Topic 1

Sets and probability spaces

Def: A <u>set</u> is a collection of objects/elements.

If x is an element of a set S then we write $x \in S$.

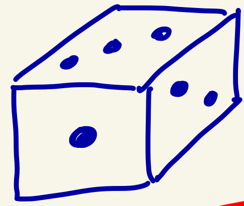
read: "x is in S"

If x is not an element of a set S then we write $x \notin S$.

read: "x is not in S"

If S has a finite number of elements then the size of S is denoted by $|S|$.

Ex: Let's make a set that models rolling a six-sided die.



Let

$S = \{1, 2, 3, 4, 5, 6\}$ ← possible outcomes of rolling a 6-sided die

later we will call S the sample space

We have

$3 \in S$

$8 \notin S$

$|S| = 6$

Note: Order doesn't matter in a set. For example,

$\{1, 2, 3, 4, 5, 6\} = \{2, 6, 5, 1, 3, 4\}$

Note: Sets can't have duplicates

$\{1, 1, 5\}$ is not a set

General way to make a set

$\{$ description of elements in the set $|$ conditions the elements must satisfy to be in the set $\}$

read: "where"
"such that"

Some people use : instead of |

Ex: Let's make a set that models rolling two 6-sided dice, one green and one red.

$$S = \left\{ (g,r) \;\middle|\; \begin{array}{l} g=1,2,3,4,5,6 \\ r=1,2,3,4,5,6 \end{array} \right\}$$

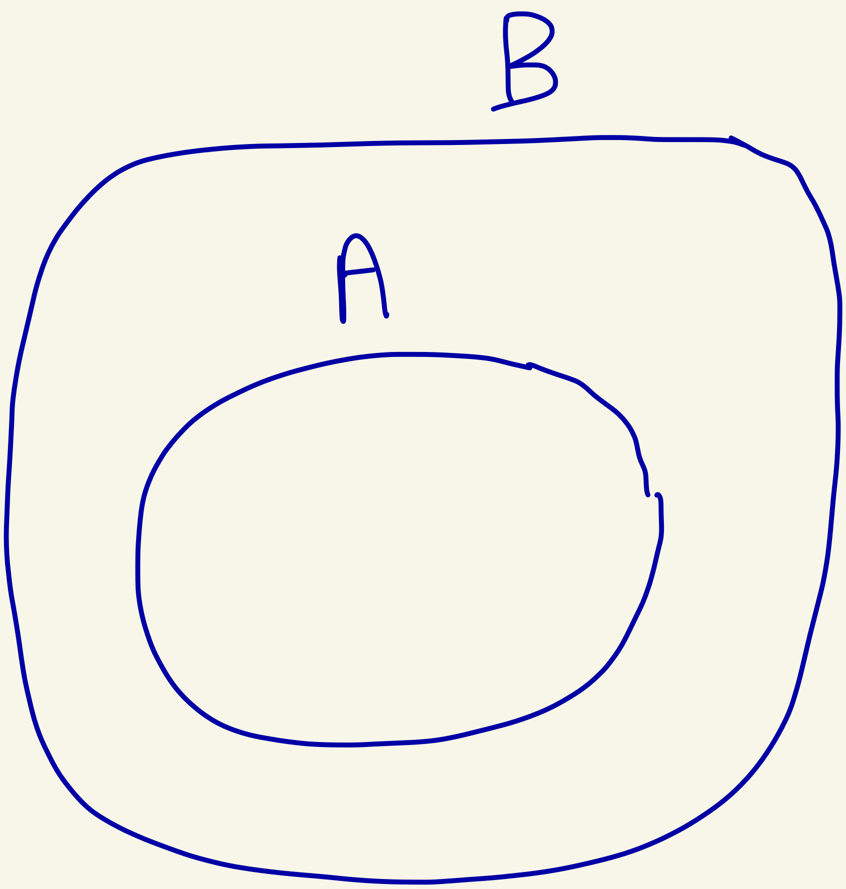
$$\begin{aligned} = \{ & (1,1),(1,2),(1,3),(1,4),(1,5),(1,6), \\ & (2,1),(2,2),(2,3),(2,4),(2,5),(2,6), \\ & (3,1),(3,2),(3,3),(3,4),(3,5),(3,6), \\ & (4,1),(4,2),(4,3),(4,4),(4,5),(4,6), \\ & (5,1),(5,2),(5,3),(5,4),(5,5),(5,6), \\ & (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \} \end{aligned}$$

$(3,4) \leftarrow$ represents green die $= 3$, red die $= 4$

$(4,3) \leftarrow$ represents green die $= 4$, red die $= 3$

Note $|S| = 36$

Def: Let A and B be sets. We say that A is a subset of B if every element of A is also an element of B. We write $A \subseteq B$ if A is a subset of B.



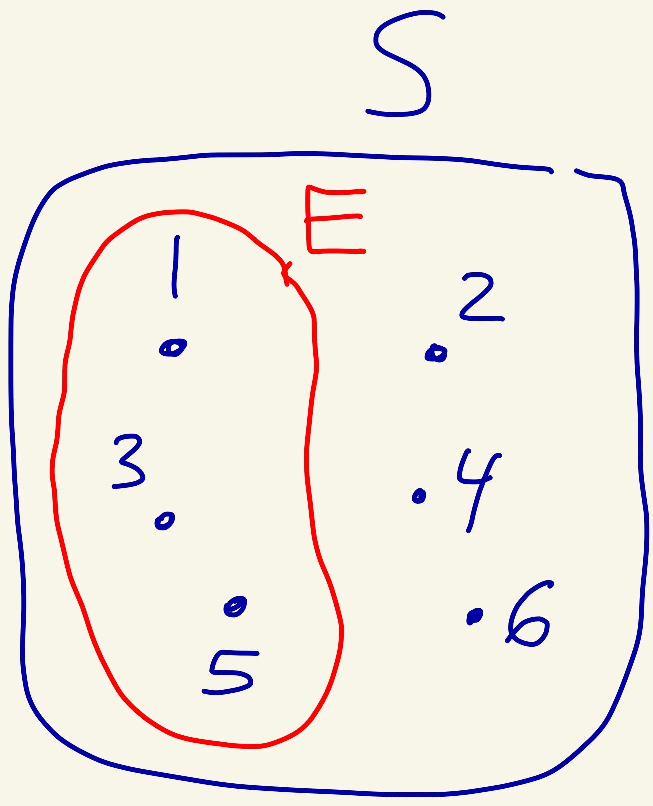
Note: Some people write $A \subset B$

Ex: Consider rolling a 6-sided die.

$S = \{1, 2, 3, 4, 5, 6\}$ ← sample space

$E = \{1, 3, 5\}$

Then $E \subseteq S$.



Later we will call E an event. We will say that E "occured" if when we roll the die either 1, 3, or 5 comes up.

Ex: Suppose we roll two 6-sided dice, one green and one red.

$$\boxed{S = \left\{ (g, r) \;\middle|\; \begin{array}{l} g = 1,2,3,4,5,6 \\ r = 1,2,3,4,5,6 \end{array} \right\}}$$ ← sample space

Let's make a subset where the dice add up to 7.

$$\boxed{E = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}}$$

Here $E \subseteq S$.

Later we will think of E as the event that the two dice add up to 7.

Note $|E| = 6$

$|S| = 36$

Ex: Suppose we flip a coin three times in a row and record each time if we get H = heads or T = tails.

