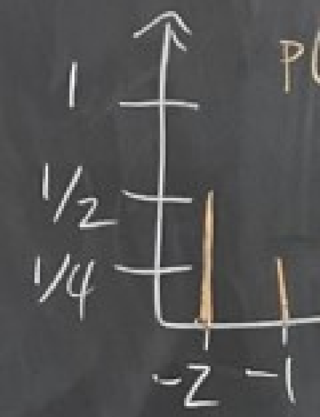


We saw $E[X] = 0$

$p(i) = P(X = i)$

$$E[X] = (-2)\left(\frac{1}{400}\right) + (-1)\left(\frac{1}{400}\right) + (0)\left(\frac{99}{100}\right) + (1)\left(\frac{1}{400}\right) + (2)\left(\frac{1}{400}\right)$$

$$= 0$$



We want a number that measures how X fluctuates from its $\mu = E[X]$.

You might try

$$\underbrace{E[\,|X-\mu|\,]}_{\text{weighted average of how far } X \text{ is from } \mu}$$

This is hard to work with because of the absolute value.

So instead we look at

$$E\left[(X-\mu)^2\right].$$

Next time we will define the variance of X to be

$$E\left[(X-\mu)^2\right] \text{ where}$$

$$\mu = E[X].$$

HW 3

9. Box with 7 red and 13 blue balls.

7 R
13 B

Take 2 balls out of the box and discard them without looking. Then draw a 3rd ball and you notice its red. What's the prob. the two discarded balls are blue?

Let BB, BR, RR be the events that the first two balls where blue/blue, blue/red, or red/red.

Let R be the event the 3rd ball is red.

We want $P(BB|R)$

$$P(BB|R) = \frac{P(BB \cap R)}{P(R)} = \frac{P(R|BB) \cdot P(BB)}{P(R)}$$

$$P(BB \cap R) = P(R|BB) \cdot P(BB)$$

$$\boxed{P(R|BB)} = \frac{P(R \cap BB)}{P(BB)} = \boxed{\frac{P(BB \cap R)}{P(BB)}}$$

7 $\cancel{8}$ $\cancel{7}$ R

11 $\cancel{12}$ $\cancel{13}$ B

$$P(BB) = \frac{\binom{13}{2}}{\binom{20}{2}} = \frac{\frac{13\cdot 12}{2}}{\frac{20\cdot 19}{2}} = \boxed{\frac{78}{190}}$$

$$\boxed{\binom{13}{2} = \frac{13!}{2!\,11!} = \frac{13\cdot 12\cdot \cancel{11!}}{2!\,\cancel{11!}} = \frac{13\cdot 12}{2}}$$

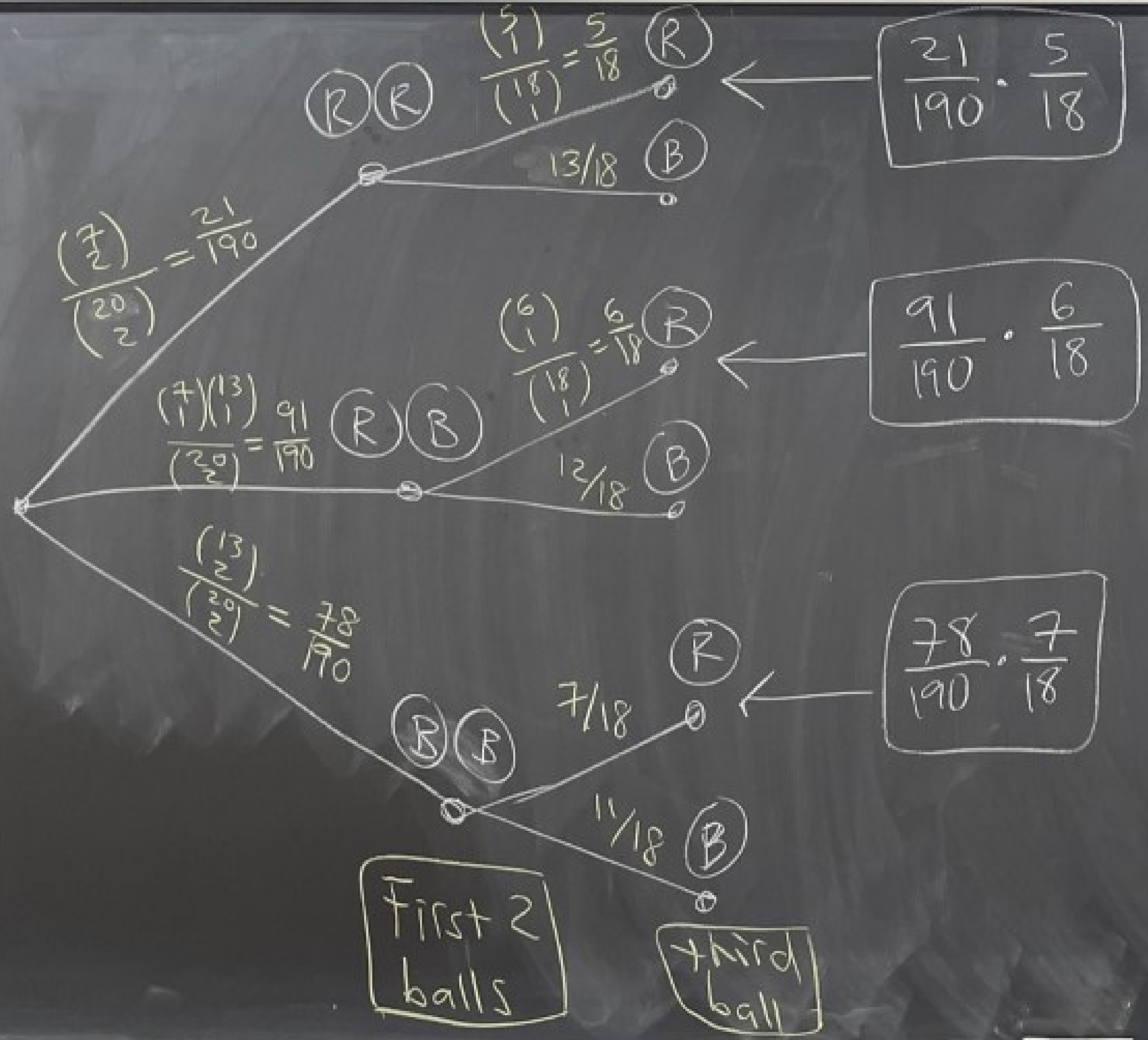
$$\boxed{\binom{n}{2} = \frac{n(n-1)}{2}}$$

$$P(R|BB) = \frac{\binom{7}{1}}{\binom{18}{1}} = \boxed{\frac{7}{18}}$$

after 2 blues taken out have

7 R

11 B



$$\frac{P(R|BB)\cdot P(BB)}{P(R)} = P(BB|R)$$

$$P(R) = \frac{21}{190}\cdot\frac{5}{18} + \frac{91}{190}\cdot\frac{6}{18} + \frac{78}{190}\cdot\frac{7}{18}$$

$$= \frac{7}{20}$$

Answer

$$P(BB|R) = \frac{P(R|BB)\;P(BB)}{P(R)} = \frac{(7/18)(78/190)}{7/20}$$

$$= \boxed{26/57} \approx \boxed{0.456}$$

Ex: Suppose you continually roll two 4-sided dice. You don't stop until the sum of the dice is either 3 or 7. What's the probability the sum of dice is 7 before the sum is 3?

Let A be the event that the sum is 7.
Let B be the event that the sum is 3.

$$P(A) = \frac{2}{4^2} = \frac{1}{8}$$

$$P(B) = \frac{2}{4^2} = \frac{1}{8}$$

$$A = \{(3,4),(4,3)\}$$

$$B = \{(2,1),(1,2)\}$$

The chances that A occurs before B is $\frac{P(A)}{P(A)+P(B)} = \frac{1/8}{1/8 + 1/8} =$

HW 4

(3) Weighted 4-sided die

roll	probability	
1	2/8	← win \$2
2	1/8	← win \$2
3	3/8	← lose \$1
4	2/8	← lose \$1

X = amount won or lost on one roll of die

$$E[X] = (\$2)\left(\frac{2}{8}+\frac{1}{8}\right) + (-\$1)\left(\frac{3}{8}+\frac{2}{8}\right)$$

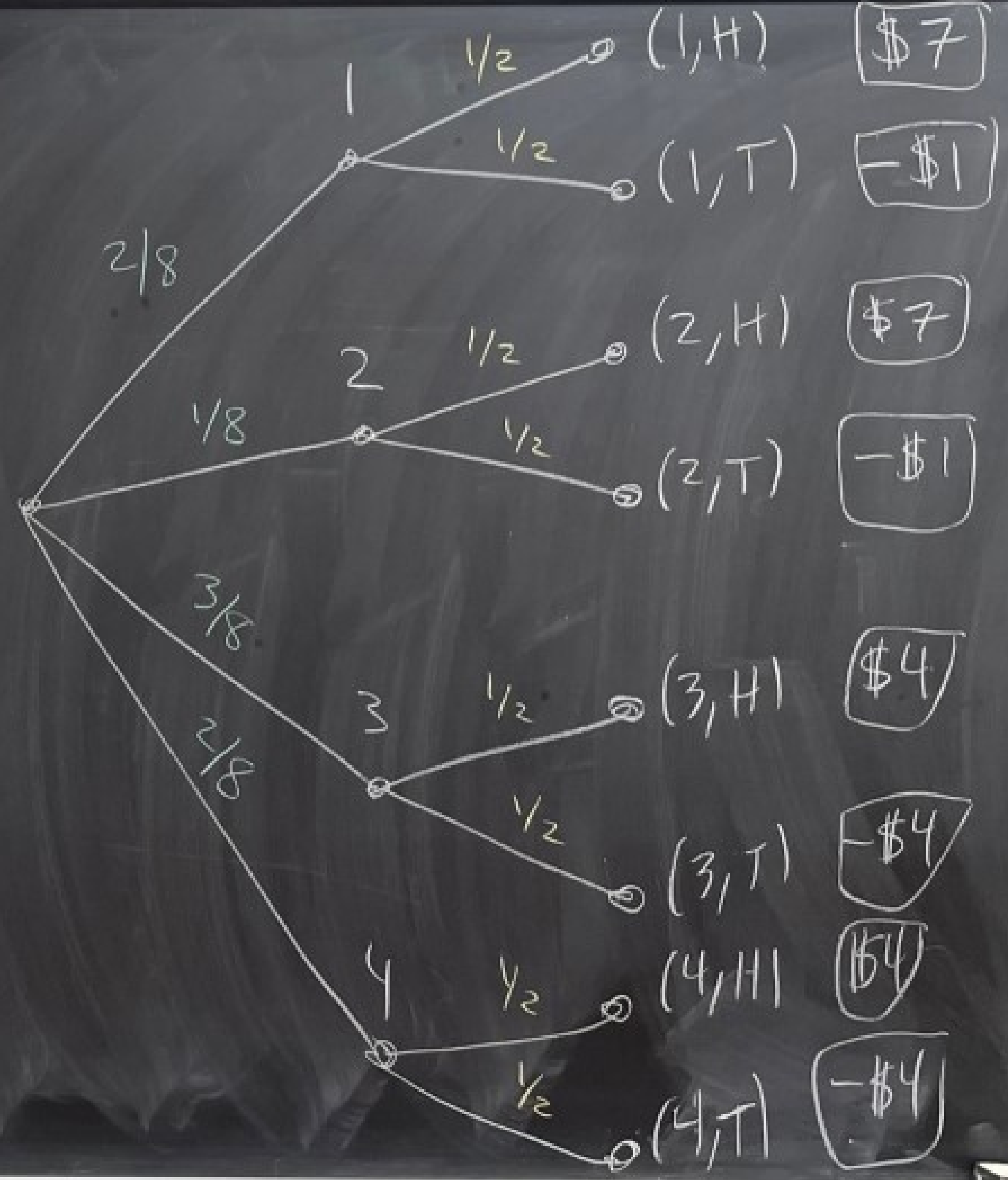
$$= \$\left(\frac{6}{8}-\frac{5}{8}\right) = \boxed{\$\frac{1}{8}} = \boxed{\$0.125}$$

What if you rolled the die from Hw 4-#3 and then flipped a normal coin.

You win $5 for heads
You lose $3 for tails
You win $2 for 1,2 on die
You lose $1 for 3,4 on die

Let X = amount won or lost.

Find $E[X]$.



$$E[X] = (\$7)\left[\frac{2}{8}\cdot\frac{1}{2} + \frac{1}{8}\cdot\frac{1}{2}\right] + (\$4)\left[\frac{3}{8}\cdot\frac{1}{2} + \frac{2}{8}\cdot\frac{1}{2}\right]$$

$$+ (-\$1)\left[\frac{2}{8}\cdot\frac{1}{2} + \frac{1}{8}\cdot\frac{1}{2}\right) + (-\$4)\left[\frac{3}{8}\cdot\frac{1}{2} + \frac{2}{8}\cdot\frac{1}{2}\right]$$

$$= (\$7)\left(\frac{3}{16}\right) + (\$4)\left[\frac{5}{16}\right] + (-\$1)\left(\frac{3}{16}\right) + (-\$4)\left(\frac{5}{16}\right)$$

$$= \$\,\frac{21+20-3-20}{16} = \$\,\frac{18}{16} = \$\left(1+\frac{1}{8}\right) = \boxed{\$1.125}$$

Def: Let X be a discrete random variable with values $x_1, x_2, x_3, x_4, \ldots$

Let p be the probability function for X.

Let $\mu = E[X]$.

