Math 4740
4/21/25

Topic 8 – Continuous random variables

<u>Def</u>: Let $f : \mathbb{R} \to \mathbb{R}$. We say that f is a <u>probability density function</u> (pdf) if

① $f(x) \geq 0$ for all x

② $\int_{-\infty}^{\infty} f(x)\,dx$ exists and

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Ex: (Standard normal distribution)

Let $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

Let's show that f is a pdf.

① $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} > 0$ for all x

② Let $I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.

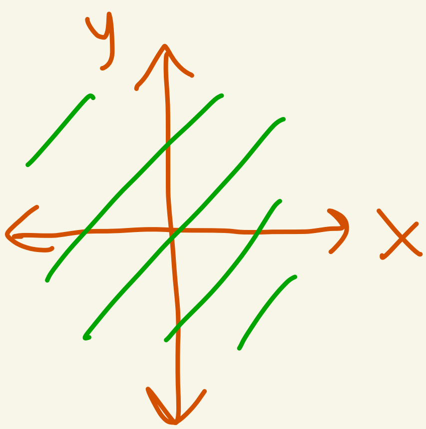
Then,

$$I^2 = \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \right) \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \right)$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \right) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-y^2/2} \, dx \, dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} \, dx \, dy$$



$0 \leq \theta \leq 2\pi$

$0 \leq r < \infty$

$x^2 + y^2 = r^2$

$dx\,dy = r\,dr\,d\theta$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2/2} \, r \, dr \, d\theta$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \left[-e^{-r^2/2} \Big|_{0}^{\infty} \right] d\theta$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \left[0 - \left(-e^{-0^2/2} \right) \right] d\theta$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} 1 \, d\theta = \frac{1}{2\pi} \theta \Big|_{0}^{2\pi} = \frac{1}{2\pi}(2\pi - 0) = 1$$

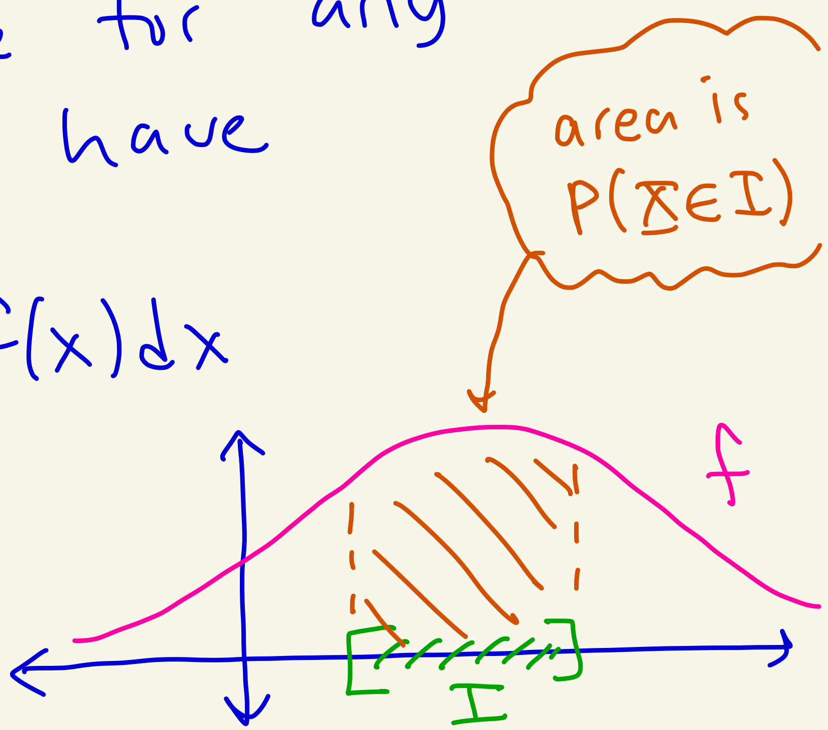
So, $I^2 = 1$. Since $I \geq 0$ we know $I = 1$.

Thus, $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = 1$.

So, f is a pdf.

