Math 4740

3/26/25

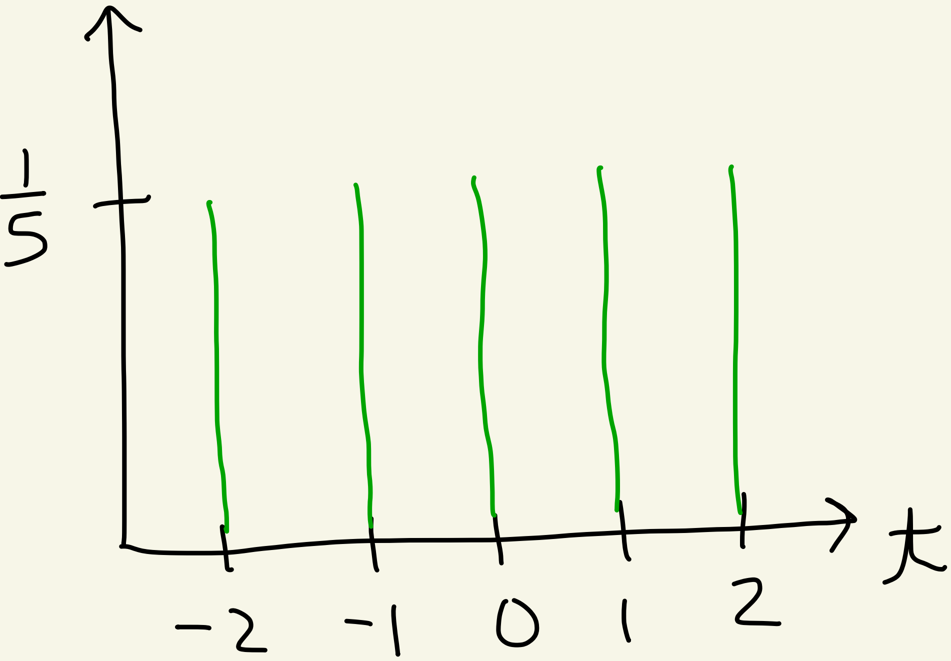
# Topic 6 – Variance, more on expected value

Expected value doesn't tell you how spread out a probability function is.

---

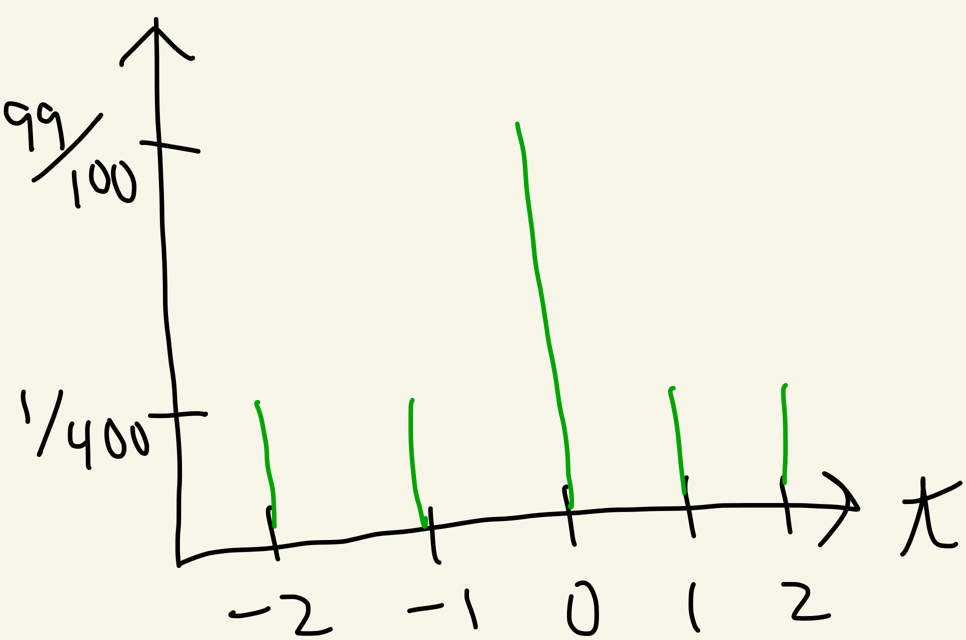
Ex: Let's see two examples of probability functions with the same expected value but that are spread out differently.

