Math 4740
4/7/25

(topic 6 continued...)

Here's another way to calculate $Var(X)$.

<u>Theorem:</u> Let X be a discrete random variable with expected value $\mu = E[X]$.

Then,

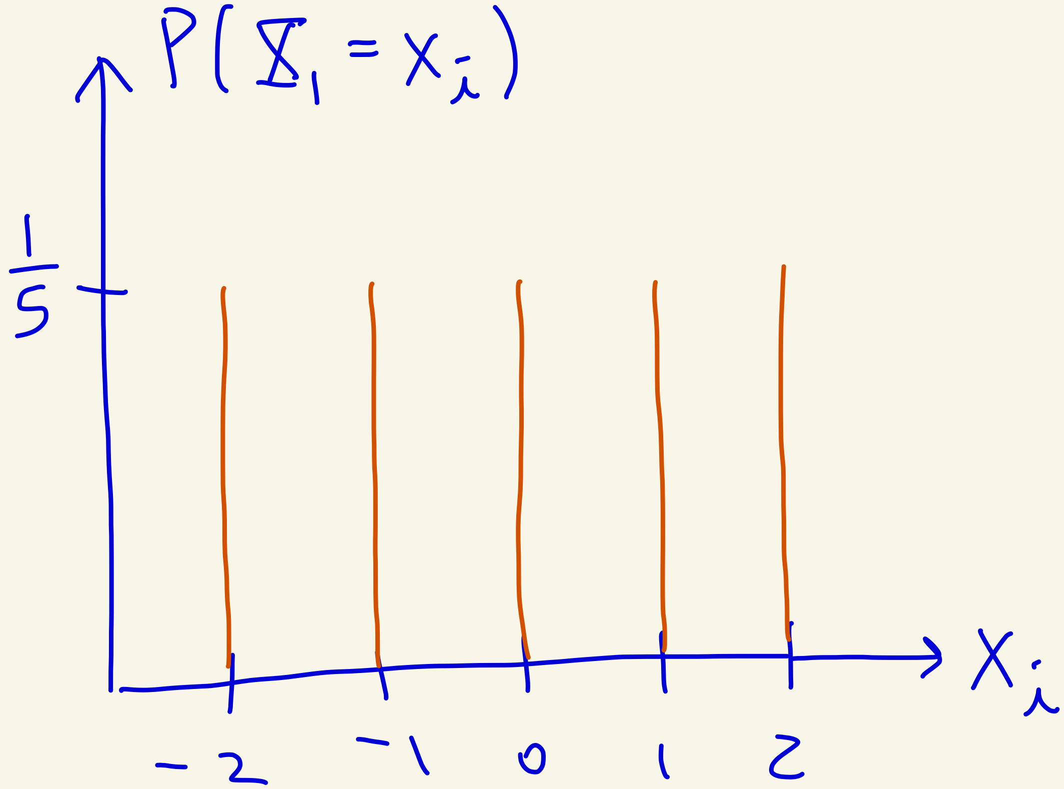
$$Var(X) = E[X^2] - \mu^2$$

where

$$E[X^2] = \sum_i x_i^2 \cdot P(X = x_i)$$

where $x_1, x_2, x_3, \ldots$ are the outputs of X.

Ex:



Previously, we calculated

$$\mu = E[X_1] = 0.$$

$$E[X_1^2] = (-2)^2 \cdot \left(\frac{1}{5}\right) + (-1)^2 \cdot \left(\frac{1}{5}\right) + (0)^2 \cdot \left(\frac{1}{5}\right) + (1)^2 \cdot \left(\frac{1}{5}\right) + (2)^2 \cdot \left(\frac{1}{5}\right)$$

$$= 10 \cdot \frac{1}{5} = 2$$

$$\mathrm{Var}(X_1) = E[X_1^2] - \mu^2$$
$$= 2 - 0^2 = 2$$
$$\sigma = \sigma_{X_1} = \sqrt{2} \approx 1.414\ldots$$

Ex:

Previously, we got $\mu = E[X_2] = 0$.

$$E[X_2^2] = (-2)^2\left(\frac{1}{400}\right) + (-1)^2\left(\frac{1}{400}\right) + (0)^2\left(\frac{99}{100}\right) + (1)^2\left(\frac{1}{400}\right) + (2)^2\left(\frac{1}{400}\right)$$

$$= 10 \cdot \frac{1}{400} = \frac{1}{40} \approx 0.025$$

$$Var(X_2^2) = E[X_2^2] - \mu^2 = \frac{1}{40} - 0^2 = \frac{1}{40}$$

$$\sigma = \sigma_{X_2} = \sqrt{1/40} \approx 0.158..$$

Theorem: Let X be a binomial random variable with parameters n and p.