<u>Def:</u> Let X be a random variable. We say that X is a <u>continuous random variable</u> if there exists a pdf f where for any interval I we have

$$P(X \in I) = \int_I f(x)\, dx$$



For example,

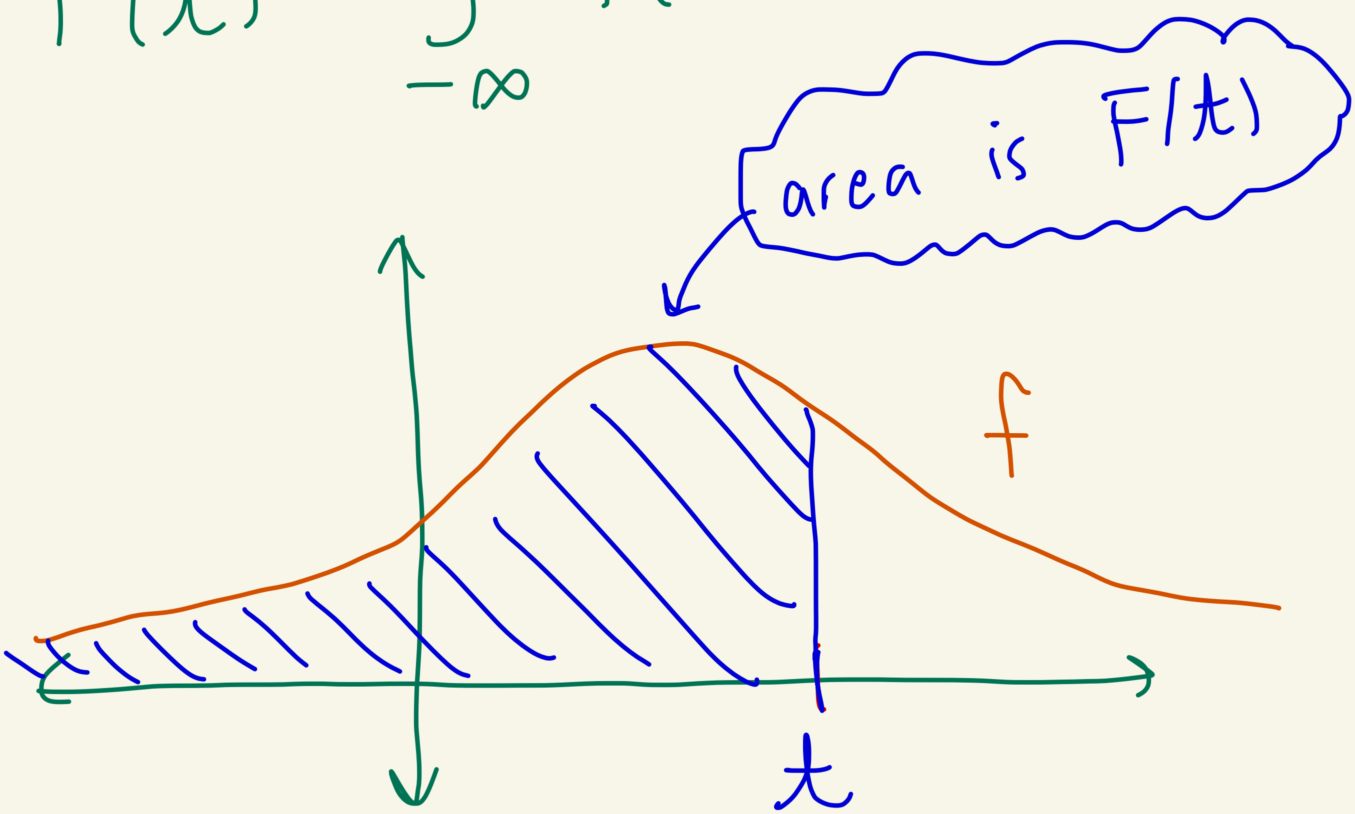
$$P(a \le X \le b) = \int_a^b f(x)\,dx$$