The <u>variance</u> of X is $E[(X-\mu)^2]$

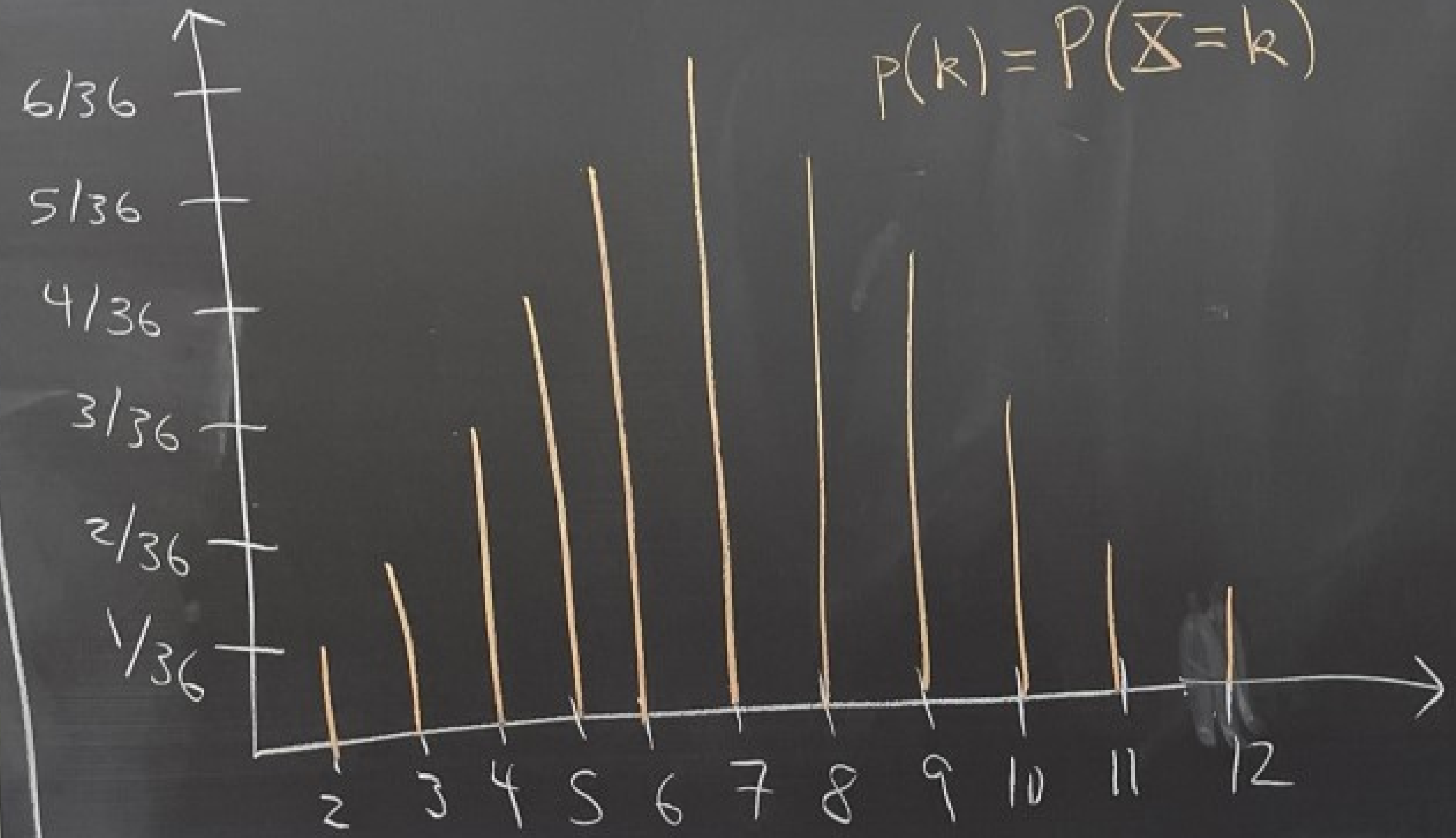
The <u>standard deviation</u> of X is $\sigma_X = \sigma = \sqrt{Var(X)}$

<u>Note:</u> By a previous thm, $\boxed{Var(X) = \sum_i (x_i - \mu)^2 p(x_i)}$

$$\boxed{E[f(X)] = \sum_i f(x_i)p(x_i)}$$

$$\boxed{\sigma \text{ sigma}}$$

Ex: Consider the experiment of rolling two 6-sided dice. Let X be the sum of the dice. Let p be the probability function for X.



$\mu = E[X] = 7$

(calculated previously)

$$\mathrm{Var}(X) = (2-7)^2 \cdot \frac{1}{36} + (3-7)^2 \cdot \frac{2}{36} + (4-7)^2 \cdot \frac{3}{36}$$

$$\boxed{\mathrm{Var}(X) = \sum_i (x_i - \mu)^2 p(x_i)}$$

$$\boxed{\mu = 7}$$

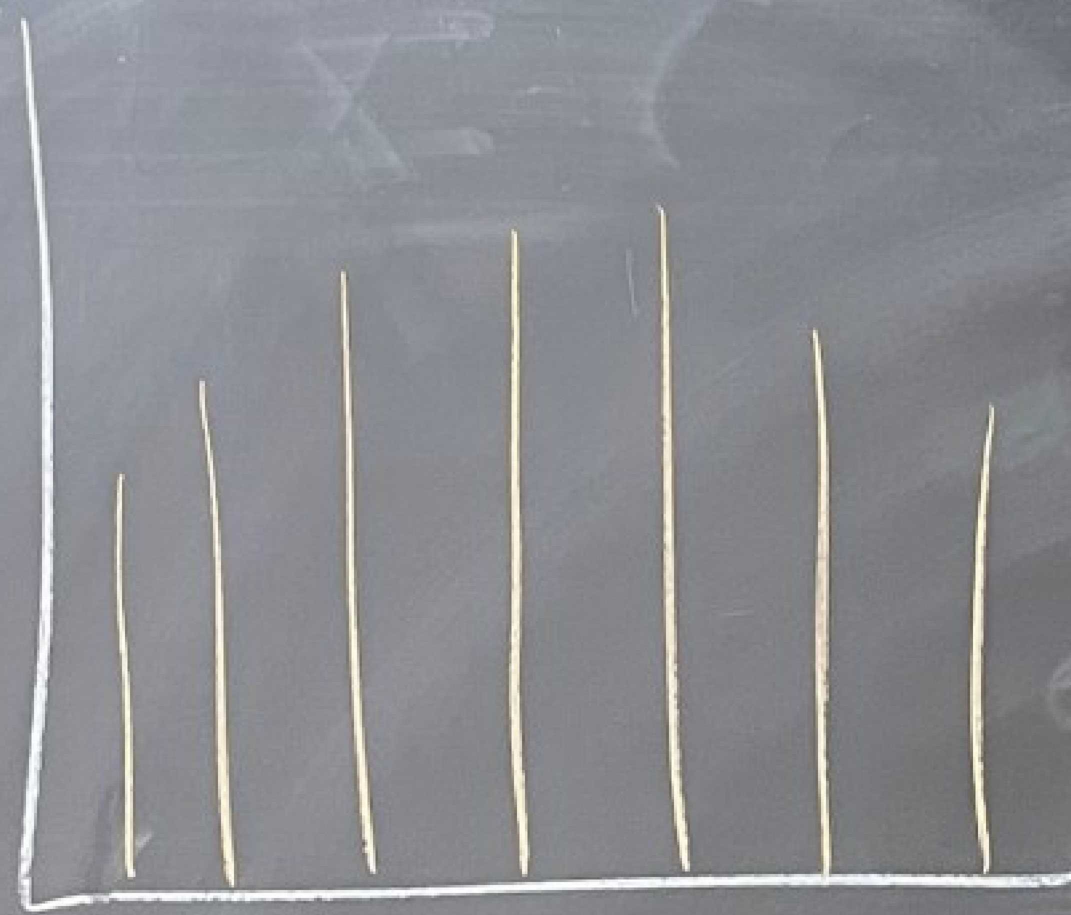
$$+ (5-7)^2 \cdot \frac{4}{36} + (6-7)^2 \cdot \frac{5}{36} + (7-7)^2 \cdot \frac{6}{36}$$

$$+ (8-7)^2 \cdot \frac{5}{36} + (9-7)^2 \cdot \frac{4}{36} + (10-7)^2 \cdot \frac{3}{36}$$

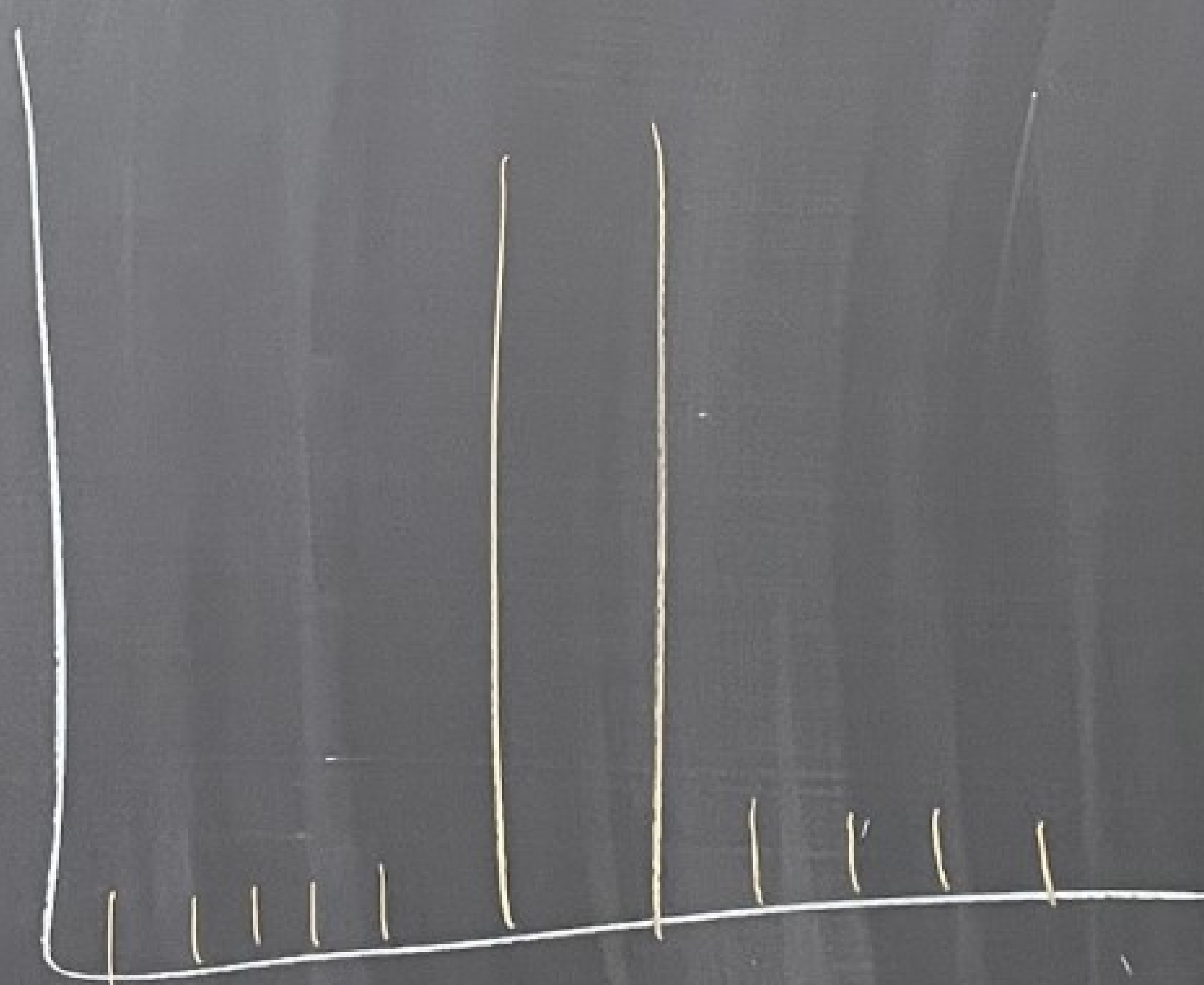
$$+ (11-7)^2 \cdot \frac{2}{36} + (12-7)^2 \cdot \frac{1}{36}$$

$$= \boxed{\frac{35}{6} \approx 5.83\ldots} = \mathrm{Var}(X)$$

$$\sigma = \sigma_X = \boxed{\sqrt{\mathrm{Var}(X)} = \sqrt{\frac{35}{6}} \approx 2.415}$$



bigger σ



smaller σ

X has values
$x_1, x_2, x_3, \dots$

$$E[X^2] = \sum_i x_i^2 \cdot p(x_i)$$

Thm: Let X be a discrete random variable. Let $\mu = E[X]$.
Then,

$$\mathrm{Var}(X) = E[X^2] - \left(E[X]\right)^2.$$
$$= E[X^2] - \mu^2$$

$$E[\alpha_1 f_1(X) + \dots + \alpha_n f_n(X)] = \alpha_1 E[f_1(X)] + \dots + \alpha_n E[f_n(X)]$$

Proof: $\mathrm{Var}(X) = E[(X-\mu)^2] = E[X^2 - 2\mu X + \mu^2]$
$$= E[X^2] - 2\mu E[X] + E[\mu^2]$$

$$E[c] = c$$

$$= E[X^2] - 2\mu\mu + \mu^2 = E[X^2] - \mu^2 \quad \blacksquare$$

We c

$\mu =$

Nev

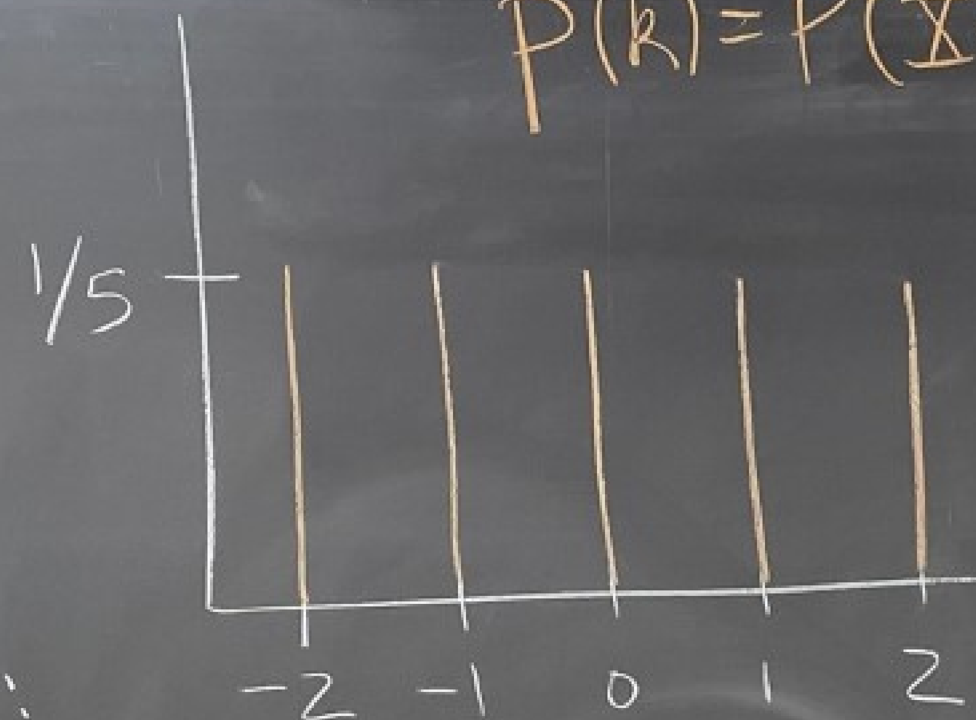
E[

Va

$\cdot\, p(x_i)$

Ex:

$P(k) = P(X = k)$



We calculated:

$\mu = E[X] = 0$

New way for Var(X):

$$E[X^2] = (-2)^2 \cdot \frac{1}{5} + (-1)^2 \cdot \frac{1}{5} + (0)^2 \cdot \frac{1}{5} + (1)^2 \cdot \frac{1}{5} + (2)^2 \cdot \frac{1}{5} = \frac{10}{5} = 2$$

$$\mathrm{Var}(X) = E(X^2) - \mu = 2 - 0 = 2$$

$$\sigma_X = \sqrt{2} \approx 1.414\ldots$$

Ex:

$p(k) = P(X = k)$

We calculated $\mu = E[X] = 0$

$$E[X^2] = (-2)^2 \cdot \frac{1}{400} + (-1)^2 \cdot \frac{1}{400} + 0^2 \cdot \frac{99}{100} + (1)^2 \frac{1}{400} + (2)^2 \cdot \frac{1}{400} = \frac{10}{400} = \frac{1}{40} \approx 0.025$$

$$\mathrm{Var}(X) = E[X^2] - \mu = \frac{1}{40} - 0 = \frac{1}{40} \approx 0.025$$

$$\sigma_X = \sqrt{\frac{1}{40}} \approx 0.158$$

Theorem: Let X be a binomial random variable with parameters n and p. Then, $Var(X) = np(1-p)$

$$\sigma_X = \sqrt{np(1-p)}$$

Theorem (Markov's inequality)

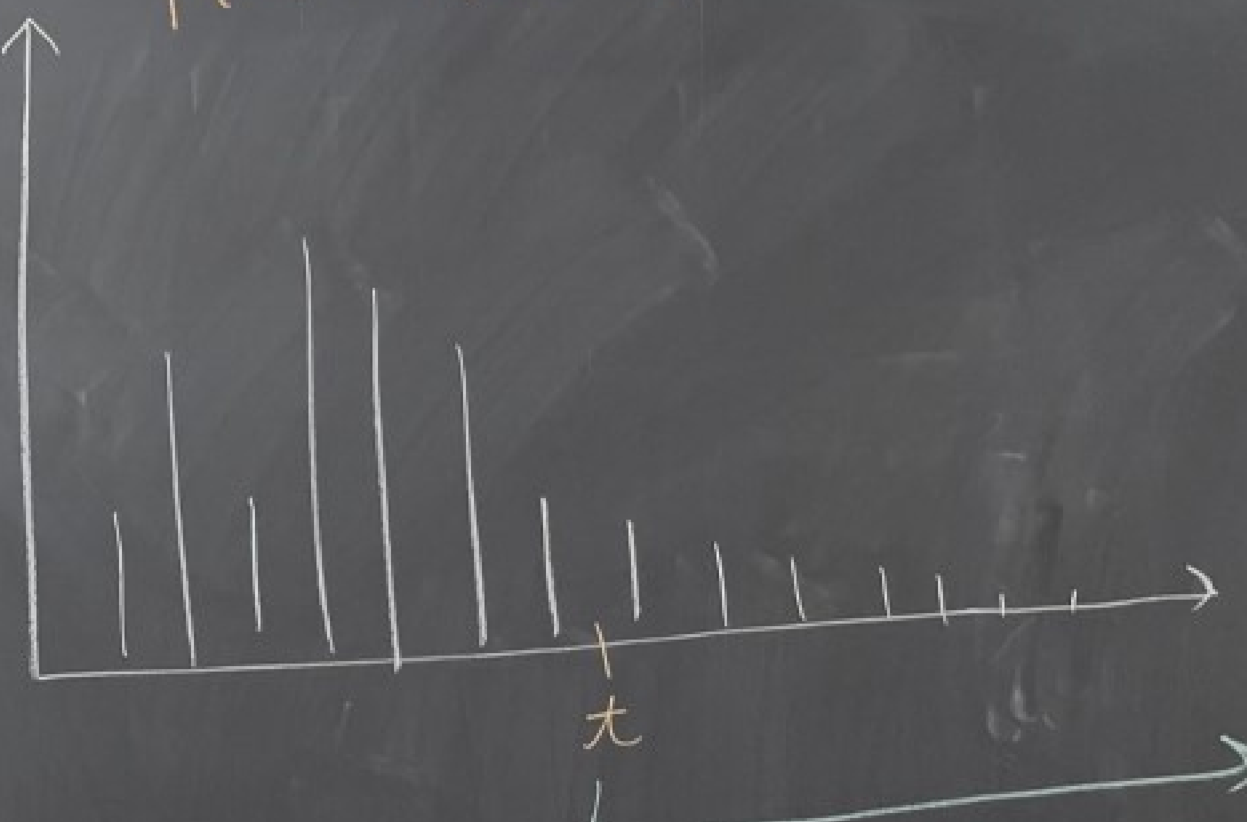
Let X be a non-negative discrete random variable with $\mu = E[X]$.

$\boxed{\text{non-negative means } X(\omega) \geq 0 \text{ for all } \omega \in S}$

Then for any real number $t > 0$ we have that

$$P(X \geq t) \leq \frac{\mu}{t}$$

$P(k) = P(X = k)$



Sum these heights to get $P(X \geq t)$

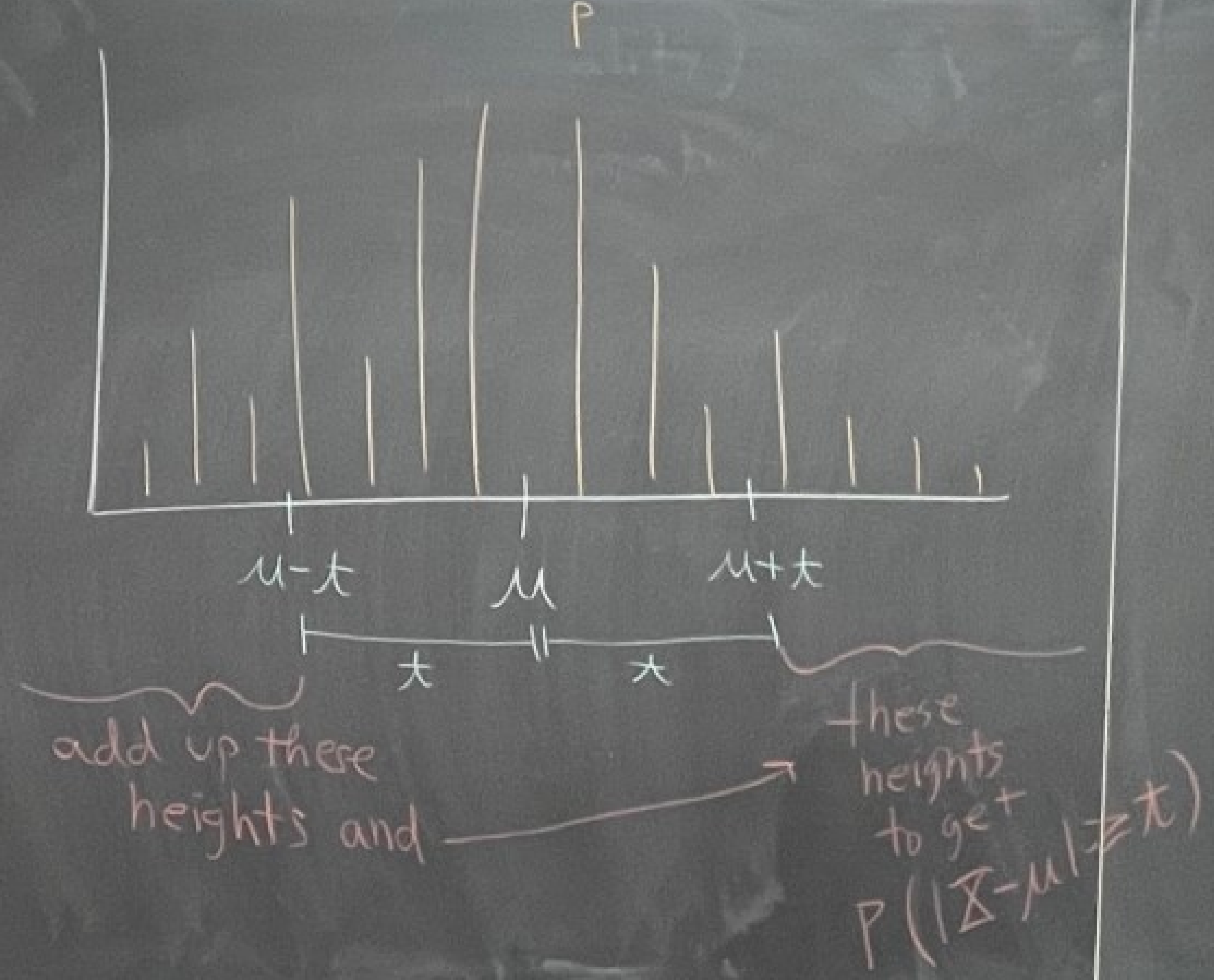
Theorem (Chebyshev's inequality)

Let X be a discrete random variable with $\mu = E[X]$ and standard deviation σ.

Then for any $t > 0$,

$$\underbrace{P(|X-\mu| \geq t)} \leq \frac{\sigma^2}{t^2}$$

means: $P(\{\omega \mid \omega \in S \text{ where } |X(\omega)-\mu| \geq t\})$



Plan

M	W
finish HW 7 (part on final)	stuff not on final
stuff not on final	Review
	FINAL 2:30-4:30

Study guide for final is on website

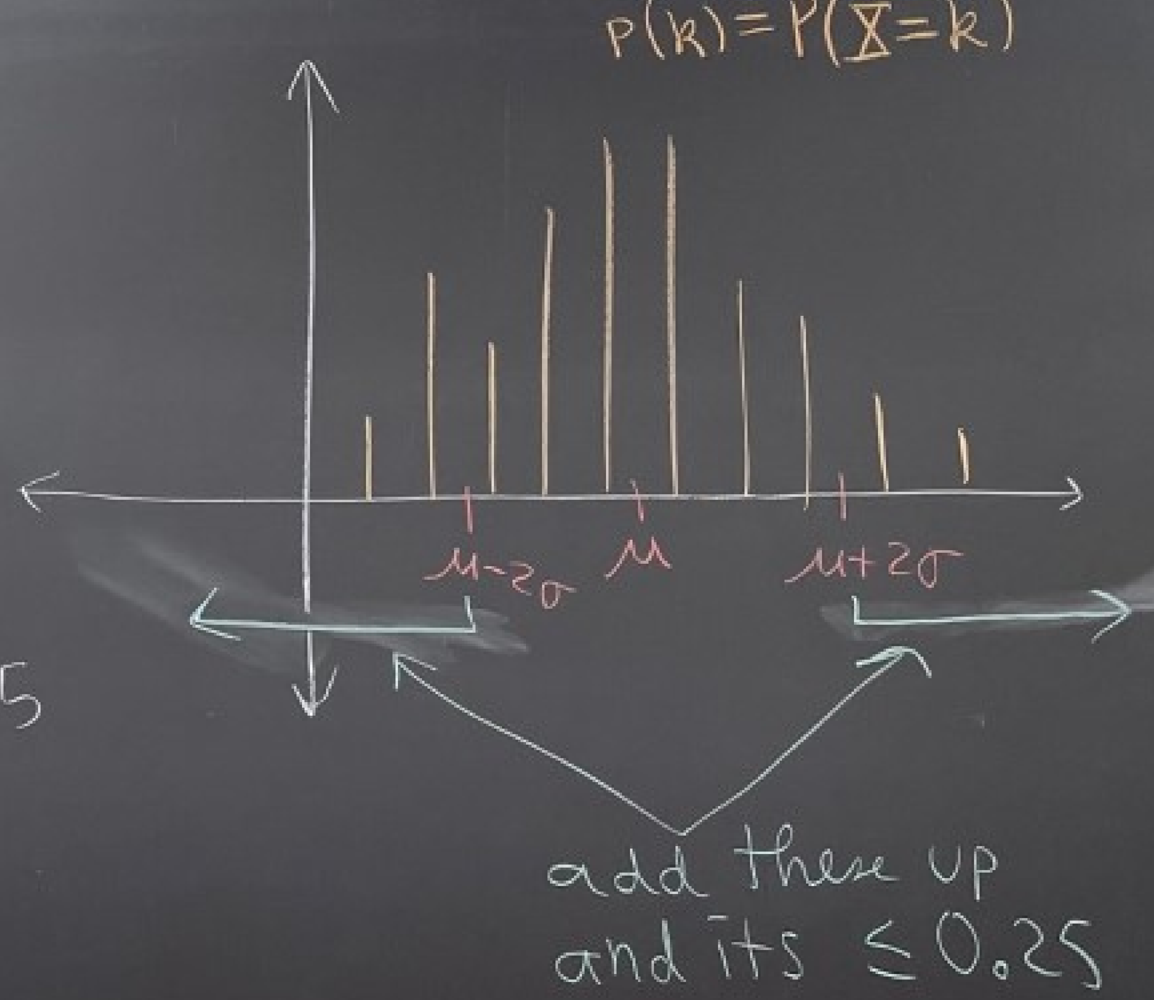
Last time

(Chebyshev's inequality)

$$P(|X-\mu| \geq t) \leq \frac{\sigma^2}{t^2}$$

$\boxed{\text{HW 6 \#5(b)}}$ Plug in $t = 2\sigma$

$$P(|X-\mu| \geq 2\sigma) \leq \frac{\sigma^2}{(2\sigma)^2} = \frac{\sigma^2}{4\sigma^2} = \frac{1}{4} \approx 0.25$$