Let's make a sample space to model this experiement.

sample space means all possible outcomes

$$S = \{ (H,H,H), (H,H,T), (H,T,H), (H,T,T), (T,H,H), (T,H,T), (T,T,H), (T,T,T) \}$$

Here (H,T,H) means:

1st flip = H
2nd flip = T
3rd flip = H

We use parenthesis to denote that order matters.

(Same example continued...)

$$E = \{(H,T,T), (T,H,T), (T,T,H)\}$$

This E would represent the event that exactly one H=head occured in the three flips.

Note

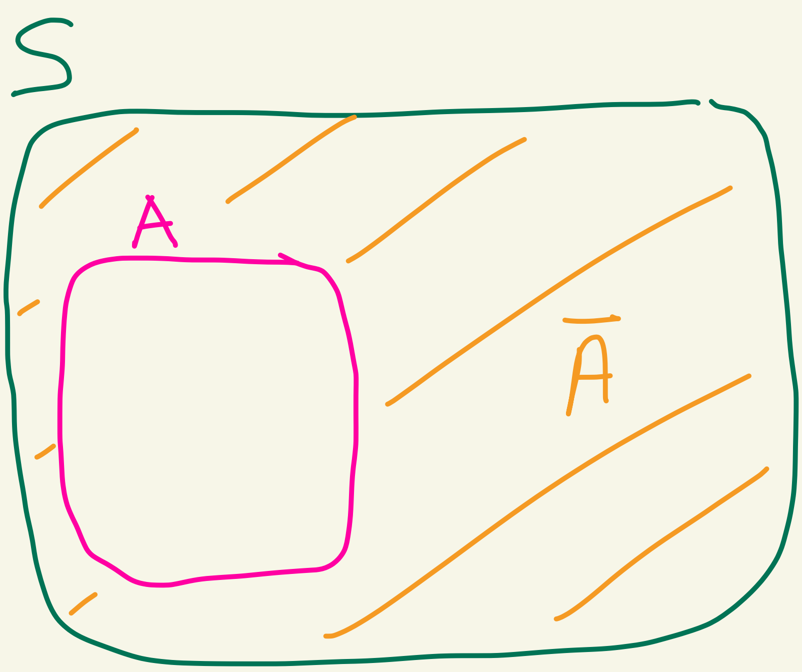
$$|S| = 8$$

$$|E| = 3$$

<u>Def:</u> Suppose S is some set and suppose $A \subseteq S$. The <u>complement</u> of A in S is defined to be

$$\overline{A} = \{ x \mid x \in S \text{ and } x \notin A \}$$

<u>read:</u> $\overline{A}$ consists of all x where x is in S and x is not in A.



Two other notations for $\overline{A}$ are

$$A^c$$

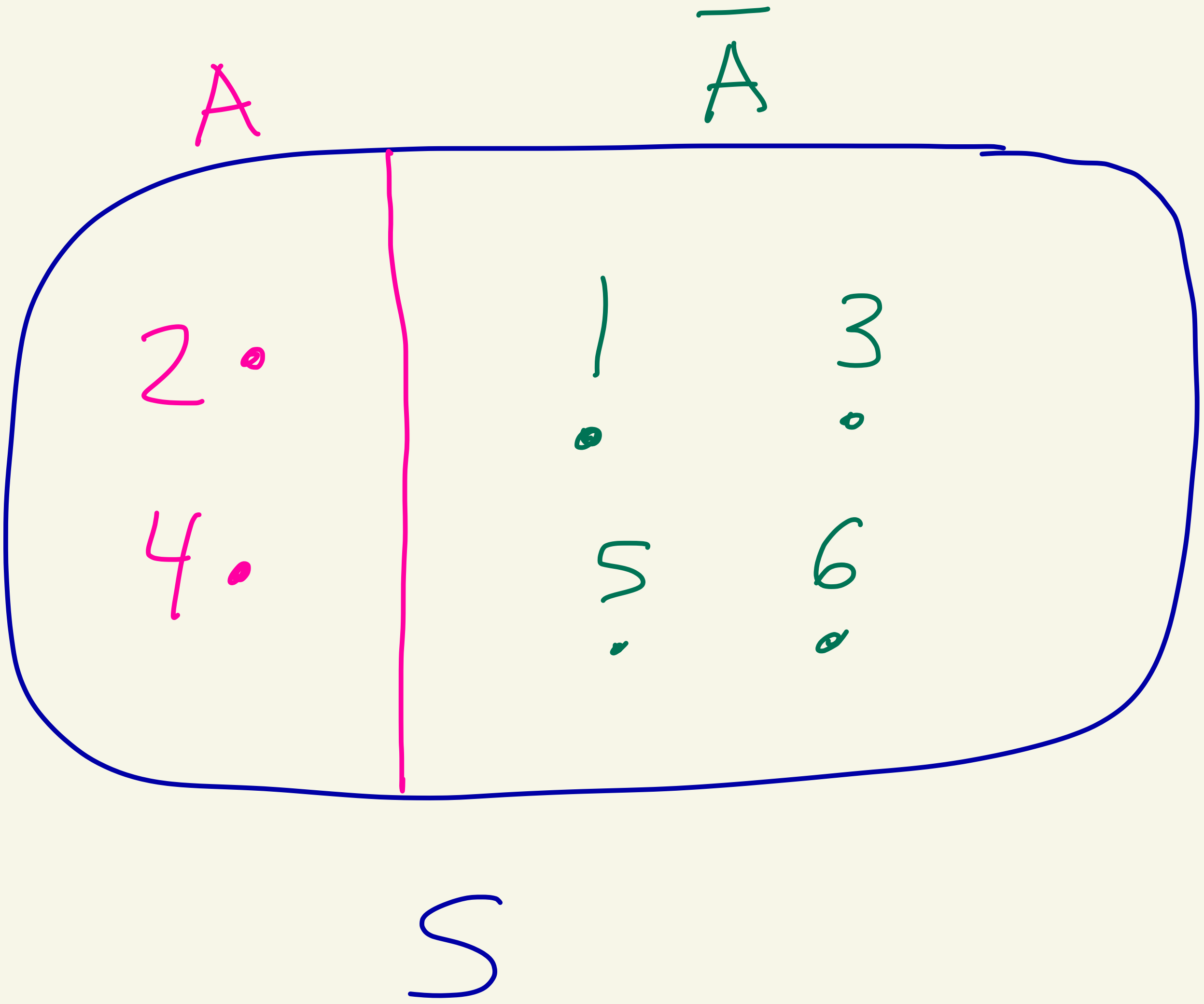
$$S - A$$

Ex: Let

$S = \{1, 2, 3, 4, 5, 6\}$

$A = \{4, 2\}$

$\bar{A} = \{$

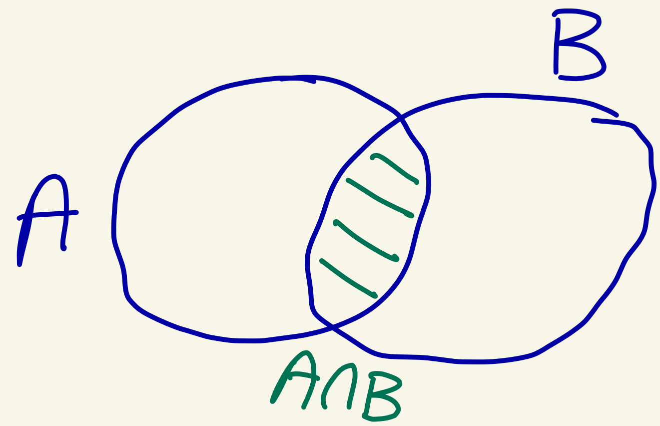


Def: Let A and B be sets.