$$P(a \le X) = \int_a^\infty f(x)\,dx$$

$$P(X \le b) = \int_{-\infty}^b f(x)\,dx$$

The <u>cumulative distribution function</u> (cdf) of X is

$$F(t) = \int_{-\infty}^{t} f(x)\,dx$$



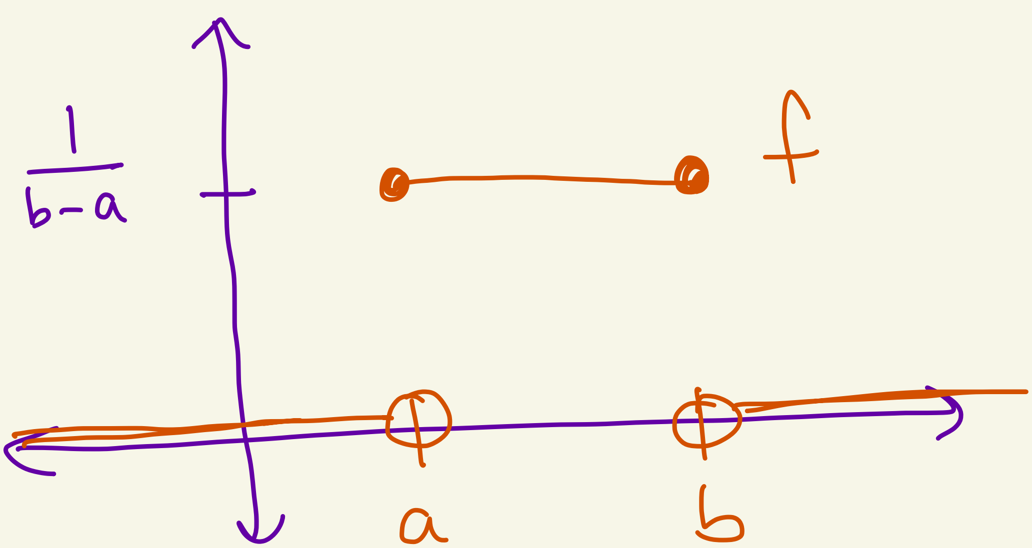
Note by the fundamental theorem of calculus, F is an anti-derivative of f, that is $F' = f$.

<u>Ex:</u> (the uniform distribution on $[a,b]$)

Let $a < b$.

Let

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

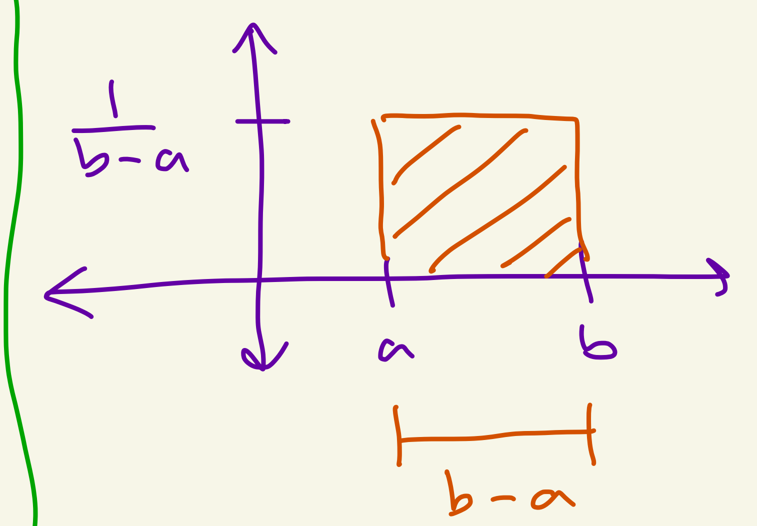


f is a pdf:

① $f(x) \geq 0$

② $\int_{-\infty}^{\infty} f(x)\,dx = \int_a^b \frac{1}{b-a}\,dx$

$$= \underbrace{\left(\frac{1}{b-a}\right)}_{\text{height}}\underbrace{(b-a)}_{\text{base}} = 1$$