Then,

$$E[X] = np$$

$$Var(X) = np(1-p)$$

$$\sigma = \sqrt{np(1-p)}$$

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proof in notes online

Ex: Suppose we flip a coin $n = 50$ times. Let X be the number of tails that occur. So, $p = 1/2$.

Then,

$$E[X] = np = (50)\left(\tfrac{1}{2}\right) = 25$$

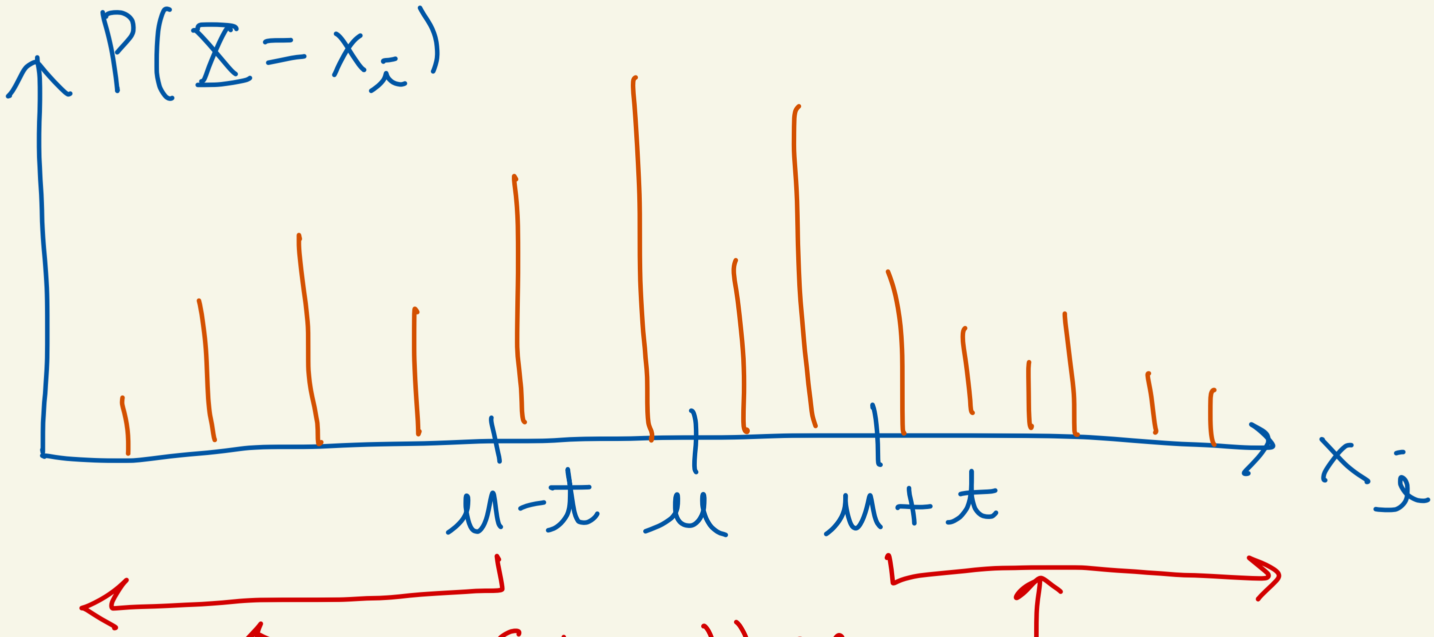
$$\mathrm{Var}(X) = np(1-p) = (50)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = 12.5$$

$$\sigma = \sqrt{12.5} \approx 3.5355$$

<u>Theorem</u> (Chebyshev's Inequality)

Let X be a discrete random variable with expected value μ. Let σ be the standard deviation. Then for any $t > 0$, we have

$$\underbrace{P(|X-\mu| \geq t)}_{\text{means: } P(\{\omega \mid |X(\omega)-\mu| \geq t\})} \leq \frac{\sigma^2}{t^2}$$