$P(X_1 = t)$




$$E[X_1] = (-2)\left(\frac{1}{5}\right) + (-1)\left(\frac{1}{5}\right) + (0)\left(\frac{1}{5}\right) + (1)\left(\frac{1}{5}\right) + (2)\left(\frac{1}{5}\right) = 0$$

$P(X_2 = t)$



$$E[X_2] = (-2)\left(\frac{1}{400}\right) + (-1)\left(\frac{1}{400}\right) + (0)\left(\frac{99}{100}\right) + (1)\left(\frac{1}{400}\right) + 2\left(\frac{1}{400}\right)$$

$$= 0$$

---

The above have the same expected value but are very different. We want a number that measures how "spread out" the graphs are.

---

We need some stuff first.

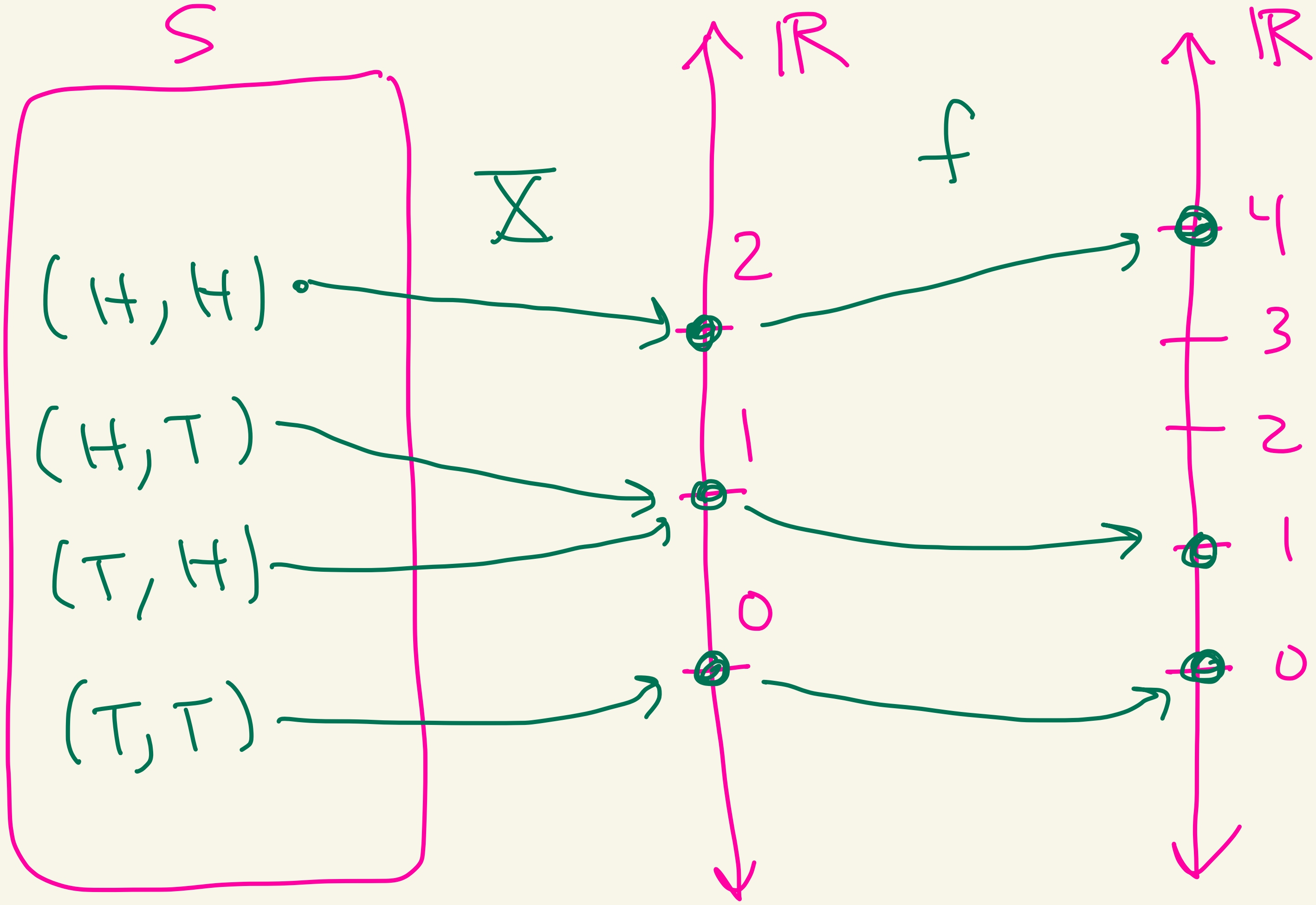
Given a random variable $X : S \longrightarrow \mathbb{R}$, if you have another function $f : \mathbb{R} \to \mathbb{R}$ and compute $f \circ X$