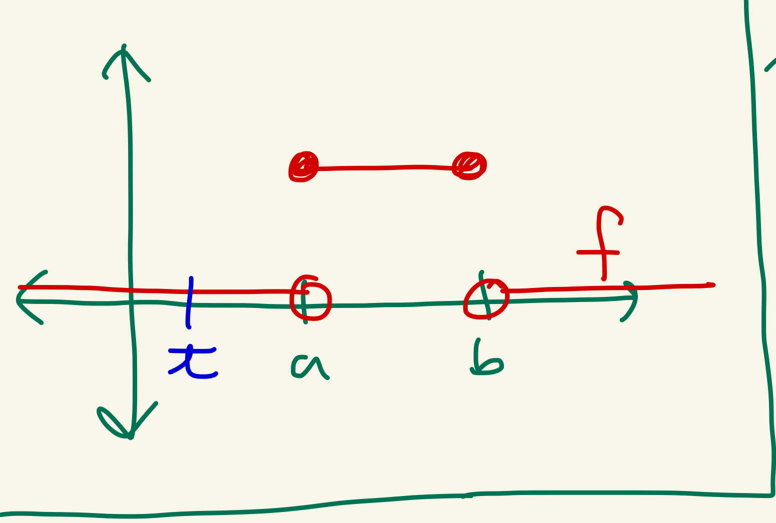


Let $F(t) = \int_{-\infty}^{t} f(x)\,dx$

Let's find a formula for F.

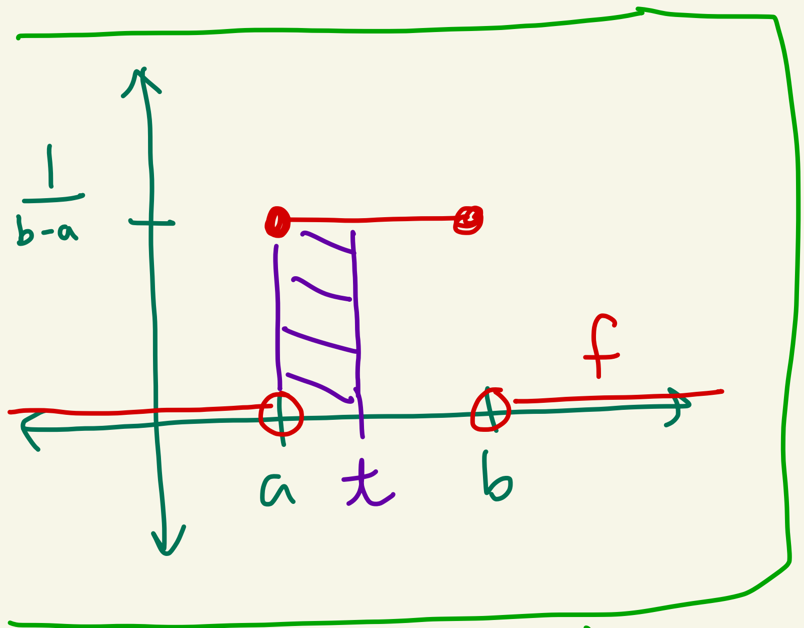
case 1: Suppose $t \leq a$. Then,

$$F(t) = \int_{-\infty}^{t} f(x)\,dx = \int_{-\infty}^{t} 0\,dx = 0.$$



Case 1 pic

Case 2: Suppose $a \leq t \leq b$.