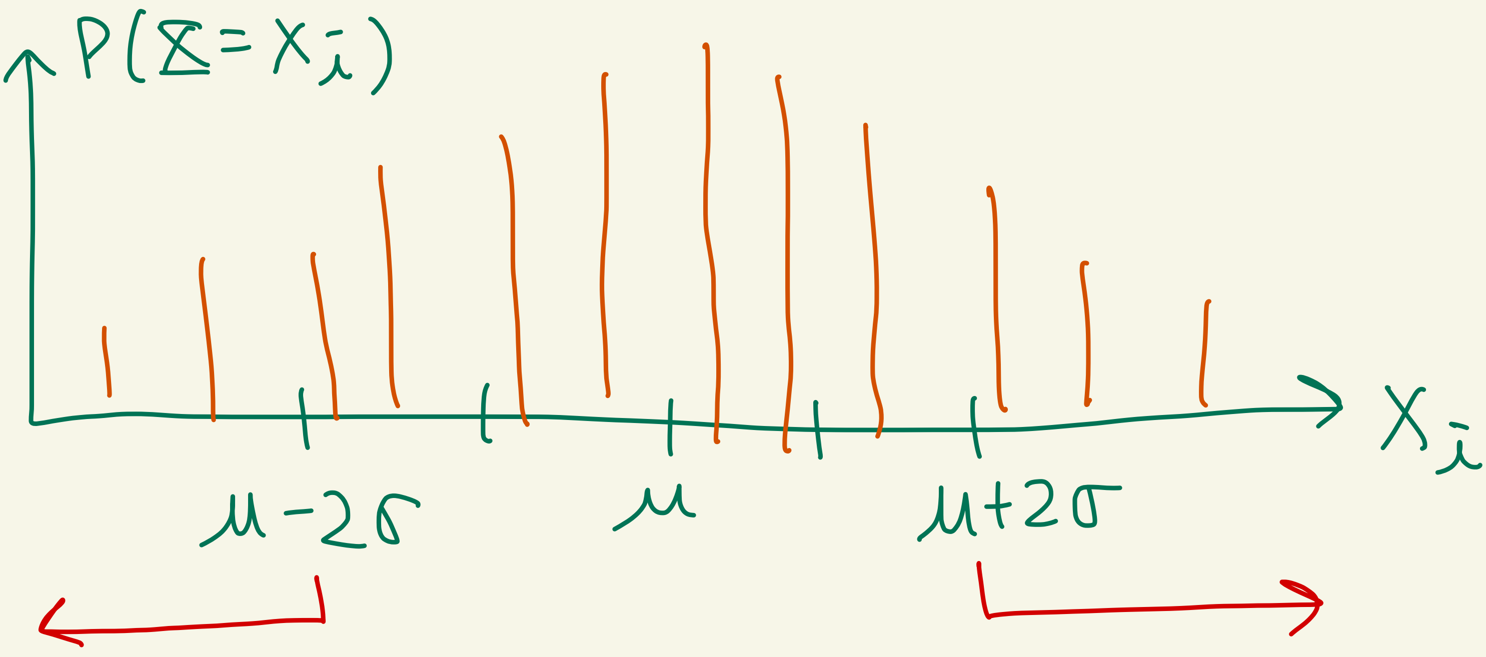


sum these

to get $P(|X-\mu| \geq t)$

Plug $t = 2\sigma$ into formula:

$$P(|X-\mu| \geq 2\sigma) \leq \frac{\sigma^2}{t^2} = \frac{\sigma^2}{(2\sigma)^2} = \underbrace{\frac{1}{4}}_{0.25}$$