The <u>intersection</u> of A and B is

$$\underbrace{A \cap B = \{ x \mid x \in A \text{ and } x \in B \}}$$

<u>read:</u> $A \cap B$ consists of all x where x is in A and x is in B

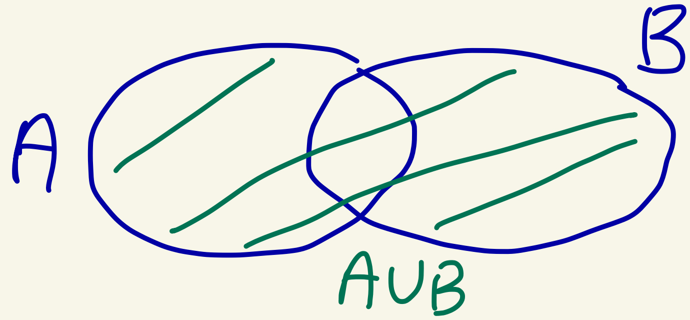


The <u>union</u> of A and B is

$$\underbrace{A \cup B = \{ x \mid x \in A \text{ or } x \in B \}}$$

<u>read:</u> $A \cup B$ consists of all x where x is in A or x is in B.

In math "or" can mean both



<u>Def:</u> The <u>empty set</u> is the set with no elements.

It's denoted by ϕ or $\{ \}$.

Ex: Let S be the sample space we made for flipping a coin three times in a row.

$$S = \{(H,H,H), (H,H,T), (H,T,H), (H,T,T), (T,H,H), (T,H,T), (T,T,H), (T,T,T)\}$$

Let

$$A = \{(H,H,T), (H,H,H), (T,T,H)\}$$

$$B = \{(T,T,T), (T,T,H), (H,H,H), (T,H,T)\}$$

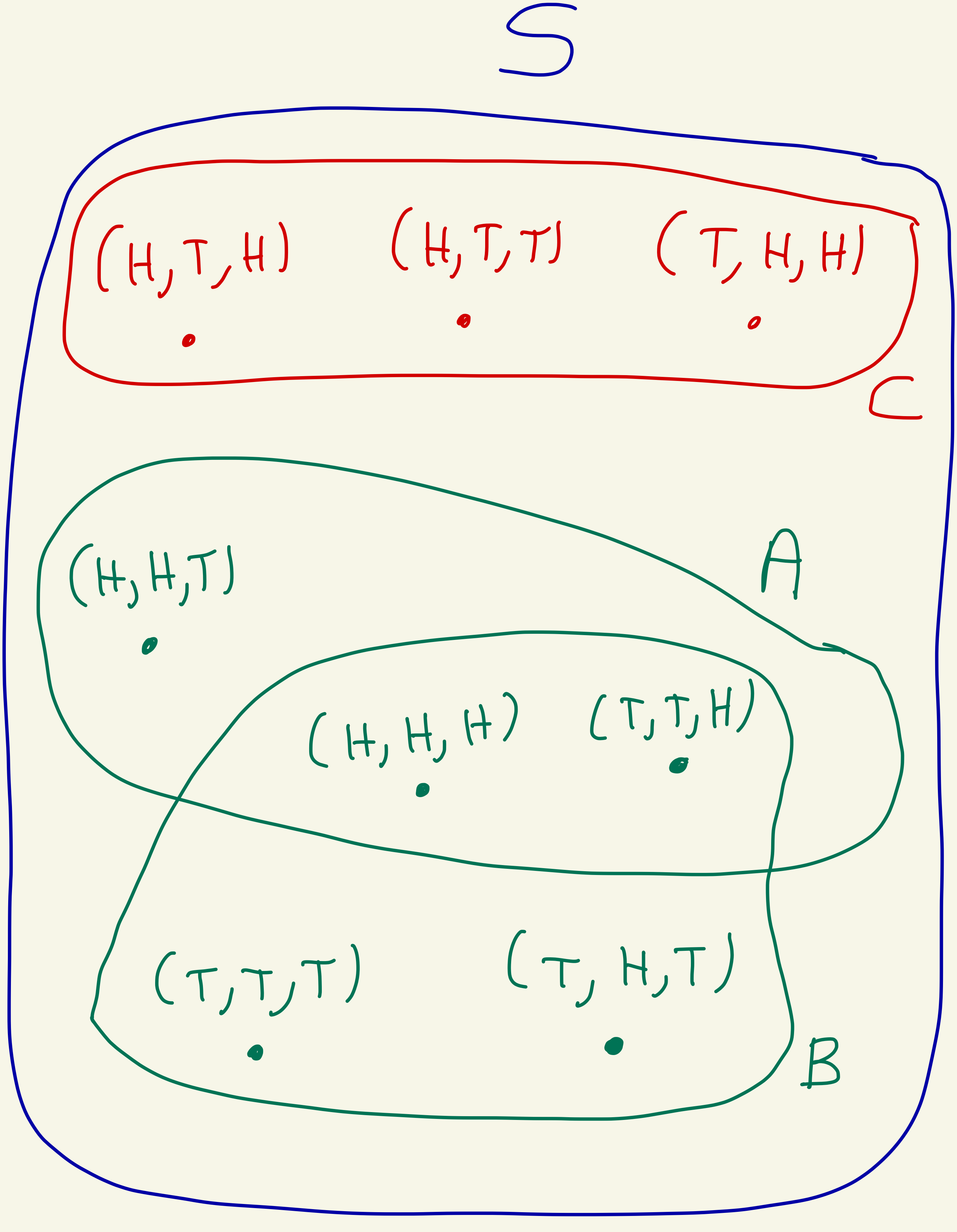
$$C = \{(H,T,H), (H,T,T), (T,H,H)\}$$

Then

$$A \cup B = \{(H,H,T), (H,H,H), (T,T,H), (T,T,T), (T,H,T)\}$$

$$A \cap B = \{(H,H,H), (T,T,H)\}$$

$$A \cap C = \phi \qquad B \cap C = \phi$$


S
(H, T, H)
(H, T, T)
(T, H, H)
C
(H, H, T)
A
(H, H, H)
(T, T, H)
(T, T, T)
(T, H, T)
B

Def: We say that two sets X and Y are **disjoint** if $X \cap Y = \emptyset$

Ex: In the previous example,

- $A \cap C = \emptyset$ so A and C were disjoint
- $B \cap C = \emptyset$ so B and C were disjoint

__Def:__ Let $A_1, A_2, \ldots, A_n$ be sets.

Define

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

$$= \underbrace{\left\{ x \mid x \in A_1 \text{ and } x \in A_2 \text{ and } \cdots \text{ and } x \in A_n \right\}}_{\text{the } x\text{'s that are in } \underline{\text{all}} \text{ the } A_i\text{'s}}$$

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

$$= \underbrace{\left\{ x \mid x \text{ is in at least one of the sets } A_1, A_2, \ldots, A_n \right\}}_{\text{put all the } A_1, A_2, \ldots, A_n \text{ together into one set.}}$$