case 2 pic

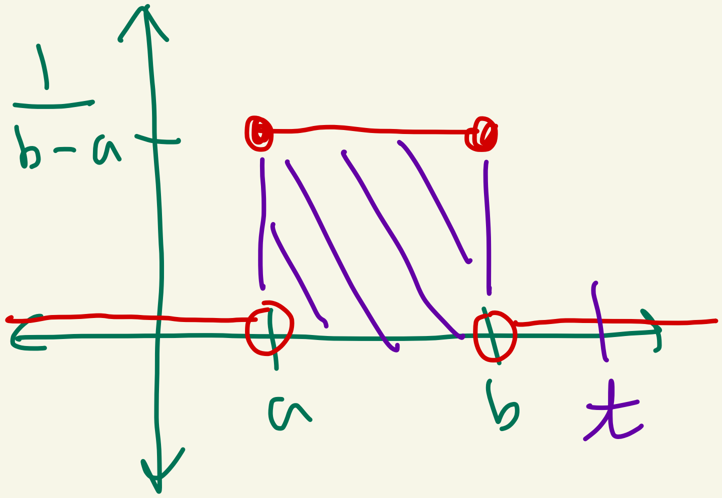
$$F(t) = \int_{-\infty}^{t} f(x)\,dx$$

$$= \int_{a}^{t} \frac{1}{b-a}\,dx$$

$$= \underbrace{\left(\frac{1}{b-a}\right)}_{\text{base}} \underbrace{(t-a)}_{\text{height}}$$

$$= \frac{t-a}{b-a}$$

Case 3: Suppose $b \leq t$.



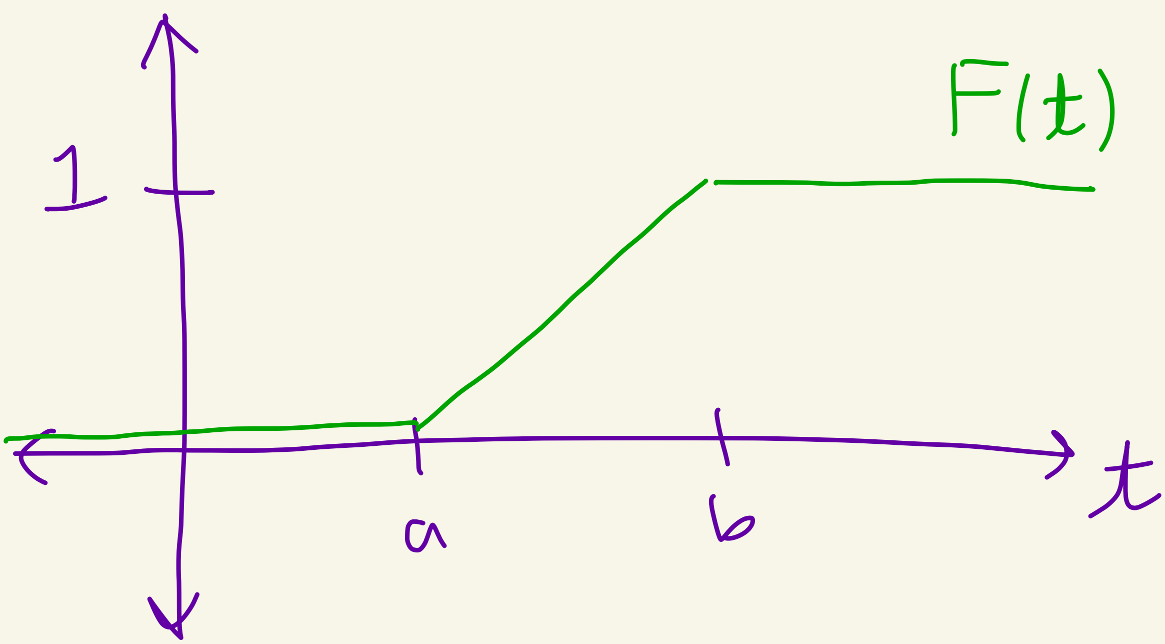
Case 3

$$F(t) = \int_{-\infty}^{t} f(x)\,dx$$

$$= 1$$

Thus,

$$F(t) = \begin{cases} 0 & \text{if } t \le a \\ \dfrac{t-a}{b-a} & \text{if } a \le t \le b \\ 1 & \text{if } b \le t \end{cases}$$