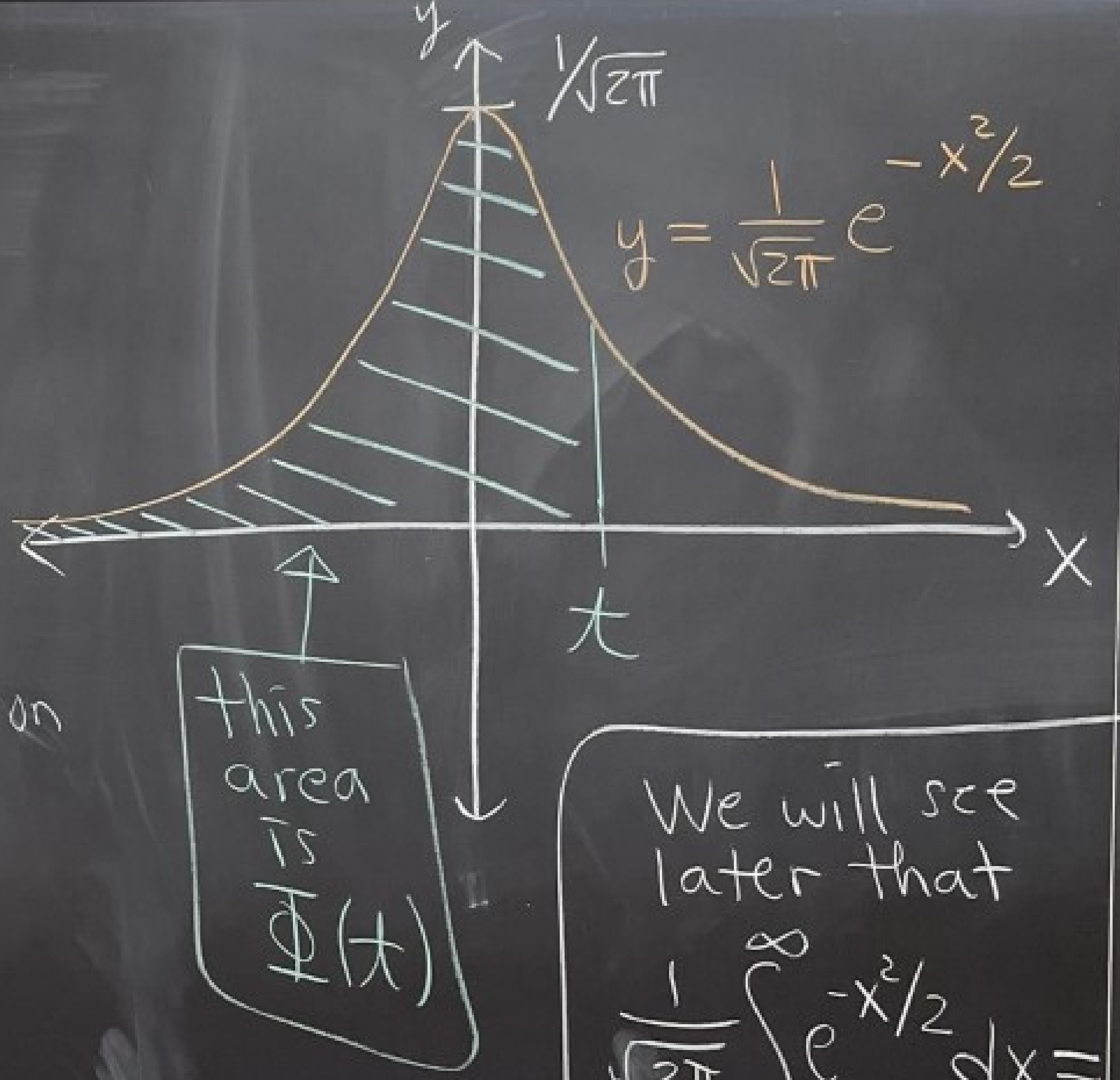


HW 7 Topic

Def: Let

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2}\, dx.$$

Φ is called the distribution function of the standard normal random variable.



We will see later that

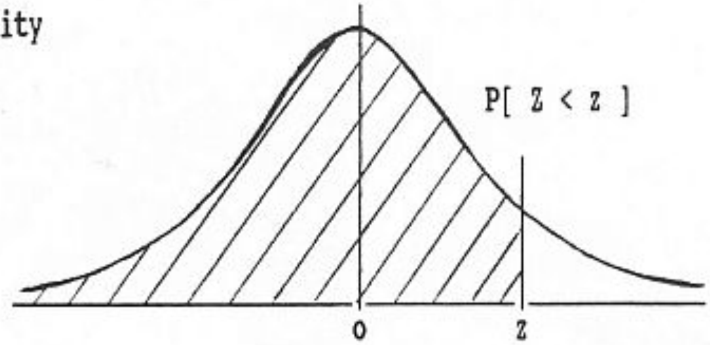
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \boxed{1}$$

1. Areas under the Normal Distribution

The table gives the cumulative probability up to the standardised normal value z

i.e.

$$P[\ Z < z\ ] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}Z^2)\ dZ$$



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5159	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7854
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8804	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9773	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9865	0.9868	0.9871	0.9874	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9924	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9980	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

z	3.00	3.10	3.20	3.30	3.40	3.50	3.60	3.70	3.80	3.90
P	0.9986	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000

Use table to calculate $\Phi(t)$ for $t \geq 0$

$$\Phi(2.25) \approx 0.9878$$

Z 0.05

⋮

$2.2 \longrightarrow \boxed{0.9878}$

$$\Phi(1.36) \approx 0.9131$$

$$\begin{array}{lcl} z & \cdots & 0.06 \\ \vdots & & \downarrow \\ 1.3 & \longrightarrow & \boxed{0.9131} \end{array}$$

If $t \geqslant 3$ we estimate

$$\Phi(t) \approx 1$$

$$\Phi(0.4) \approx 0.6554$$

$$\begin{array}{lcl} z & & 0.00 \\ \vdots & & \downarrow \\ 0.4 & \rightarrow & \boxed{0.6554} \end{array}$$

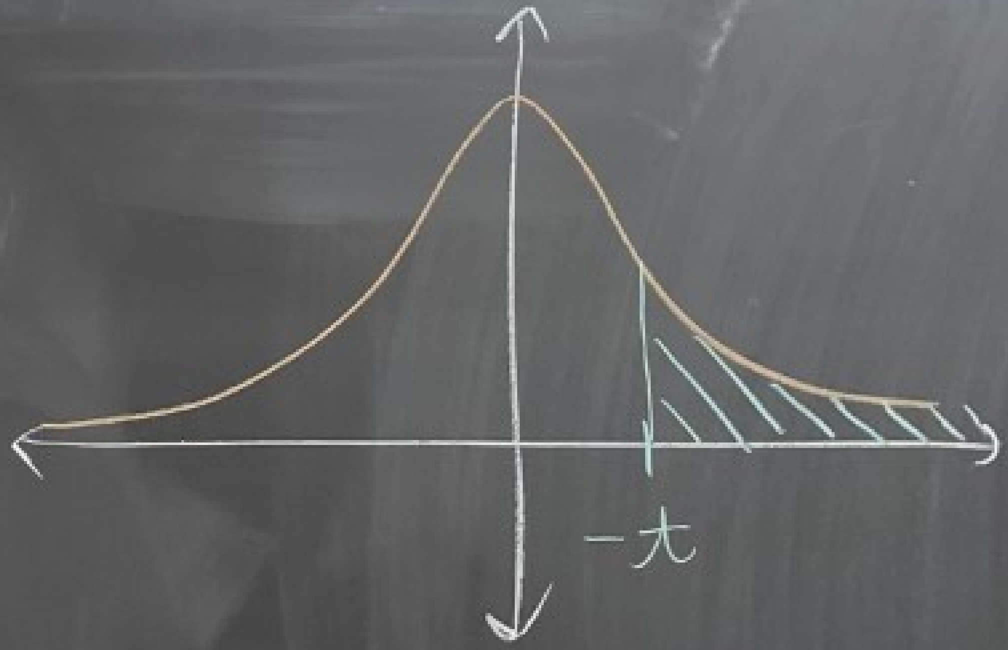
How to calculate $\Phi(t)$ when $t < 0$

want:

by symmetry

these are the same

$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is an even function

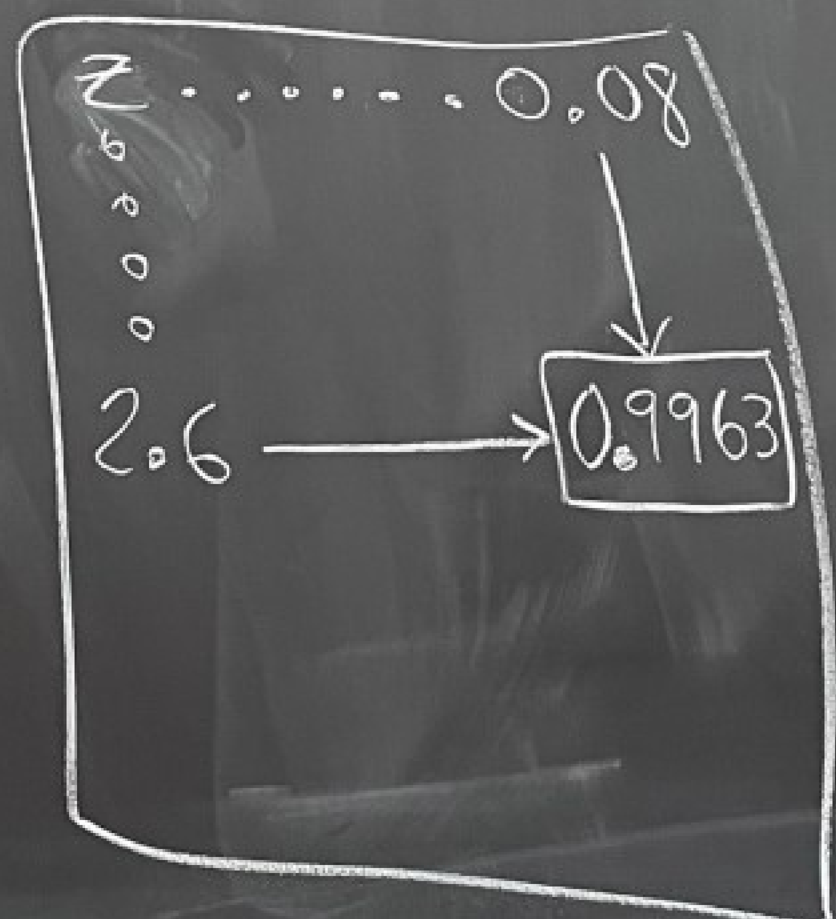


So, $\Phi(t) = 1 - \Phi(-t)$

for $t < 0$

Ex: $\Phi(-2.68) = 1 - \Phi(2.68)$

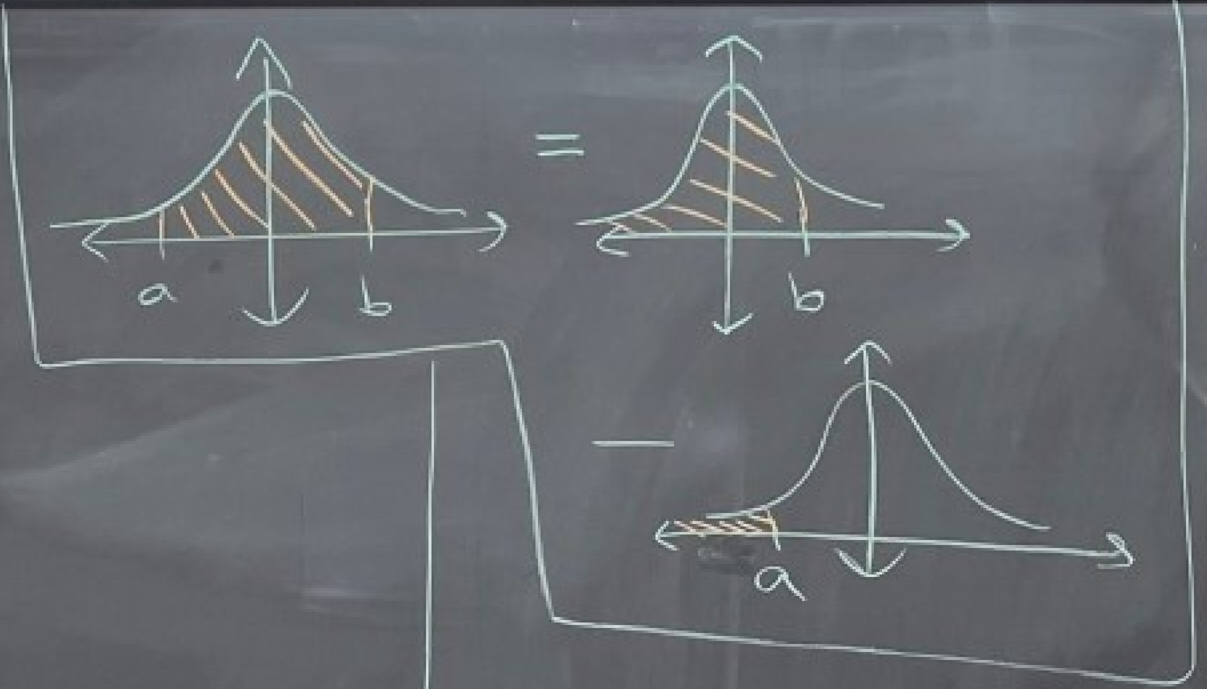
$$\approx 1 - 0.9963 \approx \boxed{0.0037}$$



Theorem: (DeMoivre–Laplace Theorem)

Let X be a binomial random variable with parameters n and p. Then for any real numbers a and b with $a<b$ we have

$$\lim_{n\to\infty} P\left(a \le \frac{X-np}{\sqrt{np(1-p)}} \le b\right) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-x^2/2}\,dx = \Phi(b)-\Phi(a)$$



Note: $E[X]=np$

$\sigma_X = \sqrt{np(1-p)}$

You can also do

$$\lim_{n\to\infty} P\left(\frac{X - np}{\sqrt{np(1-p)}} \le b\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{b} e^{-x^2/2}\,dx = \Phi(b)$$

"$a = -\infty$"

Ex: Suppose you toss a coin 10,000 times.
Let X be the number of heads that occur.