Ex: Let

$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

this could represent rolling a 12-sided die

dodecahedron

$A_1 = \{1, 2, 3\}$ $\quad A_4 = \{8, 3\}$

$A_2 = \{3, 4, 5\}$

$A_3 = \{5, 6, 7, 4\}$

Then,

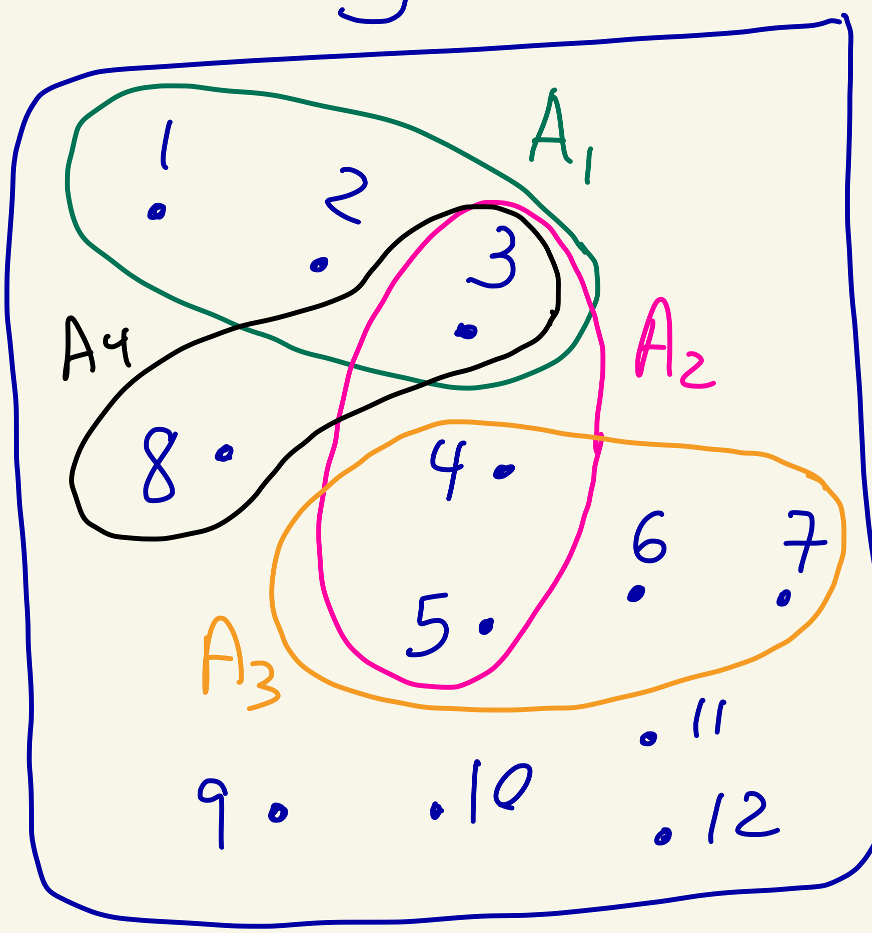
$$\bigcup_{i=1}^{4} A_i = A_1 \cup A_2 \cup A_3 \cup A_4$$
$$= \{1, 2, 3, 4, 5, 6, 7, 8\}$$

$$A_1 \cup A_2 \cup A_4$$
$$= \{1, 2, 3, 4, 5, 8\}$$

$$\bigcap_{i=1}^{4} A_i = A_1 \cap A_2 \cap A_3 \cap A_4$$
$$= \phi$$

$$A_1 \cap A_2 \cap A_4 = \{3\}$$

S



Def: Suppose we have an infinite number of sets $A_1, A_2, A_3, \ldots$

Define

$$\bigcap_{i=1}^{\infty} A_i = \{ x \mid x \text{ is in every one of the } A_i \}$$

$$\bigcup_{i=1}^{\infty} A_i = \{ x \mid x \text{ is in at least one of the } A_i \}$$

Ex: Let

$$S = \mathbb{Z}$$

$\mathbb{Z}$ is the set of integers

$$= \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

For $i \geq 1$, define

$$A_i = \left\{ n \;\middle|\; \begin{array}{l} n \text{ is an integer} \\ -i \leq n \leq i \end{array} \right\}$$

$$= \{-i, \ldots, 0, \ldots, i\}$$

For example,

$$A_1 = \{-1, 0, 1\}$$

$$A_2 = \{-2, -1, 0, 1, 2\}$$

$$A_3 = \{-3, -2, -1, 0, 1, 2, 3\}$$

$$A_4 = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$$

Then,

$$\bigcap_{i=1}^{\infty} A_i = \{-1, 0, 1\}$$

$$\bigcup_{i=1}^{\infty} A_i = \mathbb{Z}$$

Def: Let A and B be two sets. The **Cartesian product** of A and B is

$$\underbrace{A \times B}_{\text{read: "A cross B"}} = \underbrace{\left\{ (a,b) \;\middle|\; \begin{array}{l} a \text{ is in } A \\ b \text{ is in } B \end{array} \right\}}_{\text{all elements of the form } (a,b) \text{ where } a \in A \text{ and } b \in B}$$

Ex: Let $A = \{H, T\}$
$B = \{1, 2, 3, 4\}$

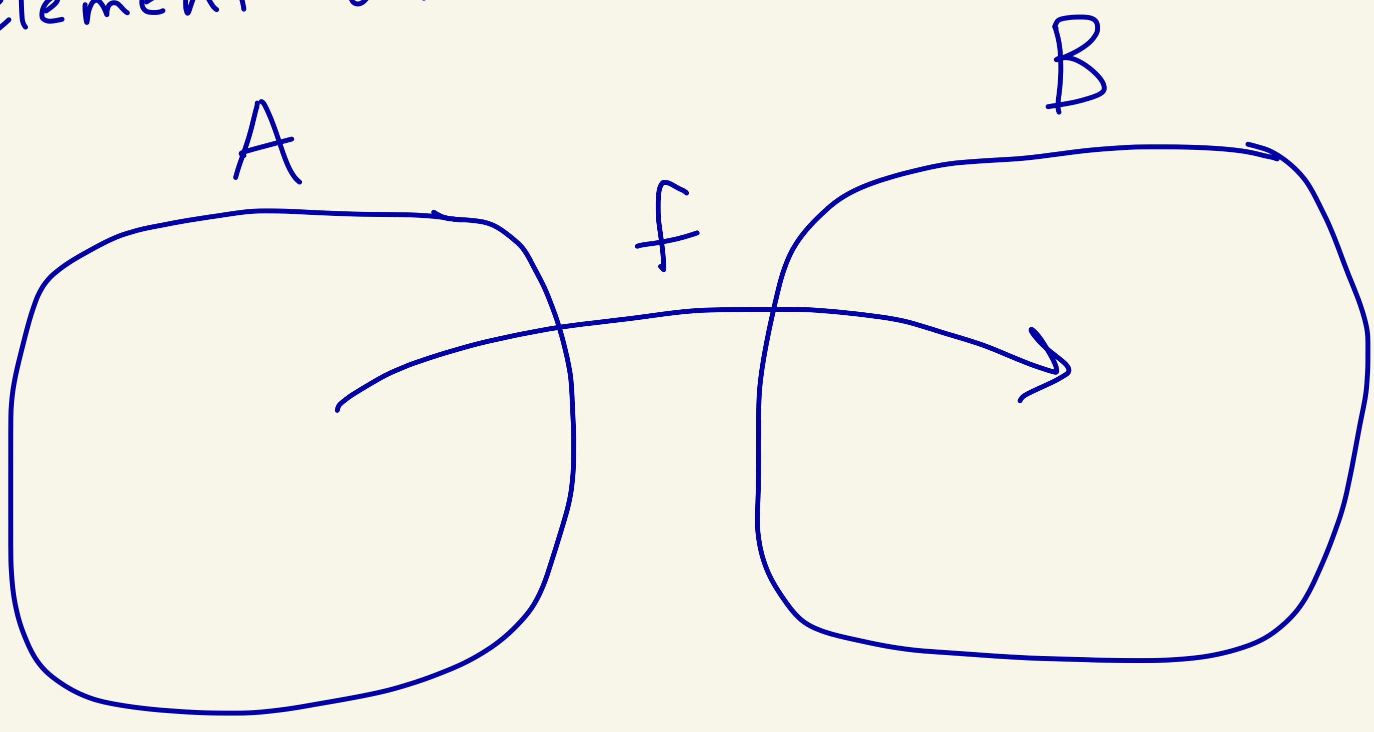
Then,

$$A \times B = \{ (H,1), (H,2), (H,3), (H,4), (T,1), (T,2), (T,3), (T,4) \}$$

$$A \times A = \{ (H,H), (H,T), (T,H), (T,T) \}$$

$$B \times B = \{ (1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4) \}$$

Def: Let A and B be sets.
A **function** f from A to B,
notated $f: A \longrightarrow B$, is
a rule that assigns to each
element of A a distinct
element of B