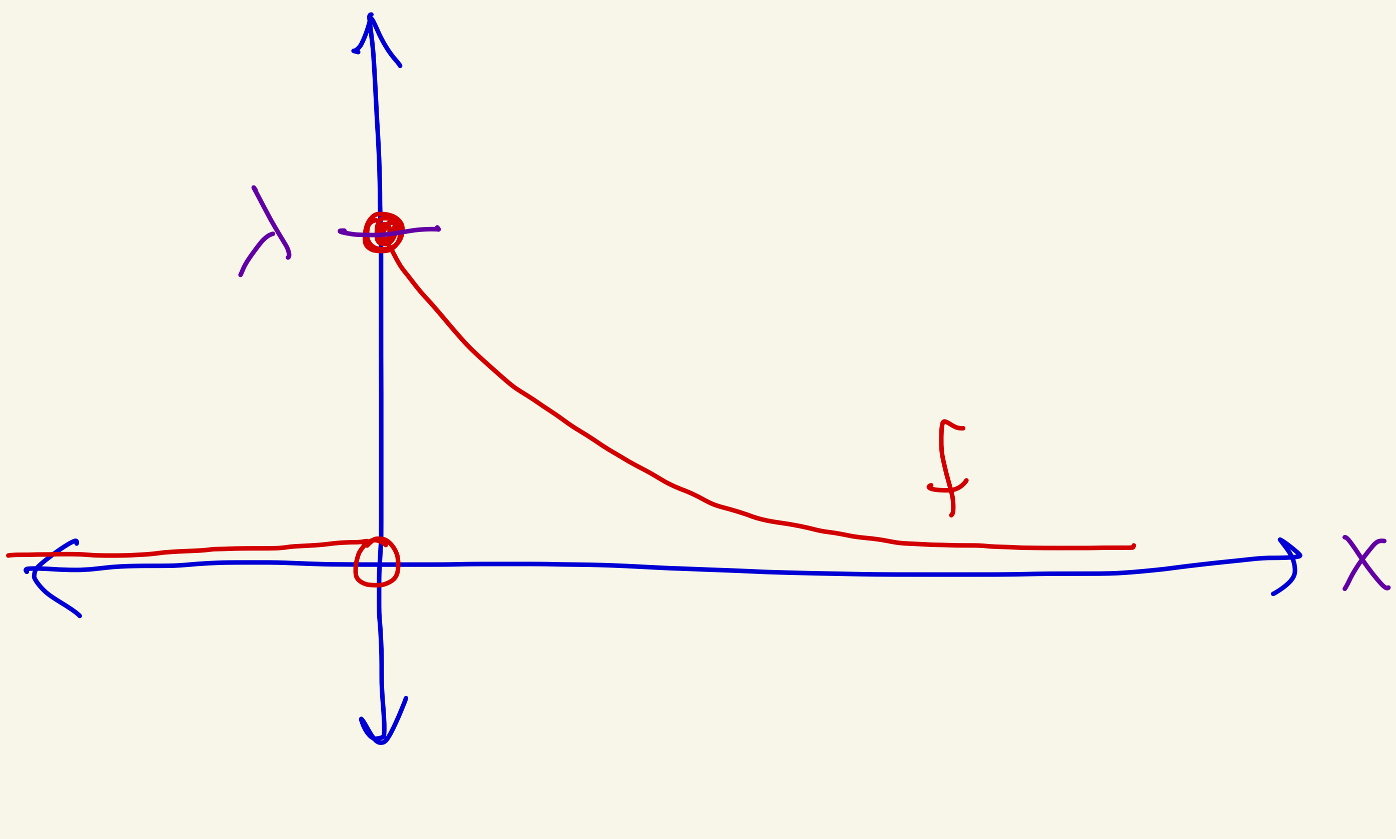


Ex: (Exponential random variable with parameter λ)

Let $\lambda > 0$.

Define

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \lambda e^{-\lambda x} & \text{if } x \geq 0 \end{cases}$$



Let's show that f is a pdf.