then under appropriate conditions, $f \circ X$ will be a random variable.

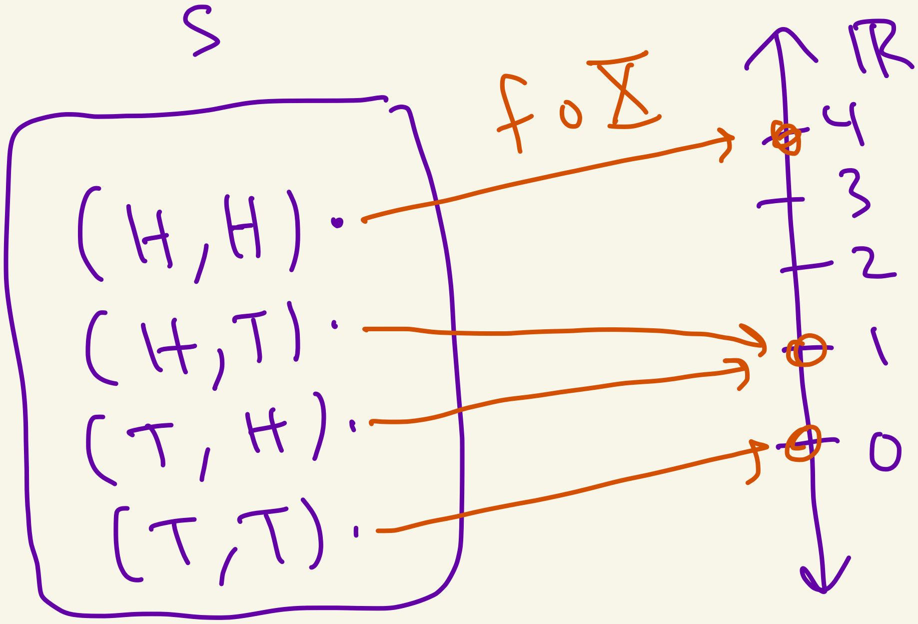
[In our class when $S$ is finite and $\Omega$ is all subsets of $S$, this is an "appropriate" condition]

---

Ex: Suppose we flip two coins. Let $X$ be the number of heads. Let $f : \mathbb{R} \to \mathbb{R}$ be $f(t) = t^2$.




$$(f\circ X)(H,H) = f(X(H,H))$$
$$= f(2) = 2^2 = 4$$




We will write $X^2$ for $f\circ X$

We want a number that measures the average magnitude of the fluctuations of the random variable from it's expected value.

Let $\mu = E[X]$

mu
greek
letter

One might try to measure

$$E[\,|X-\mu|\,].$$

I think people don't use this because formulas aren't good or its hard to work with.

So instead we measure

$$E\left[\underbrace{(X-\mu)}_{\text{square distance}}^2\right]$$

$$|X-\mu| = \sqrt{(X-\mu)^2}$$

$$|X-\mu|^2 = (X-\mu)^2$$

**Def:** Let $X$ be a discrete random variable. Define the <u>variance</u> of $X$ to be

$$\mathrm{Var}(X) = E\left[(X-\mu)^2\right]$$

where $\mu = E[X]$.