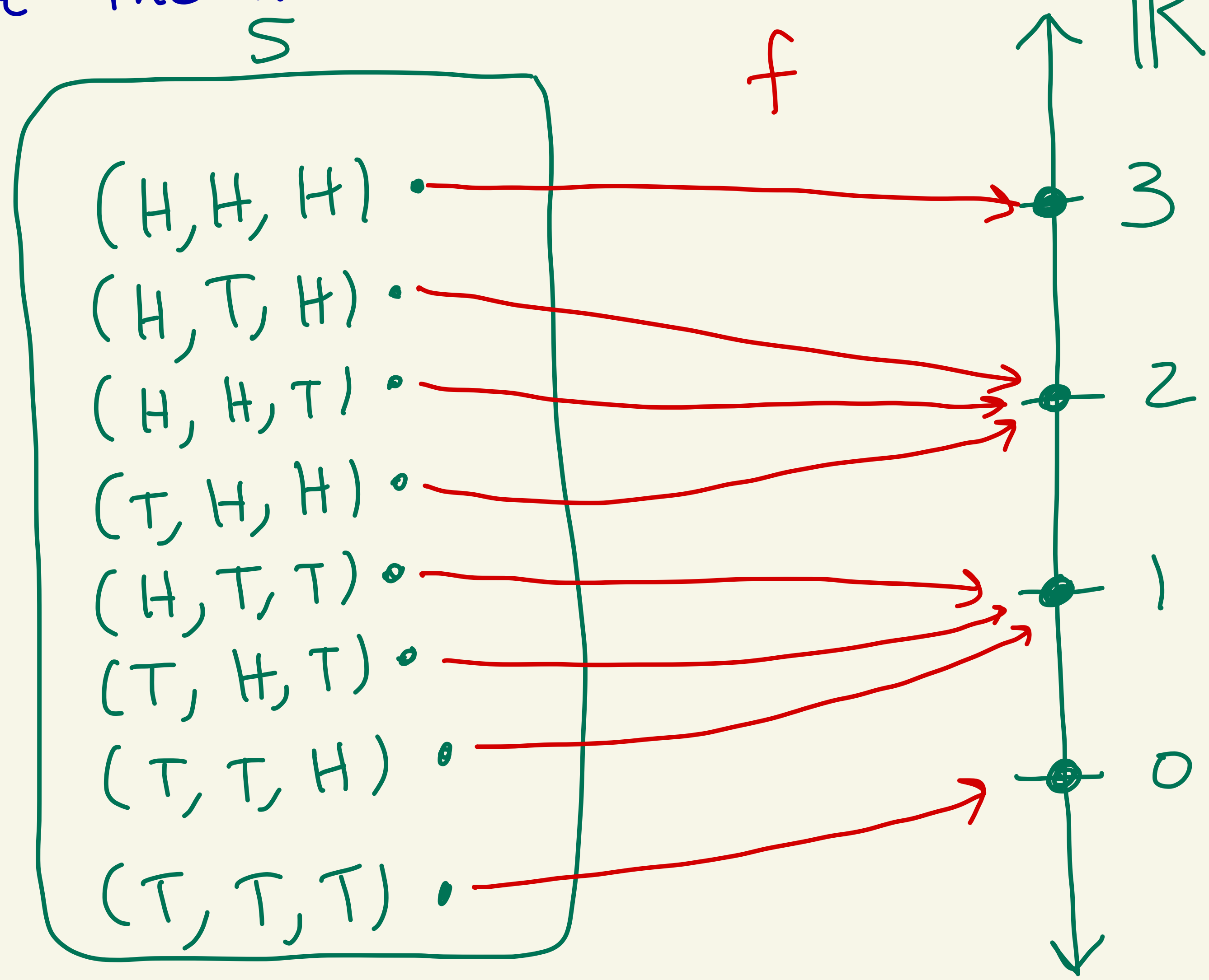
Ex: Let

$S = \{(H,H,H), (H,T,H), (H,H,T), (H,T,T), (T,H,H), (T,T,H), (T,H,T), (T,T,T)\}$

be the sample space of flipping a coin 3 times.

[$\mathbb{R}$ means set of real numbers]

Let $f : S \to \mathbb{R}$ be the number of heads that occur.



For example, $f((H,T,H)) = 2$

Example of making a probability space

Suppose we want to model the experiment of throwing/rolling one 4-side die.

$$S = \{1, 2, 3, 4\}$$

sample space
all possible of outcomes rolling the die

Omega

$$\Omega = \{\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{2,3,4\}, \{1,2,4\}, \{1,3,4\}, \{1,2,3,4\}\}$$

Ω is called the set of events
It's a set of subsets of S with special properties

Ω contains the sets that we measure the probability of

When S is finite we usually make Ω contain all the subsets of S.

What do these events mean?

ϕ ← represents that no number came up on the die

$\{3\}$ ← represents 3 came up on the die

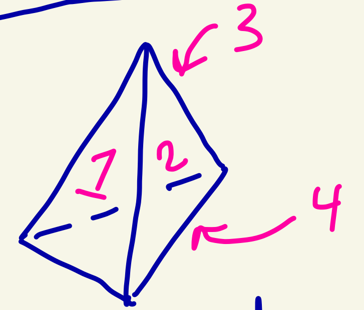
$\{1, 3\}$ ← represents 1 or 3 came up on the die

$\{2, 3, 4\}$ ← represents 2 or 3 or 4 came up on the die

$\{1, 2, 3, 4\}$ ← represents 1 or 2 or 3 or 4 came up on the die

Now we make the probability function $P : \Omega \to \mathbb{R}$.

On a normal 4-sided die each side is equally likely to occur.



4 sided die

First step is to assign the probability of each number/side individually.

$$
\left.
\begin{aligned}
P(\{1\}) &= \tfrac{1}{4} \\
P(\{2\}) &= \tfrac{1}{4} \\
P(\{3\}) &= \tfrac{1}{4} \\
P(\{4\}) &= \tfrac{1}{4}
\end{aligned}
\right\}
$$

each side is equally likely

Now we extend P across all the events by doing disjoint sums, for example, define

$$P(\{1,3\}) = P(\{1\}) + P(\{3\}) = \frac{1}{4} + \frac{1}{4}$$
$$= 1/2$$

What's the probability of not rolling a 1?

$$P(\{2,3,4\}) = P(\{2\}) + P(\{3\}) + P(\{4\})$$
$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \boxed{\frac{3}{4}}$$

$\boxed{\begin{array}{c} .75 \\ 75\% \end{array}}$

We define

$$P(\emptyset) = 0$$

We have

$$P(\{1,2,3,4\}) = P(\{1\}) + P(\{2\}) + P(\{3\}) + P(\{4\})$$
$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \boxed{1}$$

$\boxed{100\%}$

<u>Def:</u> A <u>probability space</u> consists of two sets and a function (S, Ω, P). We call S the <u>sample space</u> of our experiment. The elements of S are called <u>outcomes.</u> Ω is a set of subsets of S. The elements of Ω are called <u>events.</u> $P : \Omega \to \mathbb{R}$ is a function where for each event E from Ω we get a probability $P(E)$ of the event E.