① $f(x) \geq 0$

② $$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx$$

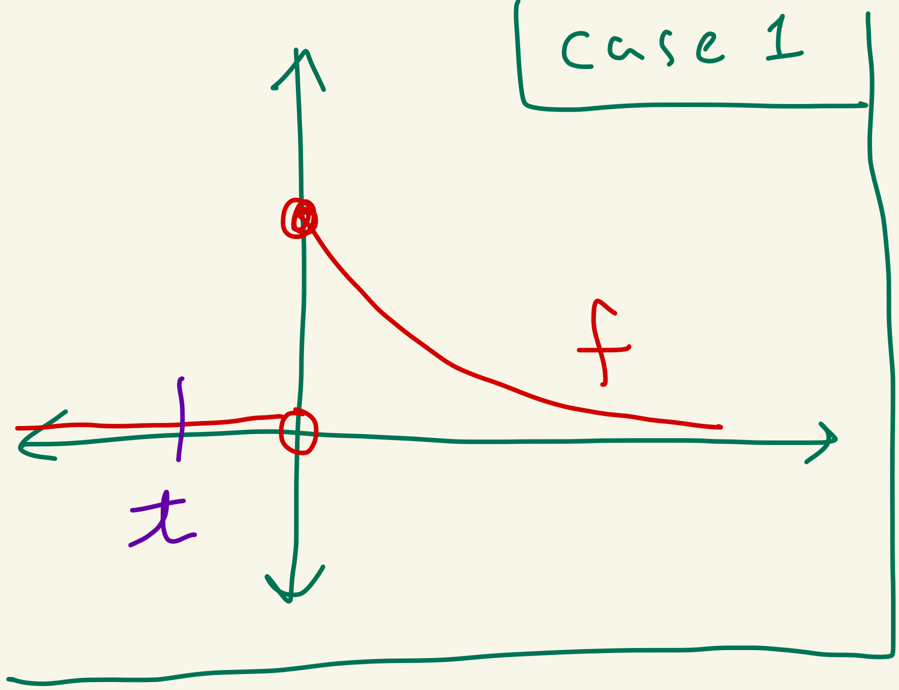
$$= -e^{-\lambda x}\Big|_0^{\infty} = \Big[-\underbrace{0}_{x \to \infty} - (-e^0)\Big]$$

$$= 0 - (-1) = 1$$

Let's find the cumulative distribution function.