Define the <u>standard deviation</u> of $X$ to be

$$\sigma_X = \sigma = \sqrt{\mathrm{Var}(X)}$$

---

**Note:** One can prove (see online notes) that if $x_1, x_2, x_3, \ldots$ are the outputs of $X$ and $f: \mathbb{R} \to \mathbb{R}$, then

$$E[f \circ X] = \sum_i f(x_i) \cdot P(X = x_i)$$

So,

$$\mathrm{Var}(X) = E[(X-\mu)^2]$$

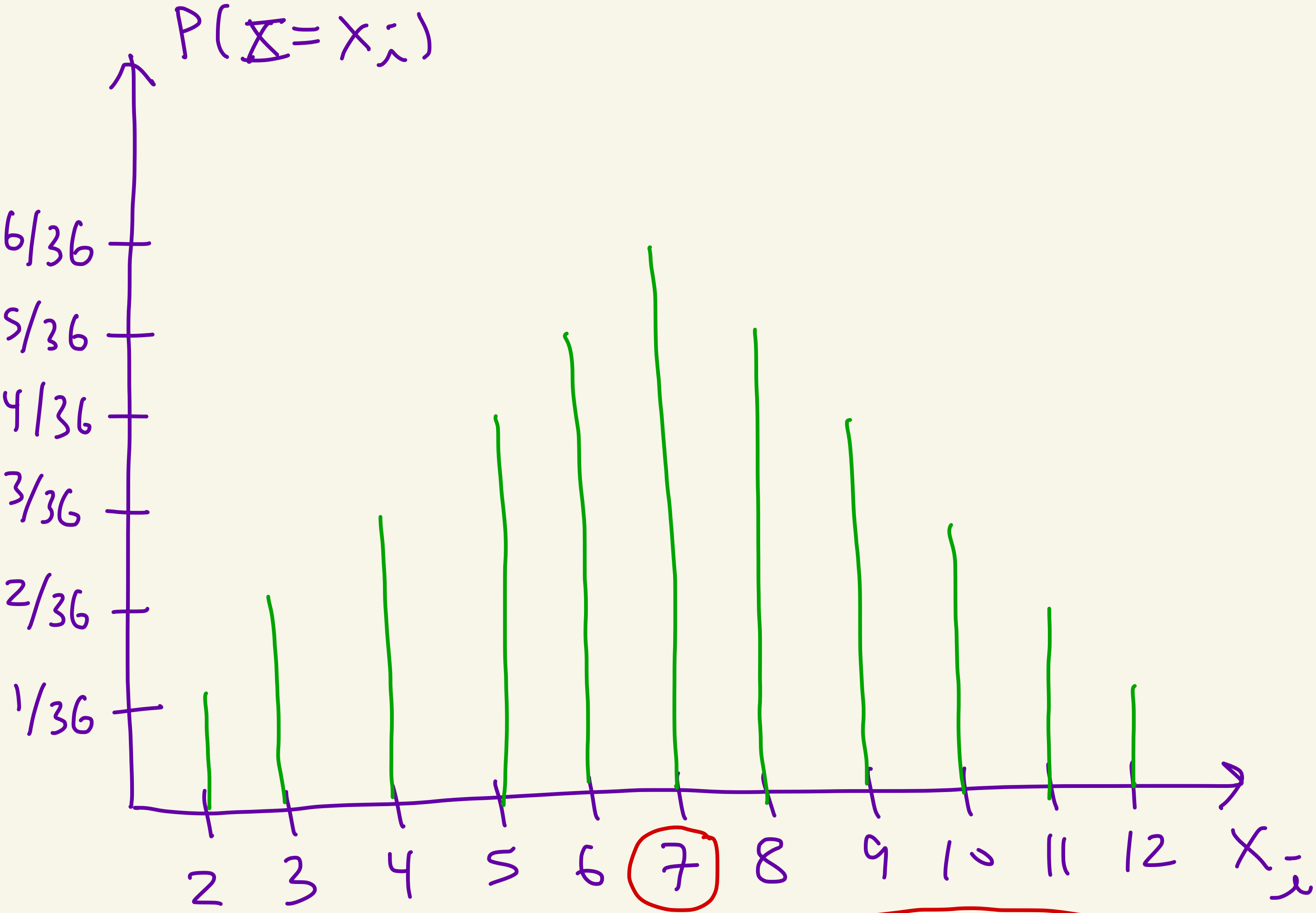
$$= \sum_i (x_i - \mu)^2 \cdot P(X = x_i)$$

$$\boxed{\begin{aligned} f(t) &= (t-\mu)^2 \\ f \circ X &= (X-\mu)^2 \end{aligned}}$$

where $\mu = E[X]$

---

<u>Ex</u>: Consider rolling two 6-sided dice. Let $X$ be the sum of the dice.