Furthermore, the following axioms must be satisfied:

① S is an event in Ω

② If E is an event in Ω then $\overline{E}$ is an event in Ω

$\overline{E}$ means the comple- -ment of E in S

③ If $E_1, E_2, E_3, \ldots$ is a finite or infinite sequence of events in Ω, then $\bigcup_i E_i$ is an event in Ω.

Andrey Kolmogorov gave this def in the 1930s.

④ $0 \leq P(E) \leq 1$ for all events E in Ω

⑤ $P(S) = 1$

⑥ If $E_1, E_2, E_3, \ldots$ is a finite or infinite sequence of events in Ω that are pair-wise disjoint [that is, $E_i \cap E_j = \phi$ if $i \neq j$]

then $P\left(\bigcup_i E_i\right) = \sum_i P(E_i)$

disjoint means no overlap

end of definition

Remark: A set Ω satisfying ①, ②, and ③ from the previous definition is called a σ-algebra or σ-field.

σ
sigma

Remark: If Ω is a σ-algebra one can show that

(a) $\phi \in \Omega$

(b) If $E_1, E_2, E_3, \ldots$ are in Ω, then $\bigcap_i E_i$ is in Ω

pf: [Skip in class - reference notes]

(a) $S \in \Omega$ by ①.
Thus, by ② $\overline{S} = \phi$ is in Ω.

(b) Suppose $E_1, E_2, E_3, \ldots$ are in Ω.

By part ②, $\overline{E_1}, \overline{E_2}, \overline{E_3}, \ldots$ are in Ω.

By part ③, $\bigcup_i \overline{E_i}$ is in Ω.

By part ②, $\overline{\bigcup_i \overline{E_i}}$ is in Ω.

But,

$$\bigcap_i E_i = \overline{\bigcup_i \overline{E_i}}$$

■

How to construct a $\boxed{\text{finite}}$ probability space

ie S is finite

Suppose S is a finite sample space that we want to make into a probability space.

Define Ω to be the set that contains all the subsets of S [includes $\emptyset$].

Ω = power set of S

For each element $\omega \in S$ pick some real number n_ω with $0 \leq n_\omega \leq 1$ and define

$$P(\{\omega\}) = n_\omega$$

n_ω is probability of ω happening

At the same time pick these numbers so that

$$\underbrace{\sum_{\omega \in S}}_{\text{means sum over all } \omega \text{ in } S} n_\omega = 1.$$

Ex: $S = \{1, 2, 3, 4\}$

$P(\{1\}) = n_1 = \frac{1}{4}$

$P(\{2\}) = n_2 = \frac{1}{4}$

$P(\{3\}) = n_3 = 1/4$

$P(\{4\}) = n_4 = 1/4$

$n_1 + n_2 + n_3 + n_4 = 1$

Now extend P to any set E in Ω.

Suppose $E = \{w_1, w_2, \ldots, w_n\}$

Define

$$\underbrace{P(E) = \sum_{i=1}^{n} P(\{w_i\})}_{\text{define } P(E) \text{ to be the sum of the probabilities of the elements of } E}$$

If $E = \phi$, define

$$P(\phi) = 0.$$

<u>Theorem:</u> The construction above creates a probability space (S, Ω, P).

<u>proof:</u> [Skip in class - reference notes]

We first prove axioms 4, 5, and 6.

4: Let E be an event from Ω.

Then, $$0 \leq \sum_{w \in E} P(\{w\}) \leq \sum_{w \in S} P(\{w\}) = 1$$

since $P(\{w\}) \geq 0$ for all $w \in S$

$E \subseteq S$

because of how P was defined

5: By the definition of P we get
$$P(S) = \sum_{w \in S} P(\{w\}) = 1.$$

6: Let
$$\begin{aligned} E_1 &= \{w_{11}, w_{12}, \ldots, w_{1n_1}\}, \\ E_2 &= \{w_{21}, w_{22}, \ldots, w_{2n_2}\}, \\ &\vdots \\ E_k &= \{w_{k1}, w_{k2}, \ldots, w_{kn_k}\} \end{aligned}$$

where $E_i \cap E_j = \phi$ if $i \neq j$.

Then $w_{il} \neq w_{jm}$ if $i \neq j$ and $l \neq m$.

So,
$$P\left(\bigcup_{i=1}^{k} E_i\right) = P\left(\{w_{11}, \ldots, w_{1n_1}, w_{21}, \ldots, w_{2n_2}, \ldots, w_{k1}, \ldots, w_{kn_k}\}\right)$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{n_i} P(\{w_{ij}\})$$

$$= \sum_{i=1}^{k} P(E_i)$$

Now we show axioms ①, ②, and ③.

Recall that Ω consists of all subsets of S.

① $S \subseteq S$ and so $S \in \Omega$.

② Suppose $E \in \Omega$.

Then $E \subseteq S$.

Thus, $\overline{E} = S - E \subseteq S$.

So, $\overline{E} \in \Omega$.

③ Let $E_1, E_2, E_3, \ldots$ are in Ω.

Then, $E_1 \subseteq S$, $E_2 \subseteq S$, $E_3 \subseteq S, \ldots$

Thus, $\bigcup_i E_i \subseteq S$.

Since ①, ②, ③, ④, ⑤, ⑥ are true we have that (S, Ω, P) is a probability space. $\boxed{}$

Ex: Suppose you have a six-sided die labeled 1, 2, 3, 4, 5, 6 and through experimentation you noticed it was a weighted die and the probabilities were roughly

number	probability
1	$2/8$
2	$1/8$
3	$1/8$
4	$1/16$
5	$1/16$
6	$3/8$

notice

$$\frac{2}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16}+\frac{3}{8}=1$$

Let's make a probability space.

Define $S = \{1, 2, 3, 4, 5, 6\}$

Define $\Omega = \{\text{all subsets of } S\}$

$= \{\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1,2\}, \{1,3\}, \ldots, \{1,2,3,4,5,6\}\}$

Ω = power set of S

Define $P: \Omega \to \mathbb{R}$ by

$P(\{1\}) = 2/8$ $\quad$ $P(\{4\}) = 1/16$

$P(\{2\}) = 1/8$ $\quad$ $P(\{5\}) = 1/16$

$P(\{3\}) = 1/8$ $\quad$ $P(\{6\}) = 3/8$

If E is an event in Ω we define $P(E) = \sum_{\omega \in E} P(\{\omega\})$

and $P(\phi) = 0$.

Note

$$\begin{aligned} P(S) &= P(\{1\}) + P(\{2\}) + P(\{3\}) + P(\{4\}) + P(\{5\}) + P(\{6\}) \\ &= 2/8 + 1/8 + 1/8 + \frac{1}{16} + \frac{1}{16} + \frac{3}{8} = 1 \end{aligned}$$

What is the probability of rolling an even number?

$$\underbrace{P(\{2,4,6\})}_{\text{probability of of rolling 2 or 4 or 6}} = P(\{2\}) + P(\{4\}) + P(\{6\})$$

$$= \frac{1}{8} + \frac{1}{16} + \frac{3}{8}$$

$$= 9/16 \approx 0.5625$$

What is the probability of rolling 1 or 6 ?

$$P(\{1, 6\}) = P(\{1\}) + P(\{6\})$$

$$= \frac{2}{8} + \frac{3}{8} = \frac{5}{8}$$

$\boxed{\text{Note:}}$ (Maybe skip in class & mention it's in notes)

You can construct a probability space when S is countably infinite, ie S is infinite and you can list the elements.