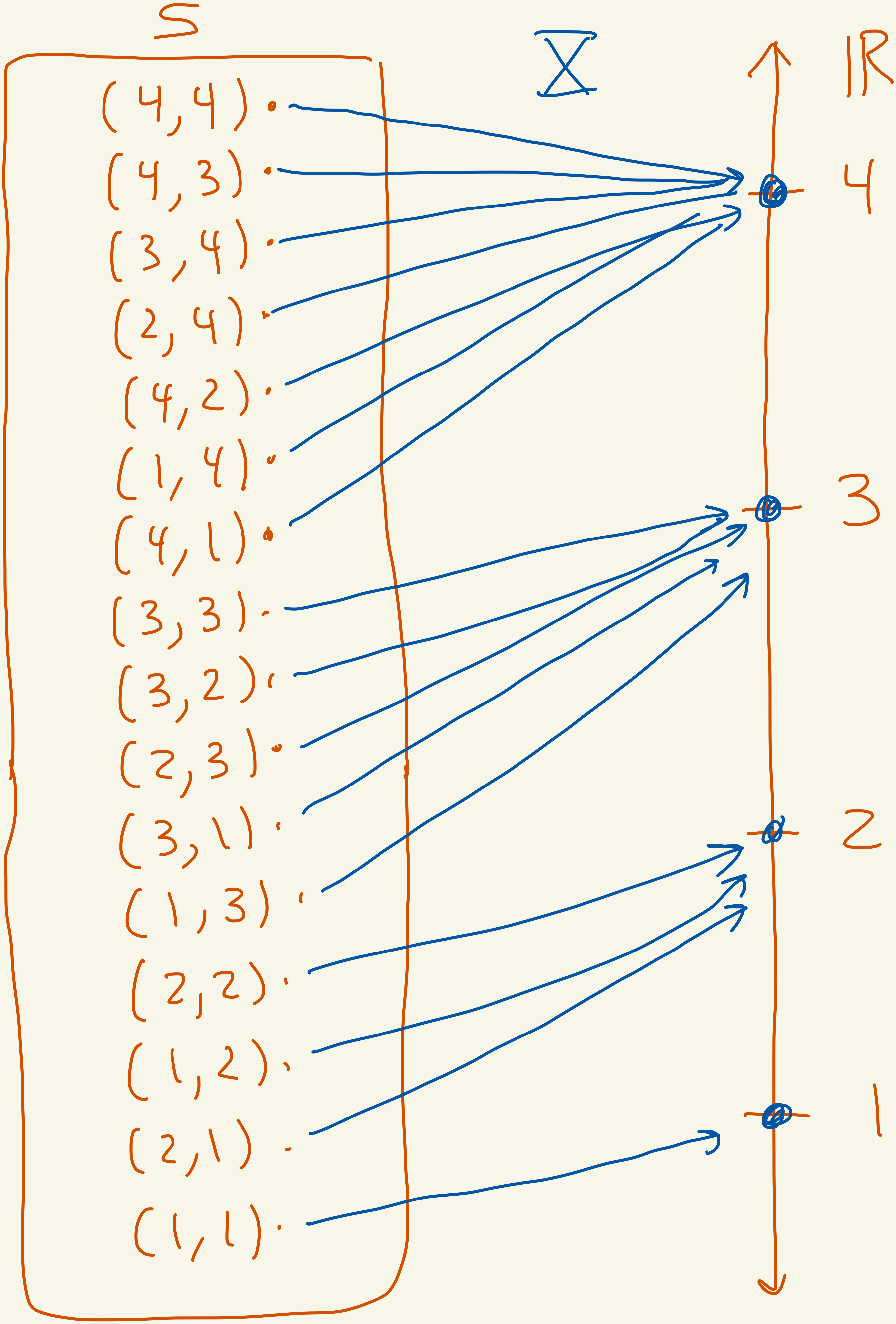
adds up to at most $1/4 = 0.25$

$\boxed{\text{HW 6}}$

③ Suppose we roll two 4-sided dice. Let X be the maximum of the two dice.