$$E[X] = 7 = \mu$$

$$Var(X) = \sum_i (x_i - \mu)^2 \cdot P(X = x_i)$$

$$= (2-7)^2 \cdot \underbrace{\left(\frac{1}{36}\right)}_{P(X=2)} + (3-7)^2 \cdot \underbrace{\left(\frac{2}{36}\right)}_{P(X=3)}$$

$$+(4-7)^2\left(\frac{3}{36}\right)+(5-7)^2\left(\frac{4}{36}\right)$$

$$+(6-7)^2\left(\frac{5}{36}\right)+(7-7)^2\left(\frac{6}{36}\right)$$

$$+(8-7)^2\left(\frac{5}{36}\right)+(9-7)^2\left(\frac{4}{36}\right)$$

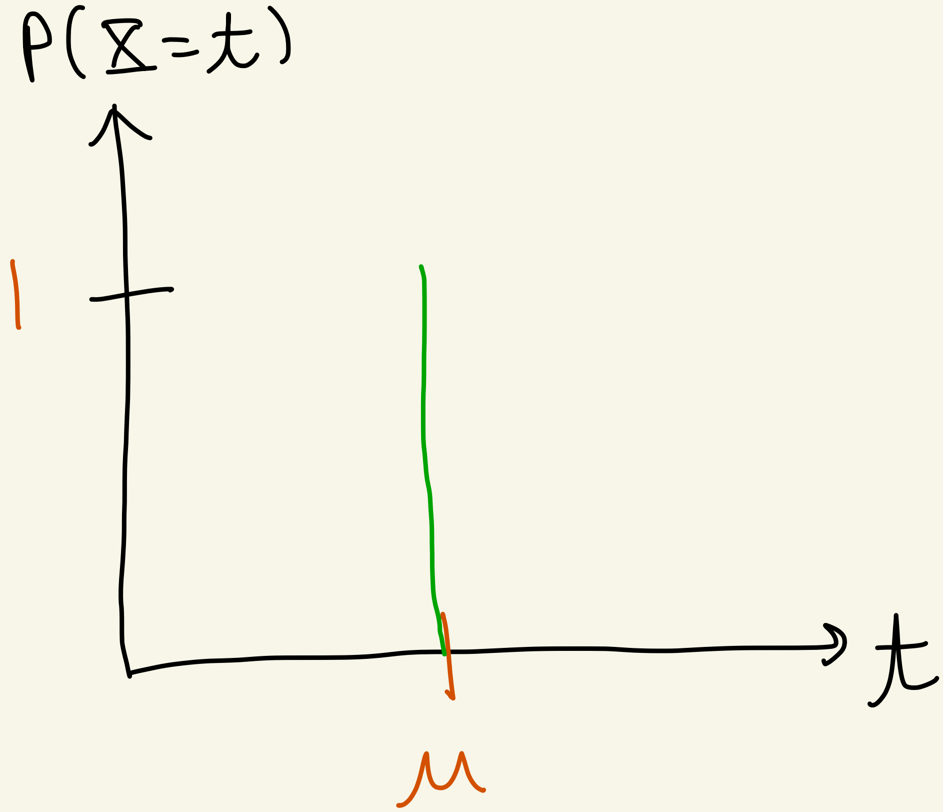
$$+(10-7)^2\left(\frac{3}{36}\right)+(11-7)^2\left(\frac{2}{36}\right)$$

$$+(12-7)^2\left(\frac{1}{36}\right)$$

$$=\boxed{\frac{35}{6}}\approx\boxed{5{,}83}$$

$$\sigma_X=\sqrt{\operatorname{Var}(X)}=\sqrt{\frac{35}{6}}\approx\boxed{2.415}$$

Ex:




$$E[X] = \mu(1) = \mu$$

$$Var(X) = (\mu - \mu)^2 \cdot (1) = 0$$

$$\sigma = \sqrt{Var(X)} = 0$$