Suppose $S = \{w_1, w_2, w_3, w_4, \ldots\}$

infinitely many more

Define Ω to be the set of all subsets of S, ie the power set of S.

Define $P(\{w_i\})$ for each $w \in S$ so that $0 \leq P(\{w_i\}) \leq 1$ and $\sum_{i=1}^{\infty} P(\{w_i\}) = 1$.

If E is an event define

$$P(E) = \sum_{w \in E} P(\{w\})$$

Theorem: This will be a probability space.

Note:

Suppose (S, Ω, P) is a probability space and S is finite.

Suppose each outcome w in S is equally weighted, that is

$$P(\{w\}) = \frac{1}{|S|} \quad \text{for all } w \text{ in } S.$$

If this is the case, its easy to calculate the probability of an event E.

Suppose $E = \{w_1, w_2, \ldots, w_n\}$ has n elements.

Then,

$$P(E) = P(\{w_1\}) + P(\{w_2\}) + \ldots + P(\{w_n\})$$
$$= \frac{1}{|S|} + \frac{1}{|S|} + \ldots + \frac{1}{|S|}$$
$$= \frac{n}{|S|} = \frac{|E|}{|S|}.$$

So, $\boxed{P(E) = \frac{|E|}{|S|}}$

Ex: Suppose we do the experiment of rolling two 6-sided dice. Suppose these are normal dice so each side has equal chance of happening.

$(a,b) \leftarrow$ denote a on die 1 and b on die 2

$$S = \{ (1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,6),(2,1),(2,2),(2,3),(2,4),(2,5),(3,6),(3,1),(3,2),(3,3),(3,4),(3,5),(4,6),(4,1),(4,2),(4,3),(4,4),(4,5),(5,6),(5,1),(5,2),(5,3),(5,4),(5,5),(6,6),(6,1),(6,2),(6,3),(6,4),(6,5) \}$$

$\Omega = \{$all subsets of $S\}$

Each outcome is equally likely.

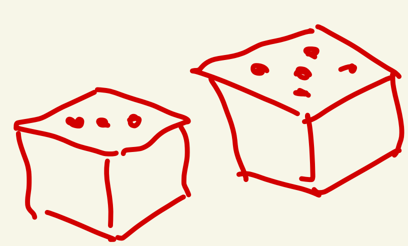
We have $|S| = 36$.

So, $P(\{(a,b)\}) = \dfrac{1}{36}$ for any a,b.

For example, $P(\{(3,5)\}) = \dfrac{1}{36}$

3 on first die

5 on 2nd die



Q: What is the probability that the sum of the dice equals 7?

Let E be the event that the sum of the dice is 7.

Then,

$$E = \{(6,1), (5,2), (4,3), (3,4), (2,5), (1,6)\}$$

6 on die 1

1 on die 2

$6+1=7$

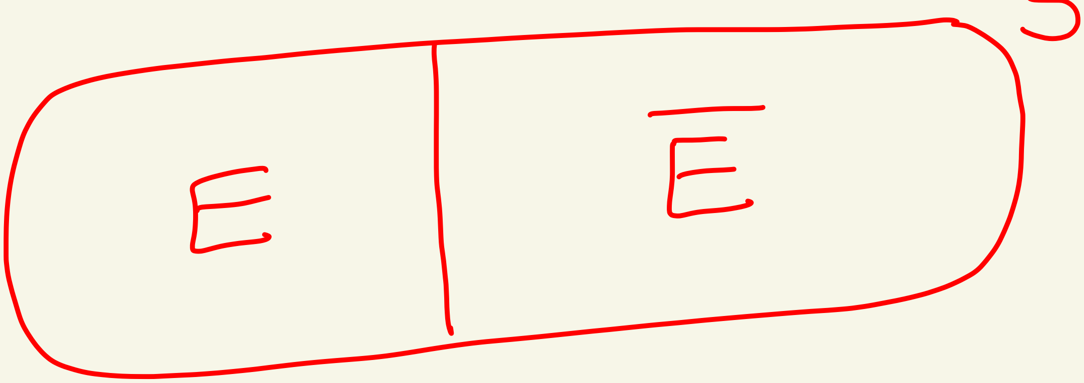
Since every outcome is equally weighted

$$P(E) = \frac{|E|}{|S|} = \frac{6}{36} = \frac{1}{6}$$

$$\approx 0.16\overline{6}$$

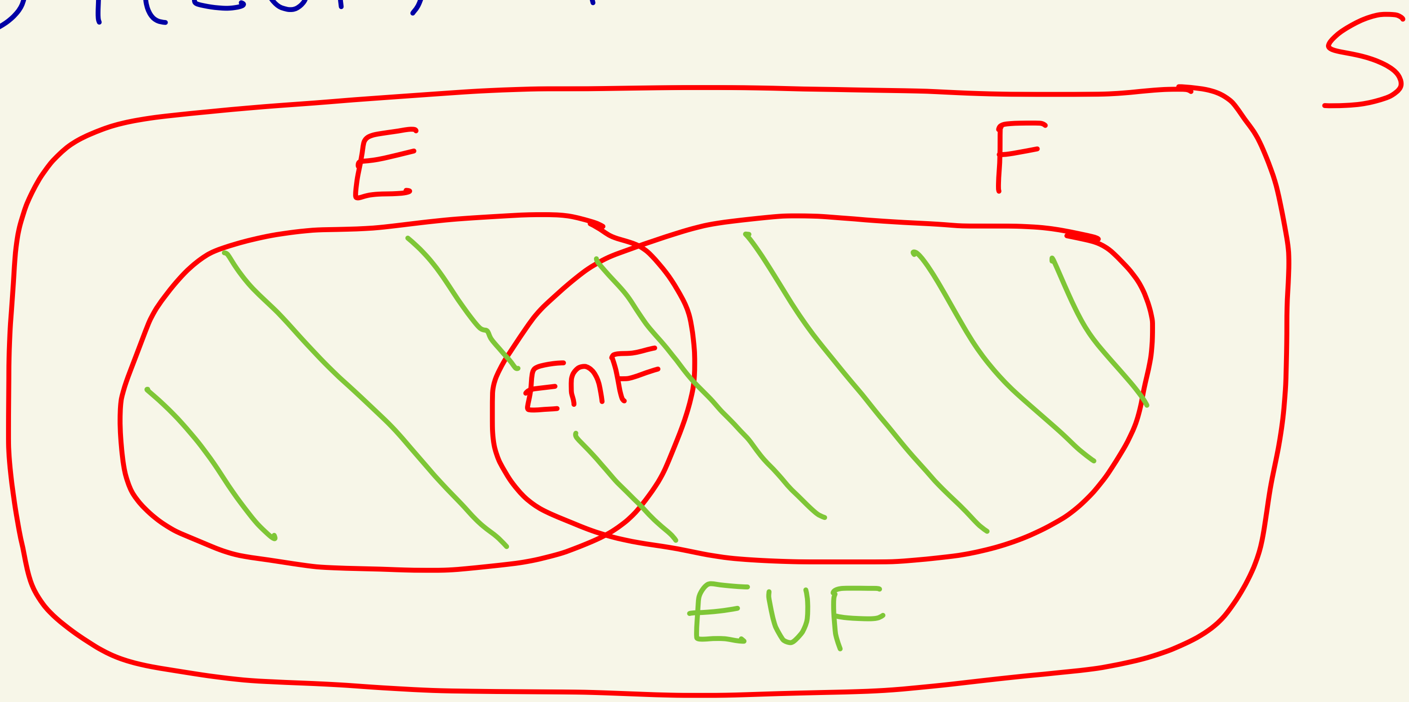
Theorem: Let (S, Ω, P) be a probability space. Let E and F be events. Then

① $P(\overline{E}) = 1 - P(E)$



② If $E \subseteq F$, then $P(E) \leq P(F)$.