(a) Draw X and $P(X = x_i)$

(b) Calculate $E[X]$, $Var(X)$, σ.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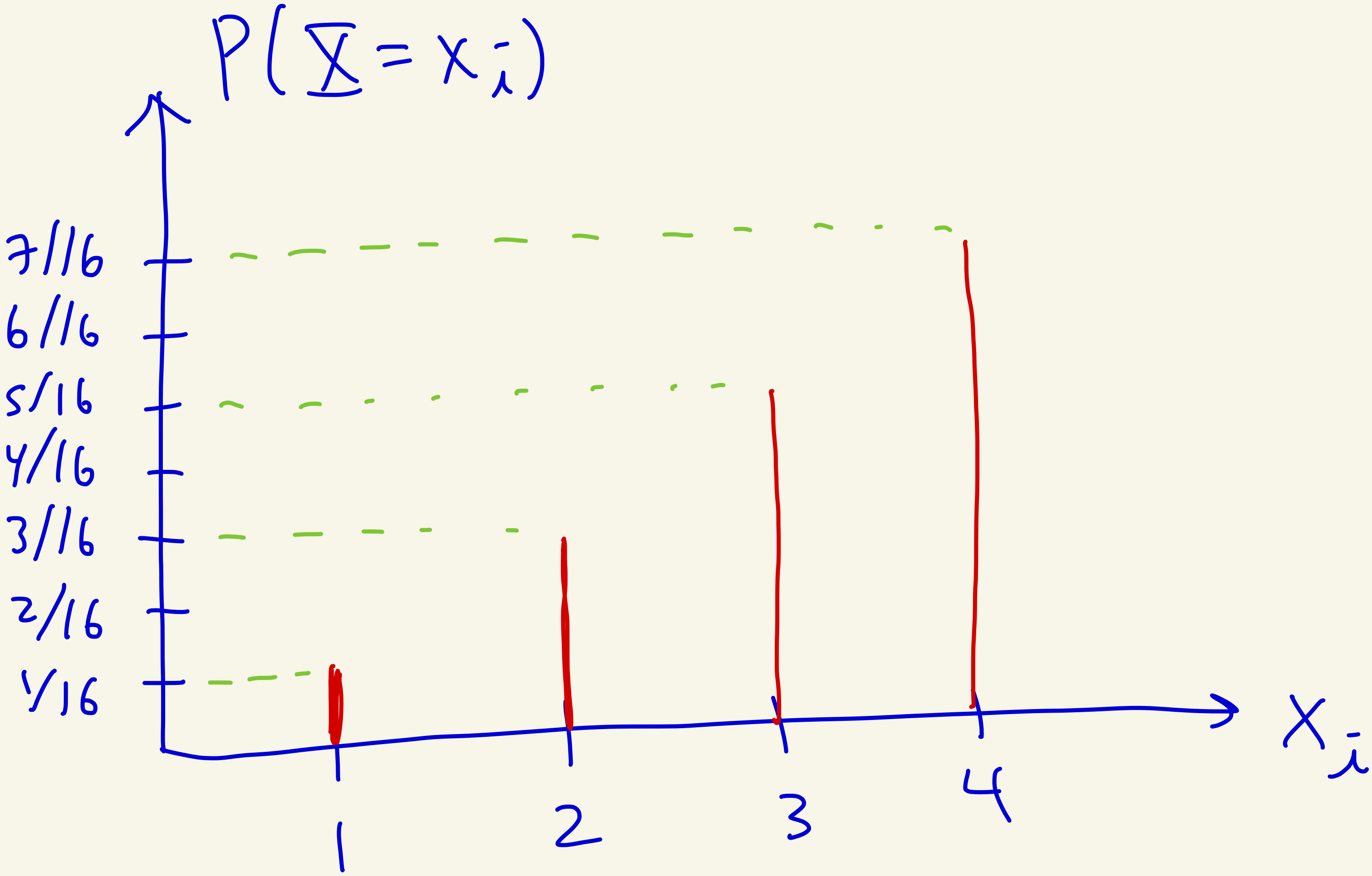

$$E[X] = (1)\left(\frac{1}{16}\right) + (2)\left(\frac{3}{16}\right) + (3)\left(\frac{5}{16}\right) + (4)\left(\frac{7}{16}\right)$$

$$= \frac{1+6+15+28}{16} = \boxed{\frac{50}{16} = 3.125}$$

μ

$$E[X^2] = (1)^2\left(\frac{1}{16}\right) + (2)^2\left(\frac{3}{16}\right) + (3)^2\left(\frac{5}{16}\right) + (4)^2\left(\frac{7}{16}\right)$$

$$= \frac{170}{16}$$

$$\mathrm{Var}(X) = E[X^2] - \mu^2$$

$$= \frac{170}{16} - \left(\frac{50}{16}\right)^2$$

$$= \frac{220}{256} \approx 0.859$$

$$\sigma \approx \sqrt{0.859} \approx 0.927$$