Recall, X is a binomial random variable with parameters $n = 10{,}000$ and $p = \frac{1}{2}$ ← probability of getting a head

Approximate the probability that $5000 \leq X \leq 5002$.
[That is, probabily you get between 5000 and 5002 heads].

$$\boxed{\begin{aligned} np &= (10{,}000)\left(\tfrac{1}{2}\right) = 5{,}000 \\ np(1-p) &= (10{,}000)\left(\tfrac{1}{2}\right)\left(1-\tfrac{1}{2}\right) = 2{,}500 \end{aligned}}$$

We want

$$P(5{,}000 \le X \le 5{,}002) = P\left(\frac{5{,}000-5{,}000}{\sqrt{2{,}500}} \le \overbrace{\frac{X-5{,}000}{\sqrt{2{,}500}}}^{\frac{X-np}{\sqrt{np(1-p)}}} \le \frac{5{,}002-5{,}000}{\sqrt{2{,}500}}\right)$$

$$= P\left(0 \le \frac{X-5{,}000}{\sqrt{2{,}500}} \le 0.04\right) \approx \Phi(0.04) - \Phi(0)$$

$\boxed{n = 10{,}000 \text{ is large}}$

$$\approx 0.5159 - 0.5 \approx \boxed{0.0159} \approx \boxed{1.59\,\%}$$

__Ex:__ Suppose you flip a coin 40 times.
Let X be the number of heads that occur.
Approximate $\underbrace{P(X=20)}_{\text{means probability we get exactly 20 heads in 40 flips}}$.

X is a binomial random variable with $n=40$ and $p=\frac{1}{2}$

$E(X)=np=40\left(\frac{1}{2}\right)=20$ and $Var(X)=np(1-p)=40\left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right)=10$

We have

$$P(X=20) = P(19.5 \le X \le 20.5)$$

X can be
$0, 1, 2, 3, \ldots, 40$
all integers

$$= P\left(\frac{19.5-20}{\sqrt{10}} \le \underbrace{\frac{X-20}{\sqrt{10}}}_{\dfrac{X-np}{\sqrt{np(1-p)}}} \le \frac{20.5-20}{\sqrt{10}}\right)$$

$$\approx P\left(-0.16 \le \frac{\bar{X}-20}{\sqrt{10}} \le 0.16\right)$$

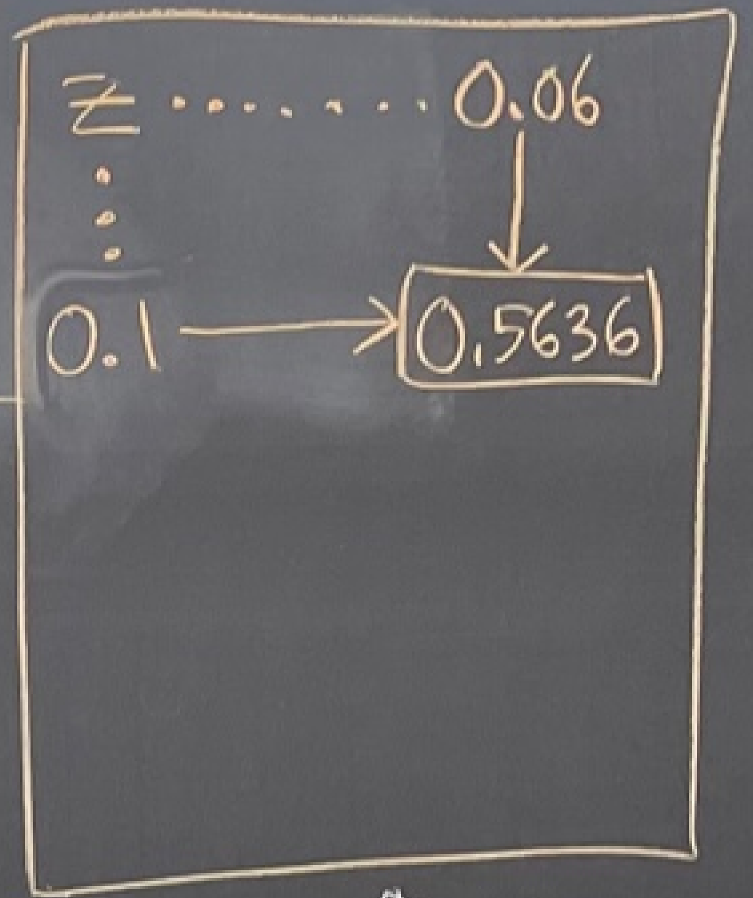
$$\approx \Phi(0.16) - \Phi(-0.16)$$

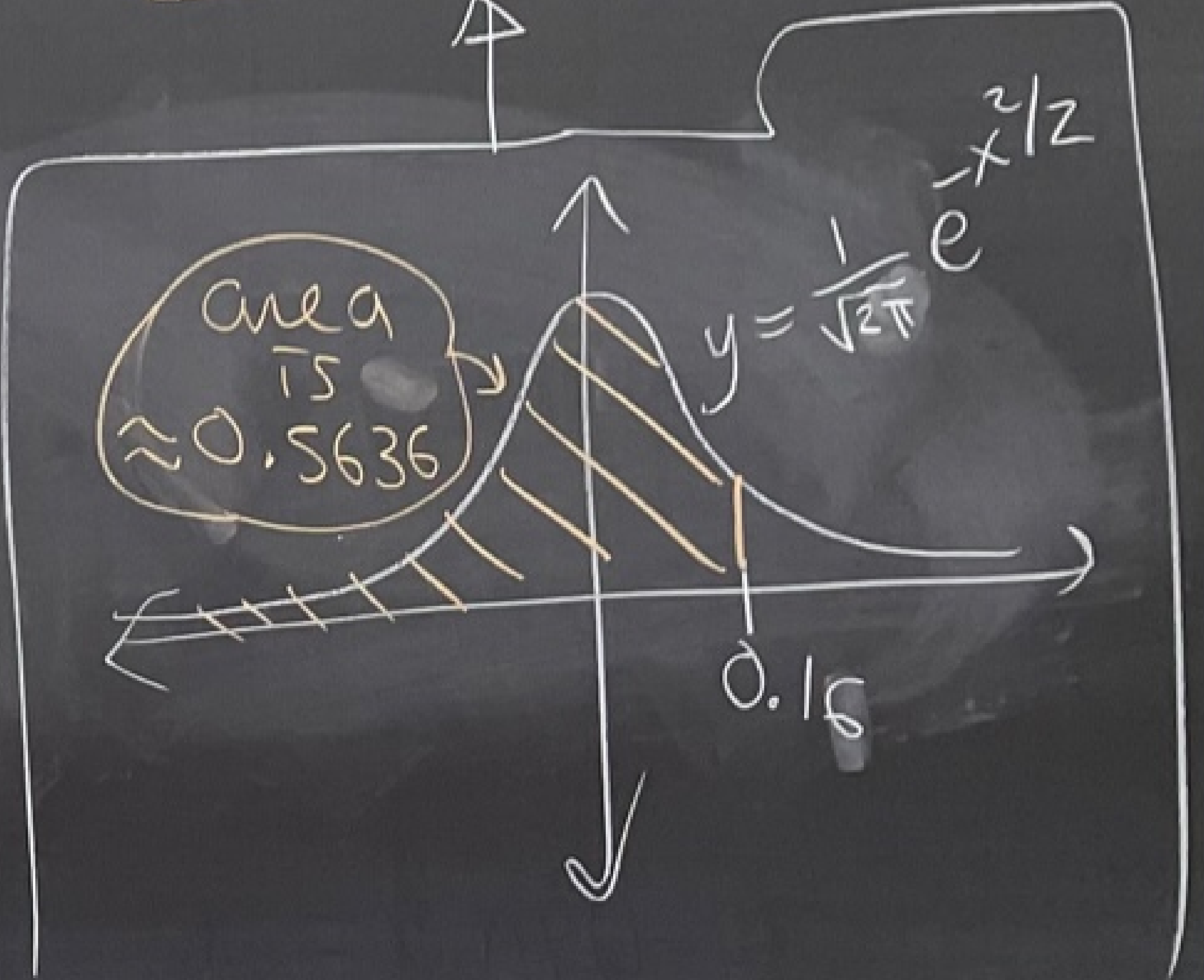
$$\approx \Phi(0.16) - \left[1 - \Phi(0.16)\right]$$

$$\Phi(-x) = 1 - \Phi(x)$$

$$\approx -1 + 2\,\Phi(0.16) \approx -1 + 2(0.5636)$$

$$\approx \boxed{0.1272} \approx 12.72\%$$





How good of an approximation is this?

The exact probability is

$$P(X=20) = \binom{40}{20}\cdot\left(\frac{1}{2}\right)^{20}\left(1-\frac{1}{2}\right)^{40-20} = \binom{40}{20}\cdot\left(\frac{1}{2}\right)^{40}$$

$$= \frac{137{,}846{,}528{,}820}{1{,}099{,}511{,}627{,}776}$$

$$\approx \boxed{0.125371}$$

$$\approx 12.537\%$$

$$\boxed{P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}}$$

binomial random variable
parameters n and p.

$n=40,\ p=1/2$ here

What if different range?

$$P(19.9 \le \bar{X} \le 20.1) = P\left(\frac{19.9-20}{\sqrt{10}} \le \frac{\bar{X}-20}{\sqrt{10}} \le \frac{20.1-20}{\sqrt{10}}\right)$$

$$\approx P\left(\frac{-0.1}{\sqrt{10}} \le \frac{\bar{X}-20}{\sqrt{10}} \le \frac{0.1}{\sqrt{10}}\right)$$

$$\approx P\left(-0.0316 \le \frac{\bar{X}-20}{\sqrt{10}} \le 0.0316\right)$$

$$\approx \Phi(0.0316) - \Phi(-0.0316)$$

$$\approx \Phi(0.032) - [1 - \Phi(0.032)]$$

$$\approx -1 + 2 \cdot \Phi(0.03)$$

$$\approx -1 + 2(0.5120) \approx \boxed{0.024} \approx 2.4\%$$

way off

Not on final – HW 8 Topic

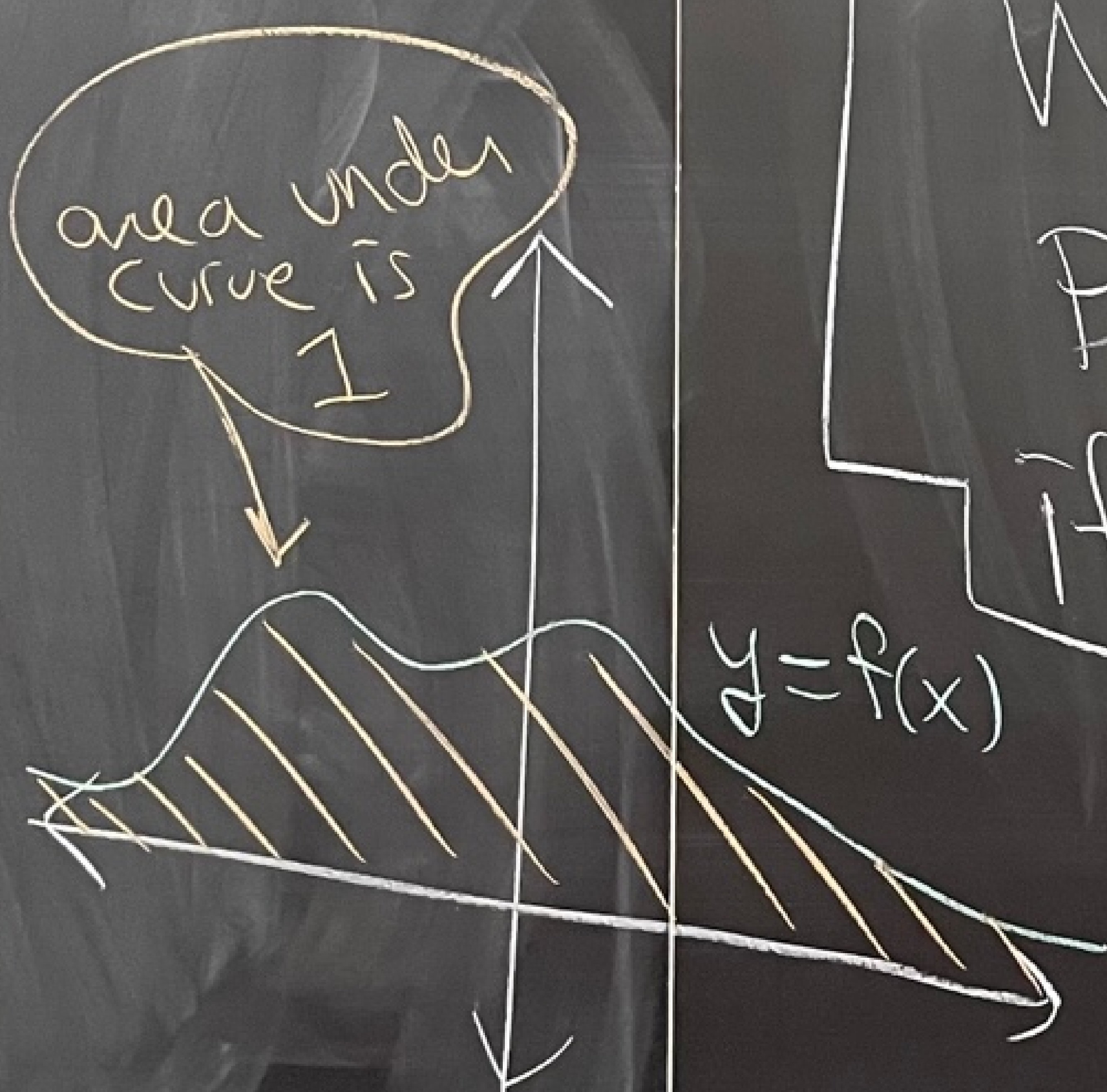
Continuous random Variables

<u>Def:</u> Let $f : \mathbb{R} \to \mathbb{R}$.

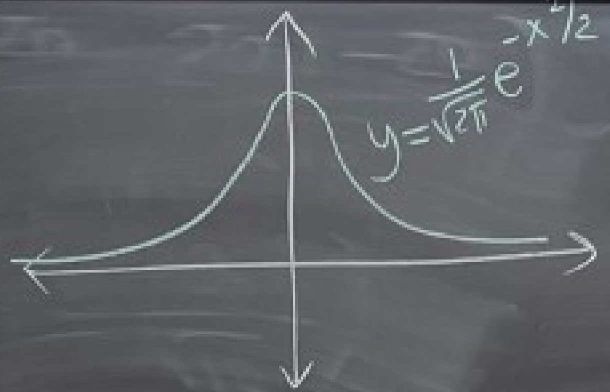
We say that f is a <u>probability density function</u> (p.d.f.)

if

1. $f(x) \geq 0$ for all x
2. $\int_{-\infty}^{\infty} f(x)\,dx$ exists and $\int_{-\infty}^{\infty} f(x)\,dx = 1$



Ex: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is a p.d.f.