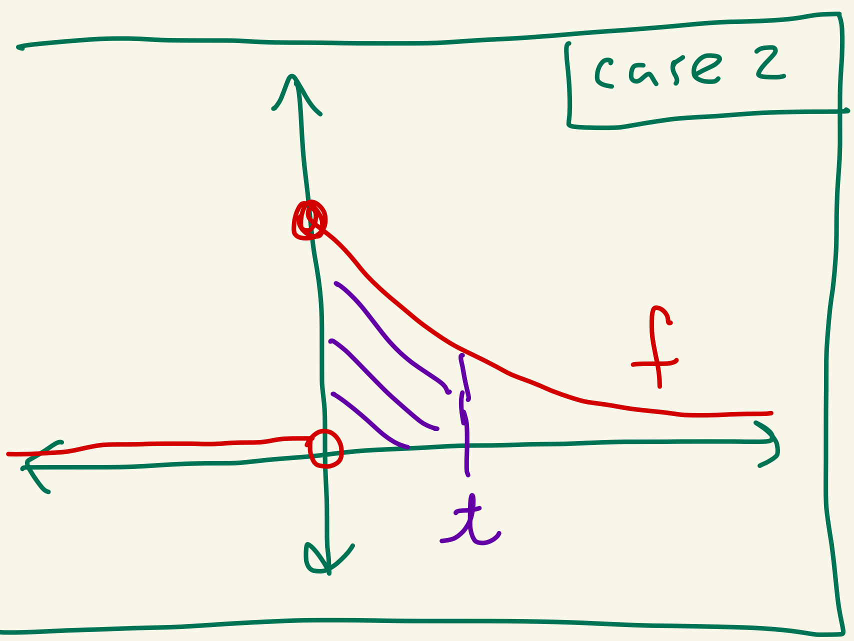
<u>case 1:</u> Suppose $t < 0$. Then,

$$F(t) = \int_{-\infty}^{t} f(x)\,dx = \int_{-\infty}^{t} 0\,dx = 0$$



case 2: Suppose $t \geq 0$. Then,

$$F(t) = \int_{-\infty}^{t} f(x)\,dx = \int_{0}^{t} \lambda e^{-\lambda x}\,dx$$



$$= -e^{-\lambda x}\Big|_0^t$$

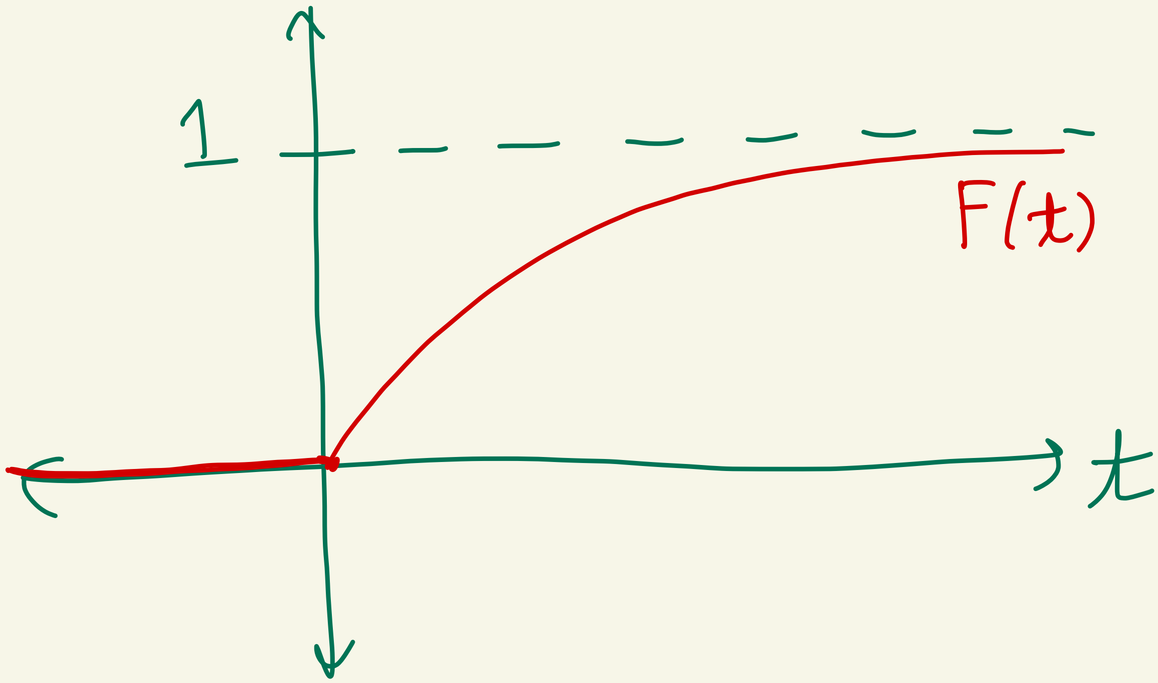
$$= -e^{-\lambda t} - (-e^{0})$$

$$= -e^{-\lambda t} + 1$$

$$= 1 - e^{-\lambda t}$$

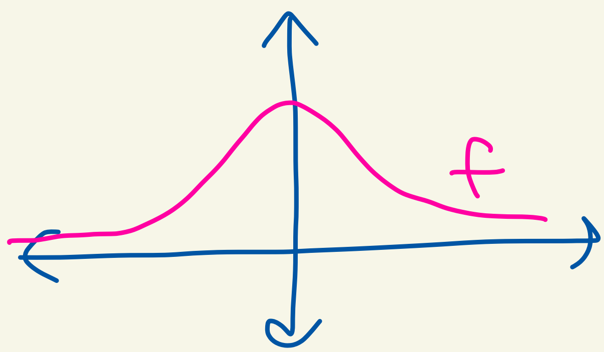
So,

$$F(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 - e^{-\lambda t} & \text{if } 0 \le t \end{cases}$$



<u>Ex:</u> (Standard normal distribution)

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$



$$F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2}\, dx = \underbrace{\Phi(t)}_{\text{topic 7}}$$

__Def:__ Let X be a continuous random variable with pdf f.
Then the __expected value__ of X is

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

Let $\mu = E[X]$.
The __variance__ is

$$\mathrm{Var}(X) = \left(\int_{-\infty}^{\infty} x^2 f(x)\, dx \right) - \mu^2$$

and the __standard deviation__ is

$$\sigma_X = \sqrt{\mathrm{Var}(X)}$$