① $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} > 0$ for all x

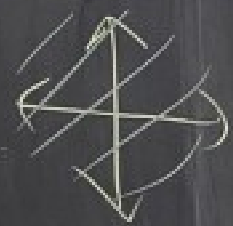
② Let $I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$

Then,

$$I^2 = \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx\right)\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy\right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\, dx\, dy$$

We are integrating over the whole xy-plane.



polar coordinates

$r^2 = x^2 + y^2$

$dx\,dy = r\,dr\,d\theta$

$0 \le r < \infty$

$0 \le \theta < 2\pi$

$\frac{1}{2\pi}$

$=$

$=$

$=$

over
e.
coordinates
x^2+y^2
$r\,dr\,d\theta$

$$= \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} e^{-r^2/2}\, r\, d\theta\, dr$$

$$= \frac{1}{2\pi}\int_0^\infty \left[r\cdot e^{-\frac{r^2}{2}}\cdot \theta \Big|_{\theta=0}^{2\pi} \right] dr$$

$$= \frac{1}{2\pi}\int_0^\infty 2\pi\cdot r\cdot e^{-r^2/2}\, dr$$

$$= \lim_{t\to\infty}\int_0^t r e^{-r^2/2}\, dr = \lim_{t\to\infty}\left[-e^{-r^2/2}\right]_0^t$$

$$= \lim_{t\to\infty}\left[\underset{\downarrow \atop 0}{-e^{-t^2/2}} - \left(-e^{0}\right)\right]$$

$$= 0 + 1 = 1.$$

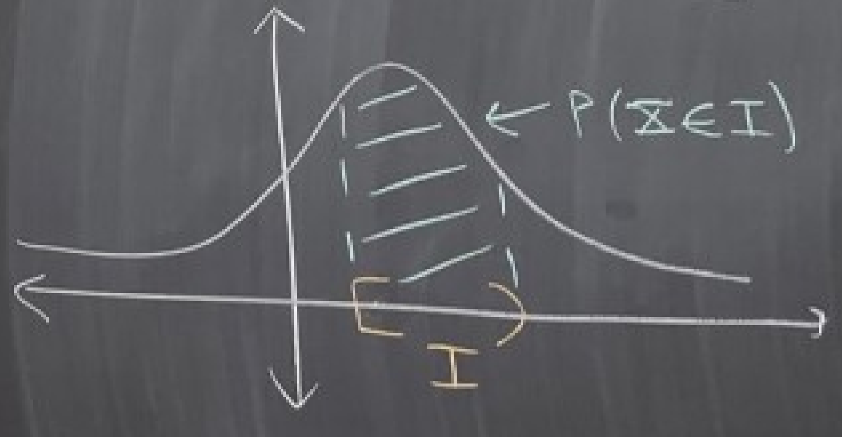
$$I^2 = 1$$

So, $I = 1$.

More stuff not on test
(HW 8)

Def: Let X be a random variable. We say that X is a <u>continuous random variable</u> if there exists a probability density function f where for any interval I in the real numbers we have

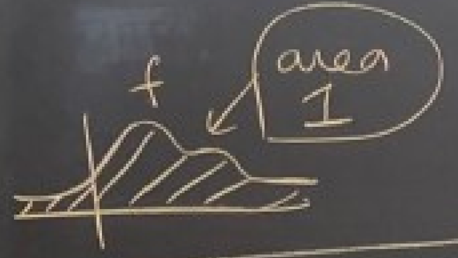
$$P(X \in I) = \int_I f(x)\,dx$$



$f : \mathbb{R} \to \mathbb{R}$

$f(x) \geq 0$

$\int_{-\infty}^{\infty} f(x)\,dx = 1$



So in particular

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$

$$P(a \le X) = \int_a^\infty f(x)\,dx$$

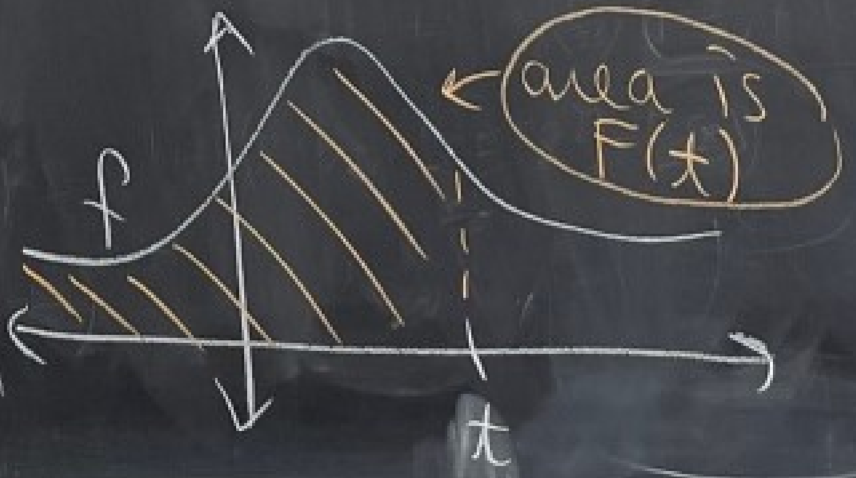
$$P(X \le b) = \int_{-\infty}^b f(x)\,dx$$

$$P(-\infty < X < \infty) = \int_{-\infty}^\infty f(x)\,dx = 1$$

The function f is called the <u>probability density function</u> (pdf) <u>of X</u>.

The <u>cumulative distribution function of X</u> (cdf) F is defined as

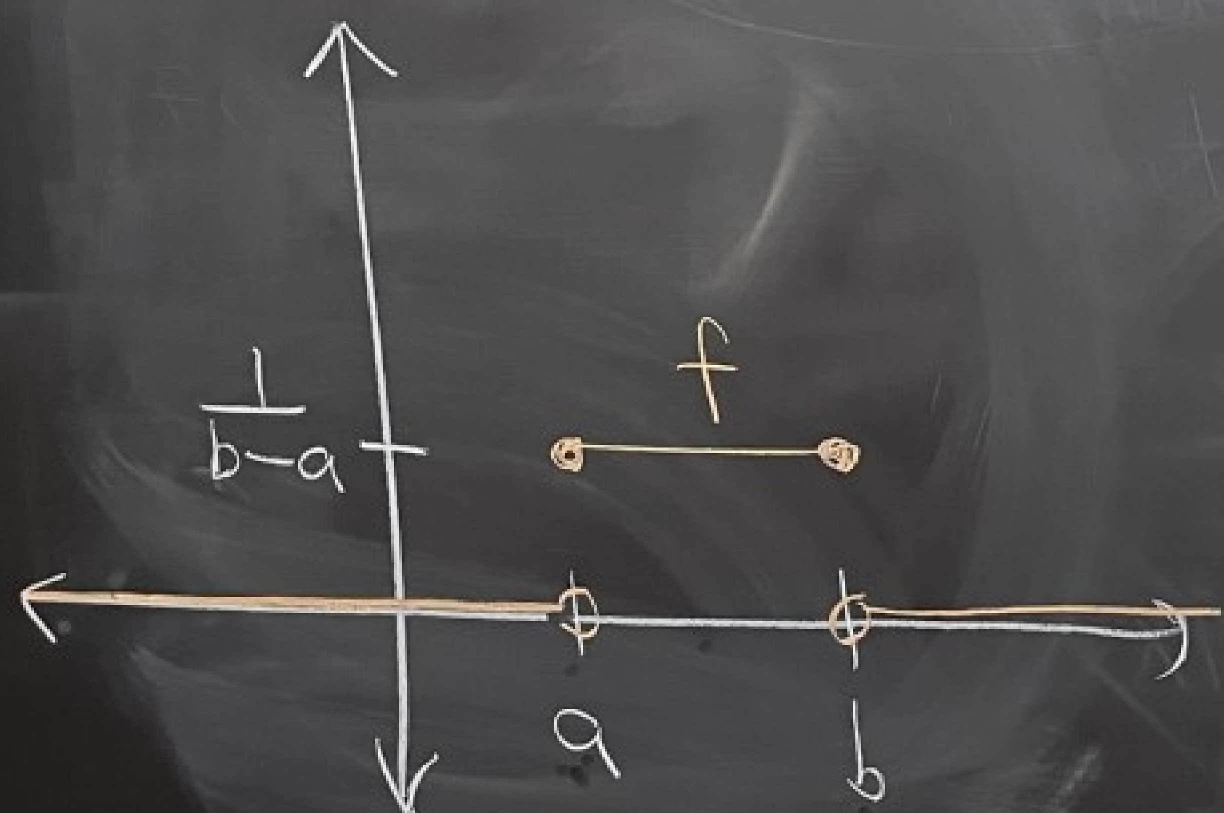
$$F(t) = P(X \le t) = \int_{-\infty}^t f(x)\,dx$$



Ex: (The uniform distribution on $[a,b]$)

Let $a<b$.

Let
$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$



f is a pdf because

① $f(x) \geq 0$ for all x

② $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_a^b \frac{1}{b-a}\,dx$

$= \dfrac{1}{b-a} \cdot (b-a) = 1$

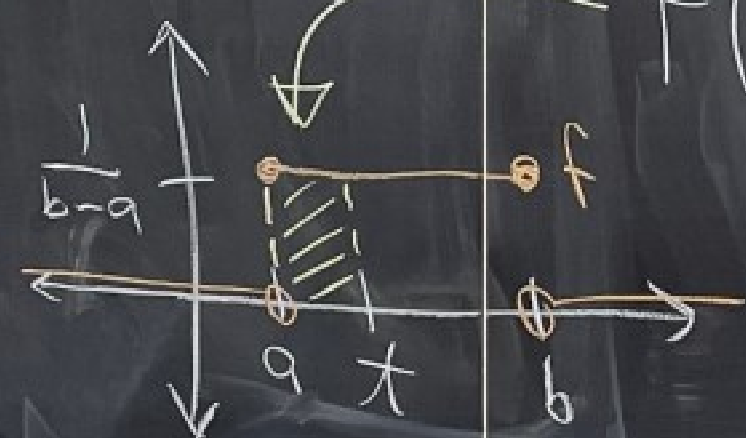
$$F(t) = \int_{-\infty}^{t} f(x)\,dx$$

case 1: Suppose $t \leq a$.

$$F(t) = \int_{-\infty}^{t} f(x)\,dx = \int_{-\infty}^{t} 0\,dx = 0.$$

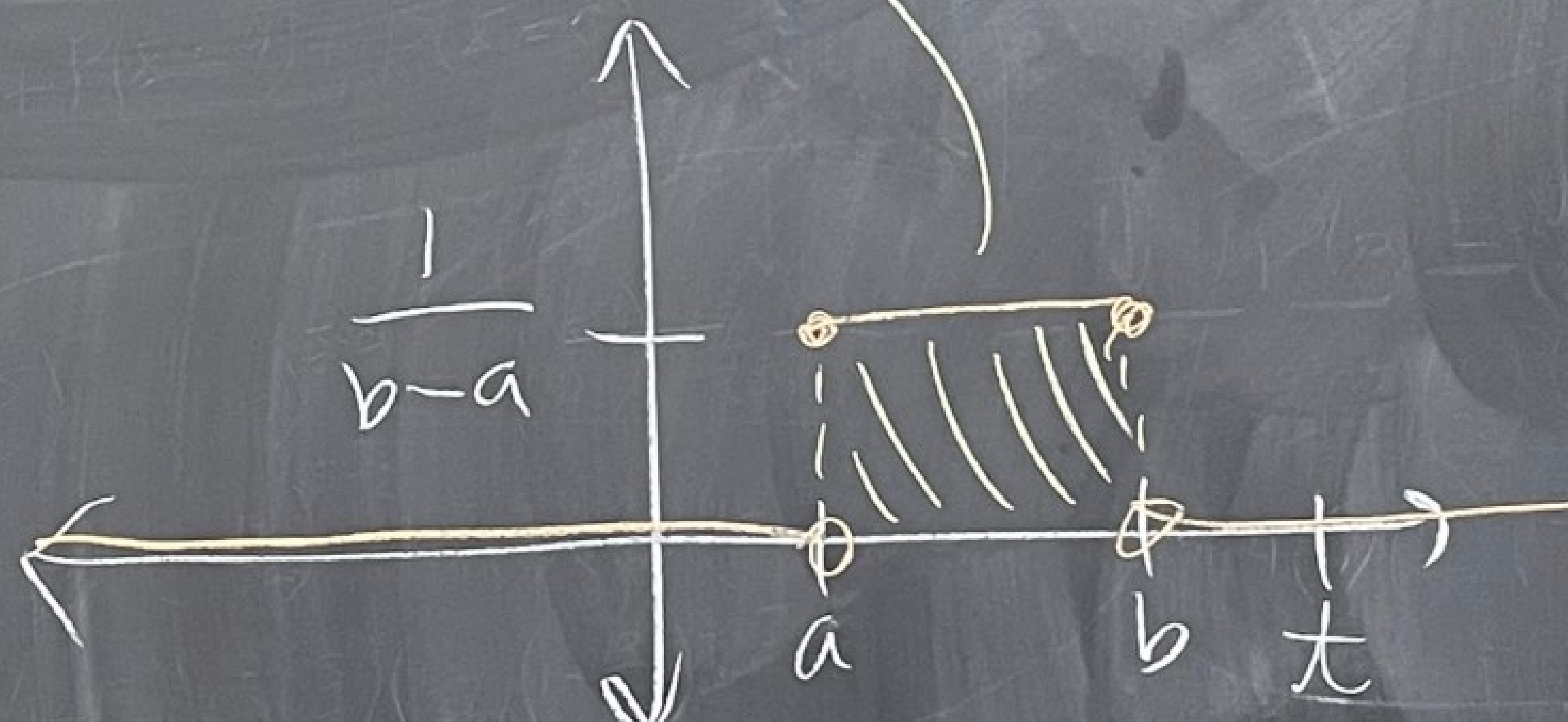
case 2: Suppose $a < t < b$.

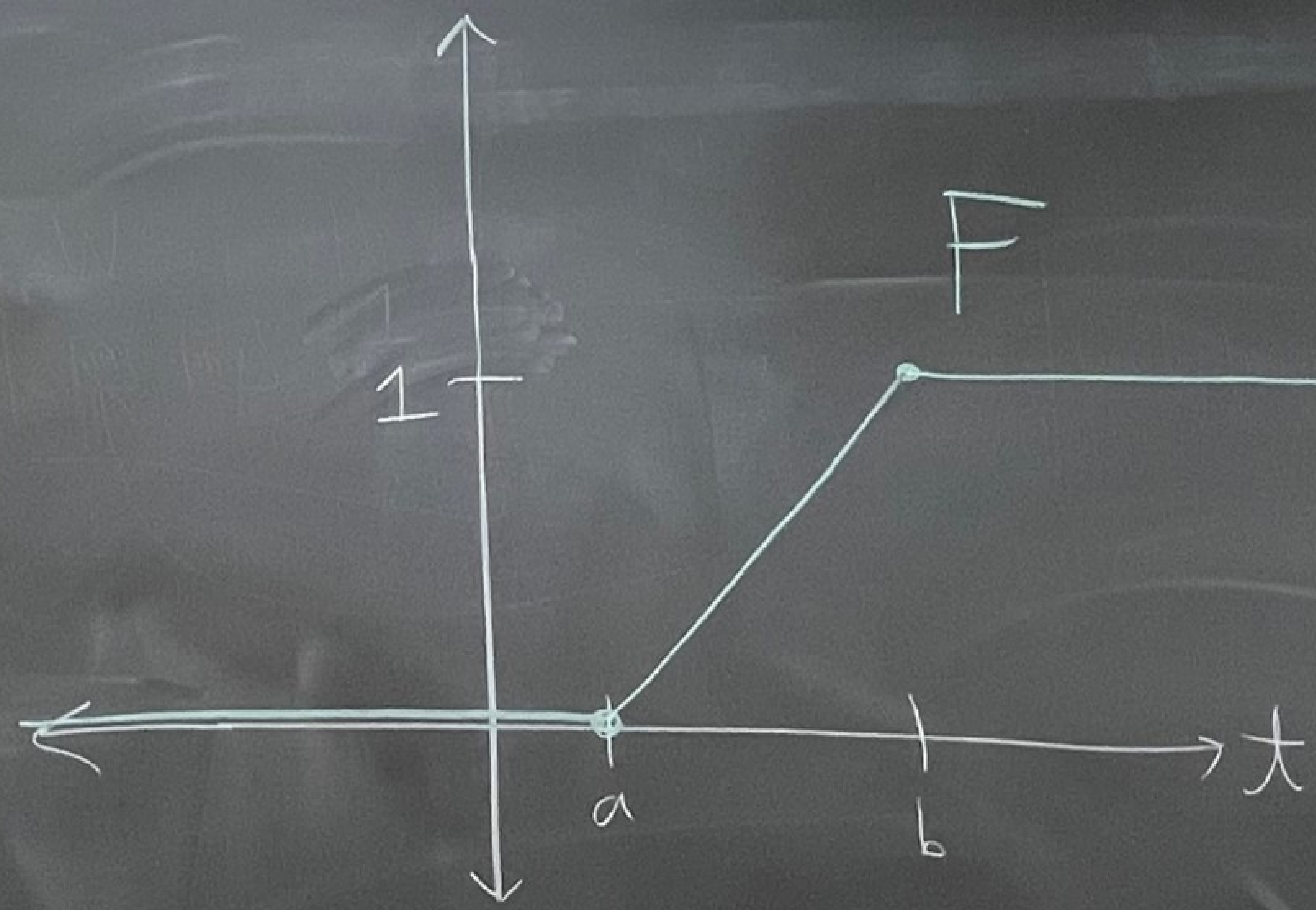
$$F(t) = \int_{-\infty}^{t} f(x)\,dx = \frac{t-a}{b-a}$$



Case 3: Suppose $b \le t$.

$$F(t) = \int_{-\infty}^{t} f(x)\,dx = 1$$



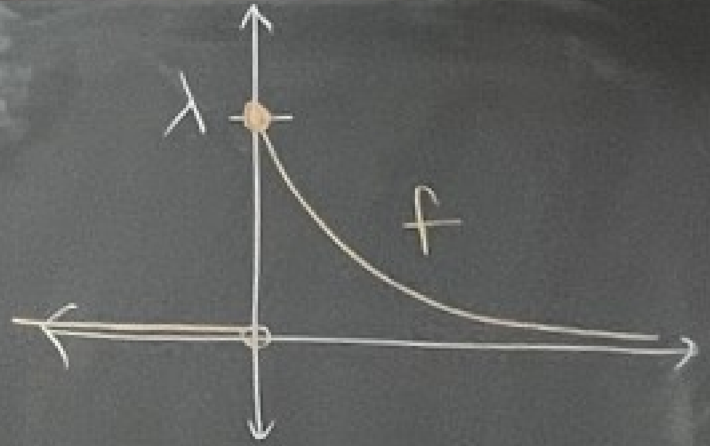

F
1
a
b
t

Ex: (Exponential random variable with parameter λ)

Let $\lambda > 0$.

Define

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



f is a pdf

① $f(x) \geq 0$ for all x

② $$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx = \lim_{u\to\infty} \int_0^u \lambda e^{-\lambda x}\,dx$$

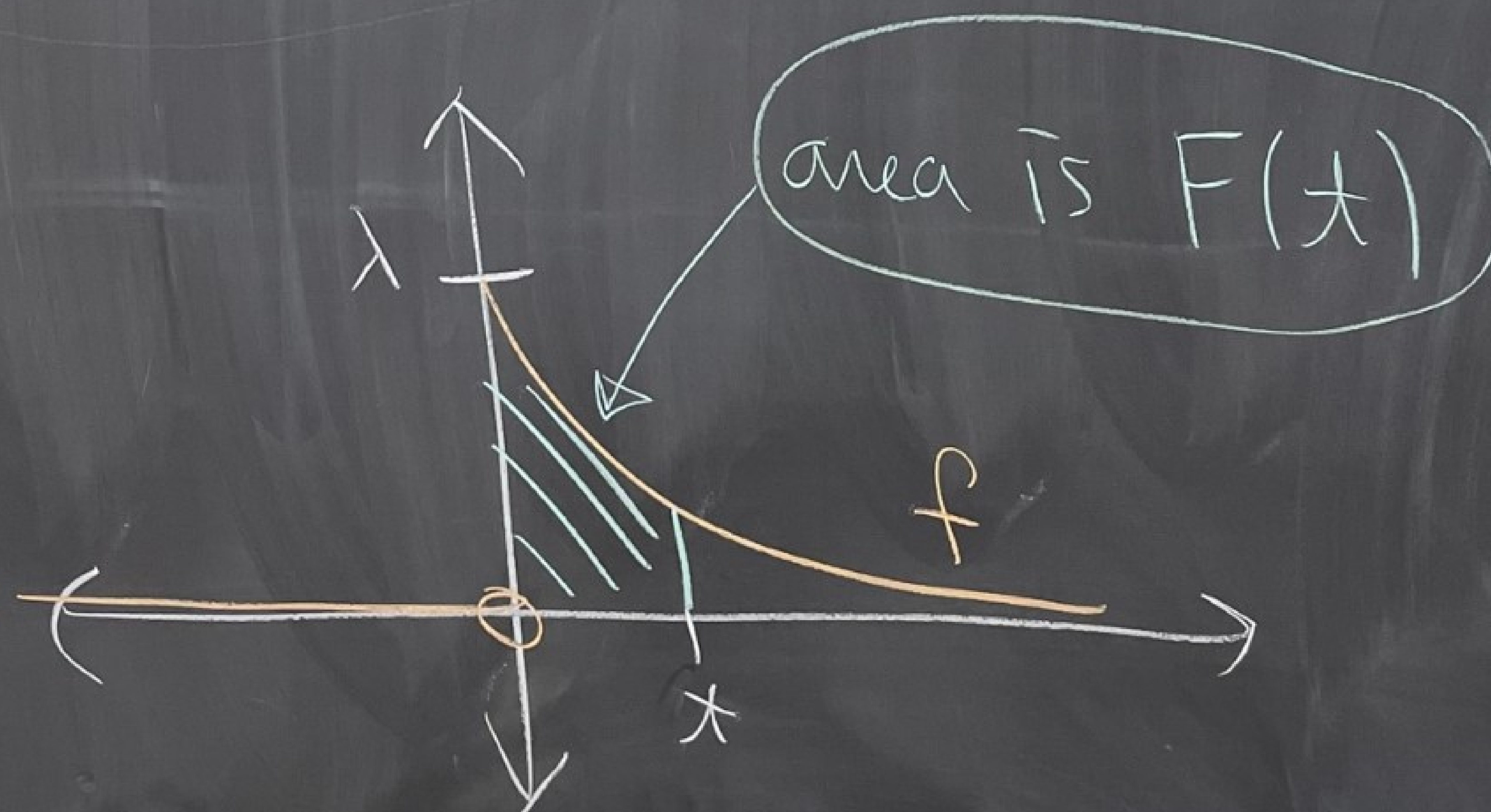
$$= \lim_{u\to\infty} \left[-e^{-\lambda x} \Big|_{x=0}^{u} \right]$$

$$= \lim_{u\to\infty} \left[-e^{-\lambda u} + e^{0} \right]$$

$$= \lim_{u\to\infty} \left[\frac{-1}{e^{\lambda u}} + 1 \right] = 0 + 1 = 1.$$

$$F(t) = \int_{-\infty}^{t} f(x)\,dx$$

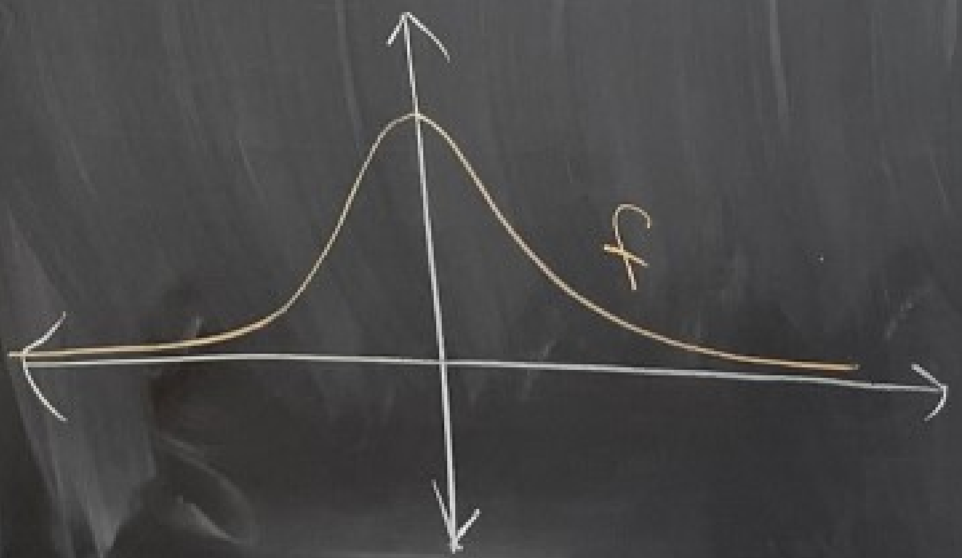
$$= \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$



Ex: (Standard normal distribution)

Let $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

We showed last week this is a pdf!



$$F(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$= \Phi(t)$$

We called this Φ

DeMoivre-Laplace thm

(binomial random variables)

$\Downarrow$

Central Limit Theorem

(more abstract thm)

Def: Let X be a random variable with pdf f.

Then,

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

Let $\mu = E[X]$.

Then,

$$\mathrm{Var}(X) = \left(\int_{-\infty}^{\infty} x^2 f(x)\, dx \right) - \mu^2$$

$$\sigma = \sqrt{\mathrm{Var}(X)}$$

Final
Weds
2:30-4:30

Hw 5

2. Roll two 6-sided dice.

Success = sum of dice is 7 or 11

X = # of successes in 10 rolls of the dice

(a) Calculate $E[X]$, $Var(X)$

(b) Calculate $P(X=5)$

(c) Probability at most 8 successes

Solution:

X is a binomial random variable with $n=10$ and $p = \underset{\boxed{\text{Sum 7}}}{\underset{\uparrow}{\frac{6}{36}}} + \underset{\boxed{\text{sum 11}}}{\underset{\uparrow}{\frac{2}{36}}} = \frac{8}{36} \approx 0.222$

(a) For binomial random variables

$$E(X) = np = 10\left(\frac{8}{36}\right) = \frac{80}{36} \approx 2.22.$$

$$Var(X) = np(1-p) = \frac{80}{36}\left(1 - \frac{8}{36}\right) = \frac{80}{36}\left(\frac{28}{36}\right)$$

$$\approx 1.7283$$

(b) $P(X=5) = \underbrace{\binom{n}{k} p^k (1-p)^{n-k}}_{n=10} = \binom{10}{5} \cdot \left(\frac{8}{36}\right)^5 \left(1-\frac{8}{36}\right)^{10-5}$

$\boxed{k=5}$

$= \frac{10!}{5!\,5!} \cdot \frac{8^5}{36^5} \cdot \frac{28^5}{36^5}$

10 rolls

(4,1), (1,2), (1,1), ..., (3,3)

sum of dice on each roll

5, 3, 2, 10, 11, 12, 8, 7, 7, 6

3 successes in 10 rolls

(c) $P(X \leq 8) = P(X=0) + P(X=1) + \cdots + P(X=8)$

$= \binom{10}{0}\left(\frac{8}{36}\right)^0\left(\frac{28}{36}\right)^{10} + \cdots + \binom{10}{8}\left(\frac{8}{36}\right)^8\left(\frac{28}{36}\right)^2$

Easier:

$P(X \leq 8) = 1 - P(X > 8) = 1 - P(X=9) - P(X=10)$

$$1-\underbrace{\binom{10}{9}}_{10}\cdot\left(\frac{8}{36}\right)^{9}\left(\frac{28}{36}\right)^{10-9}-\underbrace{\binom{10}{10}}_{1}\cdot\left(\frac{8}{36}\right)^{10}\cdot\left(\frac{28}{36}\right)^{10-10}=\boxed{1-10\cdot\left(\frac{8}{36}\right)^{9}\cdot\left(\frac{28}{36}\right)-\left(\frac{8}{36}\right)^{10}}$$

$$=\boxed{\text{Some \#}}$$

HW 6

3. roll two 4-sided dice

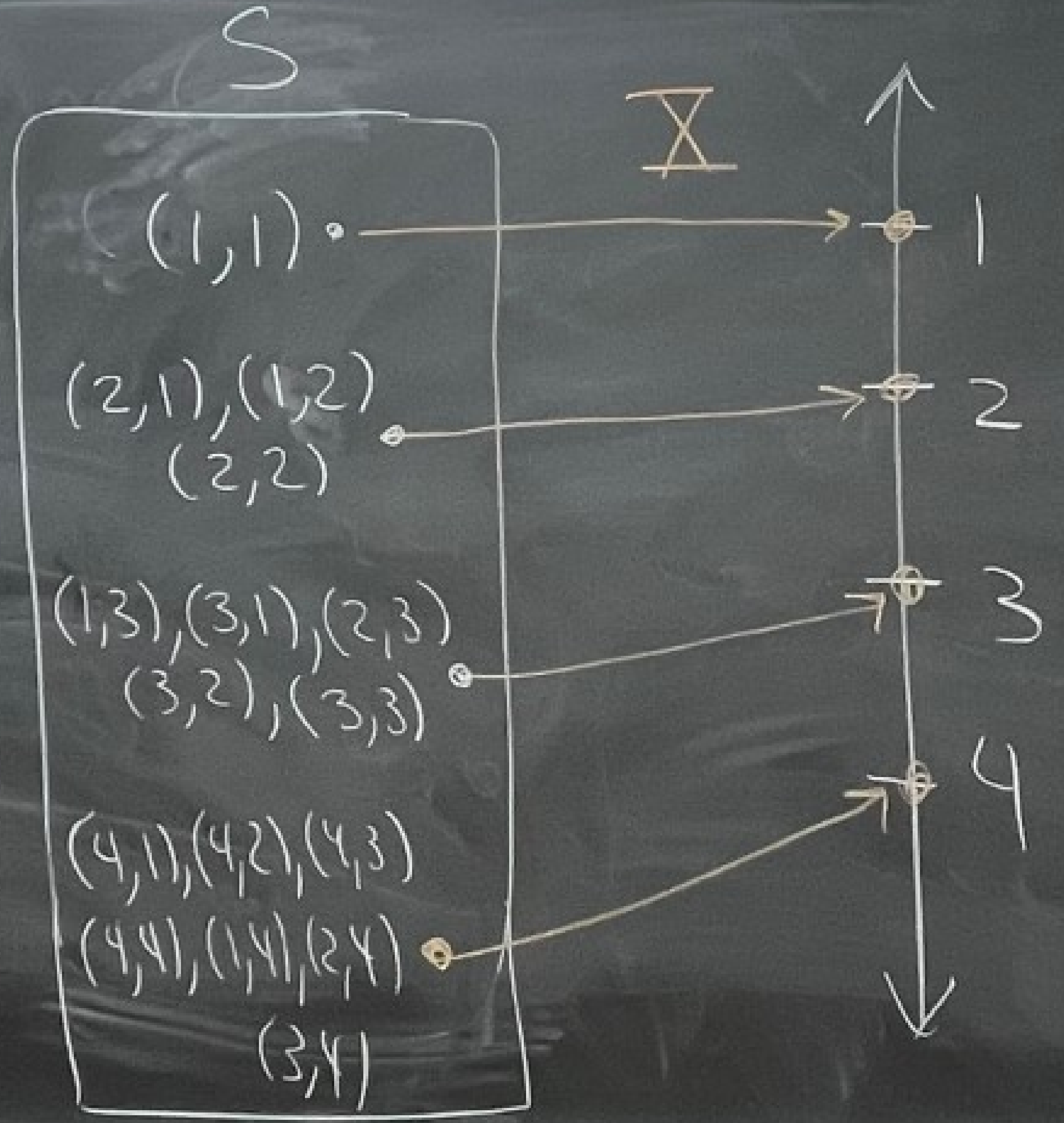
X = maximum of dice

(a) draw X and p

(b) Calculate F

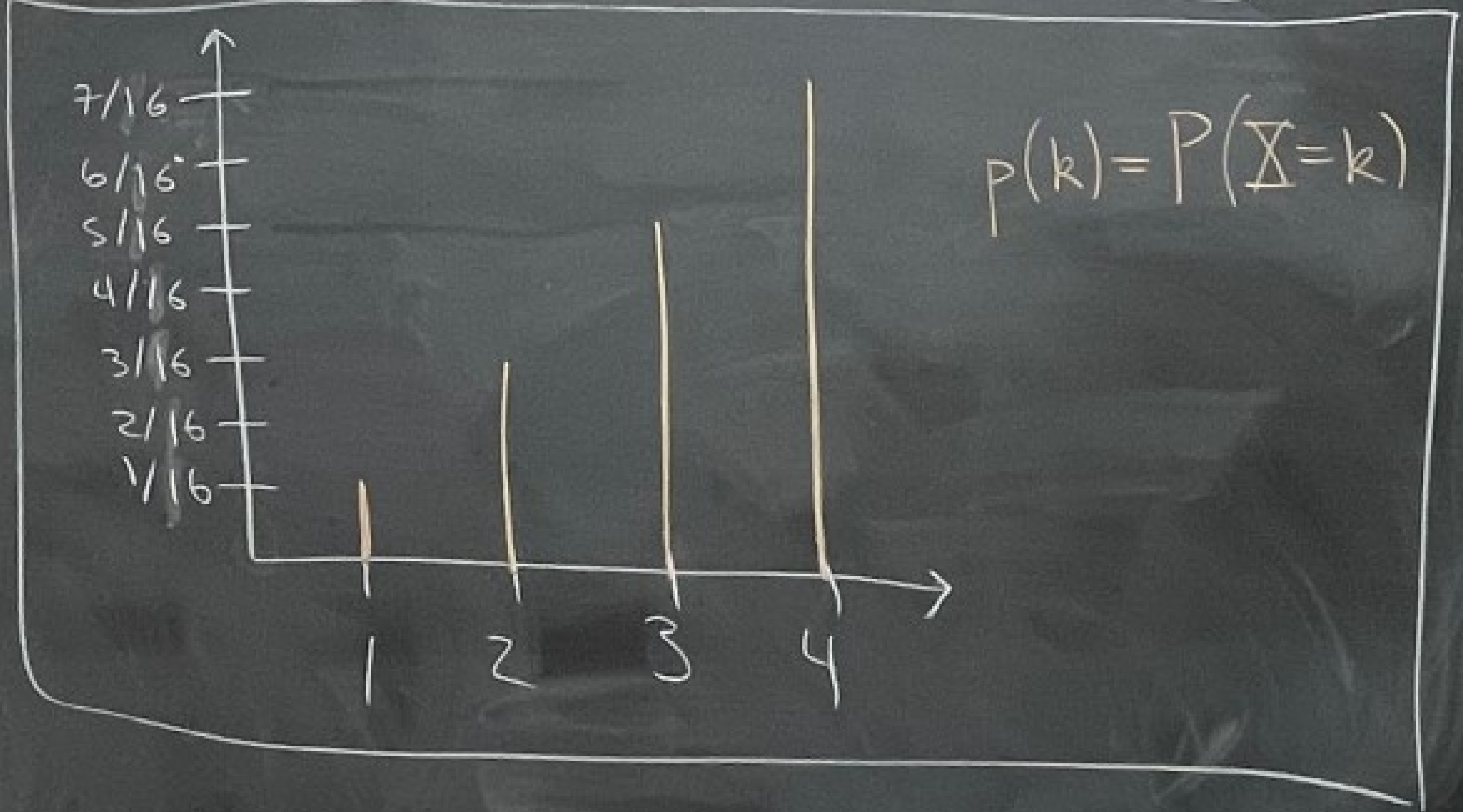
(c) Calculate $E(X)$, $Var(X)$, σ.

(a)



$P(X=1) = 1/16 \qquad P(X=3) = 5/16$

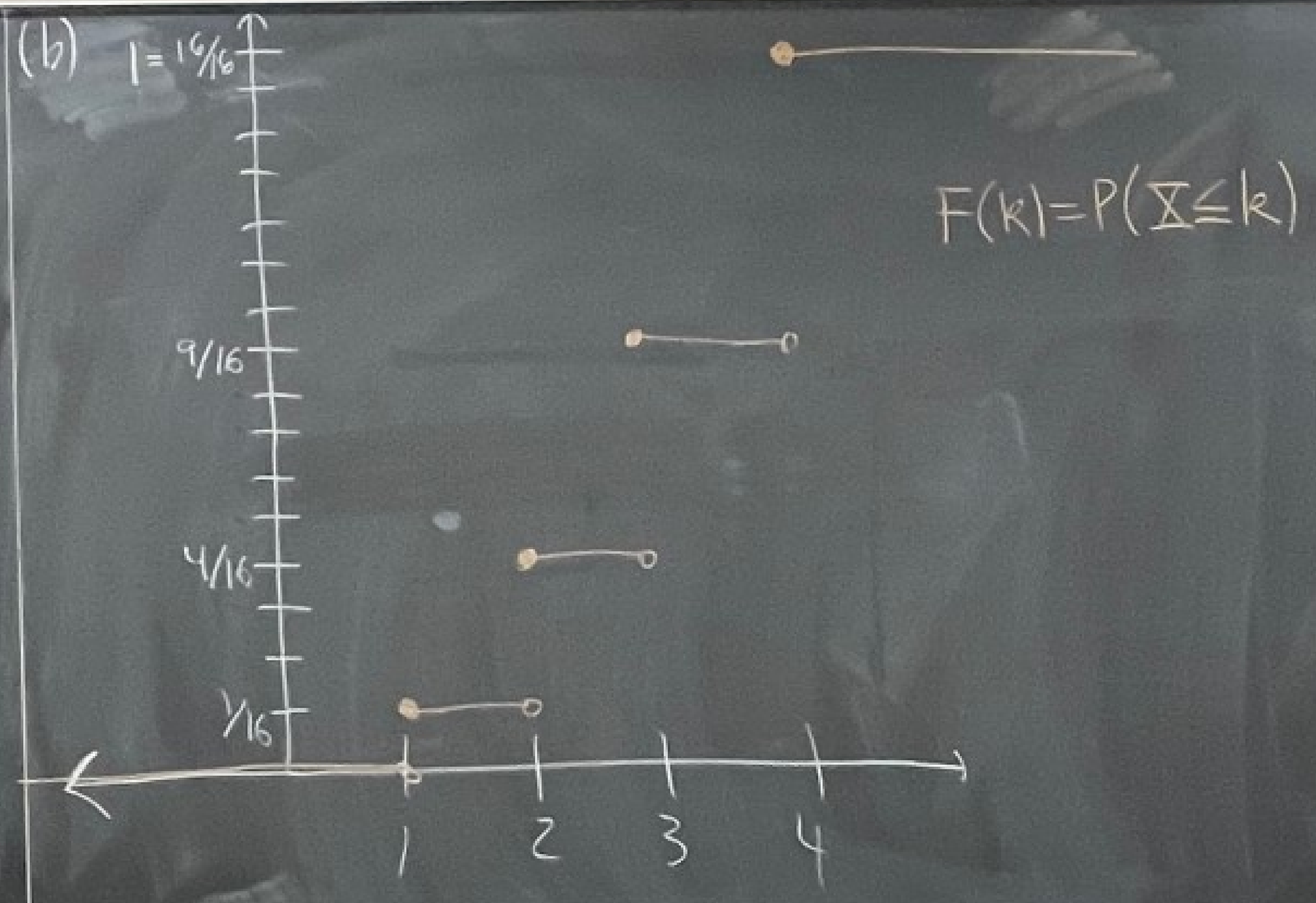
$P(X=2) = 3/16 \qquad P(X=4) = 7/16$



(b

k)

(b)



(c)

$$E[X] = (1)\left(\frac{1}{16}\right) + (2)\left(\frac{3}{16}\right) + (3)\left(\frac{5}{16}\right) + (4)\left(\frac{7}{16}\right)$$
$$= \boxed{50/16}$$

$$Var(X) = E[X^2] - \big[E[X]\big]^2$$

$$E[X^2] = (1)^2\left(\frac{1}{16}\right) + (2)^2\left(\frac{3}{16}\right) + (3)^2\left(\frac{5}{16}\right) + (4)^2\left(\frac{7}{16}\right)$$
$$= 170/16$$

$$Var(X) = \frac{170}{16} - \left(\frac{50}{16}\right)^2 = \boxed{\frac{220}{256}} = \boxed{\frac{55}{64}}$$

$$\sigma = \sqrt{Var(X)} = \sqrt{\frac{55}{64}}$$

$$E[X] = 0\cdot P(X=0) + 1\cdot P(X=1) + 2\cdot P(X=2) + \cdots + 100\cdot P(X=100)$$

$$E[X] = np = 100\left(\frac{7}{16}\right) = \frac{700}{16}$$

$$P(X=21) = \binom{100}{21}\cdot\left(\frac{7}{16}\right)^{21}\cdot\left(\frac{9}{16}\right)^{100-21} = \frac{100!}{21!\,79!}\cdot\frac{7^{21}}{16^{21}}\cdot\frac{9^{79}}{16^{79}}$$

rolling two 4-sided dice 100 times

X = # times 4 occurs as one of the dice

$p = \frac{7}{16} \rightarrow$ success $= \{(1,4),(2,4),(3,4),(4,4),$
$(4,1),(4,2),(4,3)\}$

$1-p = \frac{9}{16} \rightarrow$ failure $= \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2),(3,3),$
$(1,1),(2,2)\}$

$\mu = 11.11$
$E(X)$

$\boxed{\text{HW 7}}$

(made up problem)

Roll two 6-sided dice 50 times.
Let X = # of times the sum of the dice is 7 or 11.

Estimate $P(10 \le X \le 15)$

$n = 50$

$p = \frac{6}{36} + \frac{2}{36} = \frac{8}{36}$

$$P(10 \le X \le 15)$$

$$= P\left(\frac{10 - 50(8/36)}{\sqrt{50(8/36)(1-8/36)}} \le \underbrace{\frac{X - 50(\frac{8}{36})}{\sqrt{50(8/36)(1-8/36)}}}_{\frac{X - np}{\sqrt{np(1-p)}}} \le \frac{15 - 50(8/36)}{\sqrt{50(8/36)(1-8/36)}} \right)$$

$$\approx P\left(-0.38 \le \frac{X - np}{\sqrt{np(1-p)}} \le 1.32\right)$$

$$\approx \Phi(1.32) - \Phi(-0.38) \approx \Phi(1.32) - [1 - \Phi(0.38)]$$

$$\approx -1 + \Phi(1.32) + \Phi(0.38)$$

$$\approx -1 + 0.9066 + 0.6480 \approx \boxed{0.5546\ldots} \approx \boxed{55.46\%}$$