(3) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$



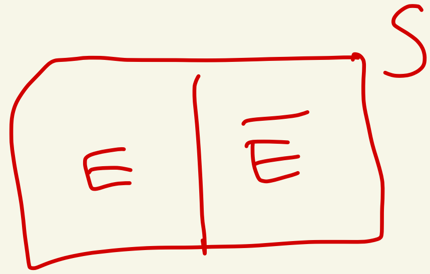
(4) If E and F are disjoint, ie $E \cap F = \phi$, then

$P(E \cup F) = P(E) + P(F)$

axiom 6 of prob. space

Proof: [Skip in class - mention to see notes]

(1) We know that $S = E \cup \bar{E}$ and that $E \cap \bar{E} = \phi$



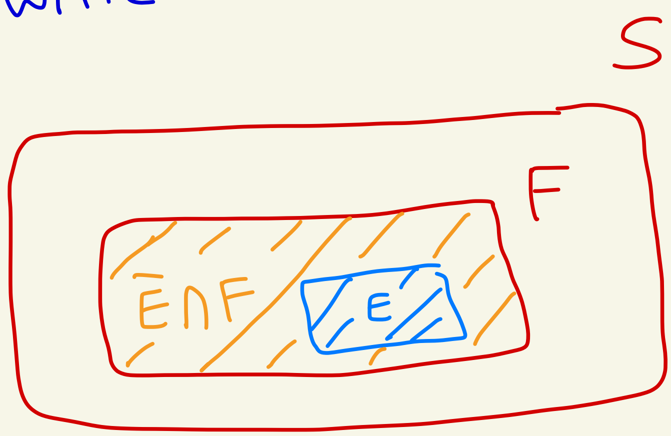
So,

$$1 \underset{\text{axiom 5}}{=} P(S) = P(E \cup \bar{E}) \underset{\text{axiom 6}}{=} P(E) + P(\bar{E}).$$

So, $P(\bar{E}) = 1 - P(E)$

② Since $E \subseteq F$ we can write $F = E \cup (\bar{E} \cap F)$

And E and $\bar{E} \cap F$ are disjoint.



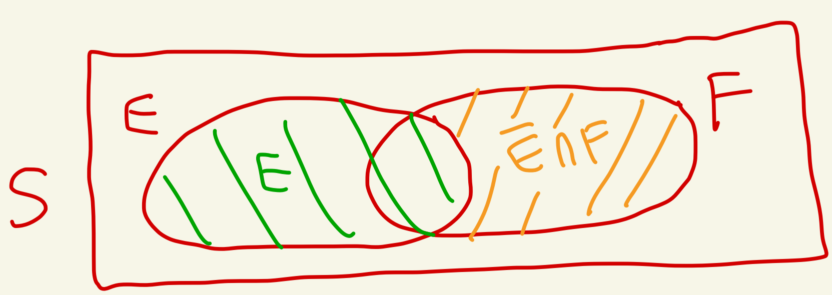
Thus,

$$P(F) = P(E \cup (\bar{E} \cap F))$$

$$\underset{\text{axiom 5}}{=} P(E) + \underbrace{P(\bar{E} \cap F)}_{\geq 0} \geq P(E).$$

So, $P(E) \leq P(F)$.

③ Note that $E \cup F = E \cup (\bar{E} \cap F)$

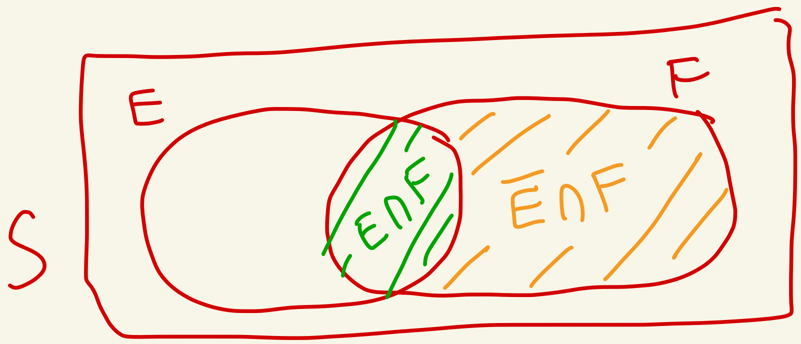


Also, E and $\bar{E} \cap F$ are disjoint.

Thus, by axiom ⑤ we get that

$$P(E \cup F) = P(E) + P(\bar{E} \cap F)$$

Furthermore, $F = (E \cap F) \cup (\bar{E} \cap F)$



And, $E \cap F$ and $\bar{E} \cap F$ are disjoint

Hence by axiom ⑤ we have that

$$P(F) = P(E \cap F) + P(\bar{E} \cap F).$$

Thus,

$$P(\bar{E} \cap F) = P(F) - P(E \cap F).$$

Therefore,

$$\begin{aligned} P(E \cup F) &= P(E) + P(\bar{E} \cap F) \\ &= P(E) + P(F) - P(E \cap F) \end{aligned}$$

$\blacksquare$

Ex: Suppose we roll two 12-sided dice. [Each number on the die are equally likely]. What is the probability that at least one of the dice is 4, 5, 6, 7, 8, 9, 10, 11, or 12?

Examples:

die 1	die 2	
$\boxed{3}$	$\boxed{7}$	have at least one of 4–12
$\boxed{8}$	$\boxed{9}$	
$\boxed{1}$	$\boxed{1}$	doesn't have a 4–12

$$S = \left\{ (a,b) \;\middle|\; \begin{array}{l} a = 1,2,\dots,12 \\ b = 1,2,\dots,12 \end{array} \right\}$$

$$= \Big\{ \underbrace{(1,1)}_{\substack{\text{die } 1 = 1 \\ \text{die } 2 = 1}},\ \underbrace{(5,9)}_{\substack{\text{die } 1 = 5 \\ \text{die } 2 = 9}},\ \underbrace{(10,11)}_{\substack{\text{die } 1 = 10 \\ \text{die } 2 = 11}},\ \underbrace{\dots}_{\text{lots more}} \Big\}$$

So, $|S| = 12 \cdot 12 = 12^2 = 144$

Let E be the event that at least one of the dice is either 4, 5, 6, 7, 8, 9, 10, 11, or 12.

lots more

$E = \{(4,4), (6,1), (12,2), (11,12), \ldots\}$

Too hard to count E.

Let's count $\overline{E}$ which is the event that neither of the dice are 4, 5, 6, 7, 8, 9, 10, 11, or 12.

So, $\overline{E}$ is the event that both dice are in the range 1, 2, 3.

So,

$$\bar{E} = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$$

We have $|\bar{E}| = 9$.

Since each outcome is equally likely with usual 12-sided dice we have

$$P(\bar{E}) = \frac{|\bar{E}|}{|S|} = \frac{9}{144} = \frac{1}{16}$$

Thus,

$$P(E) = 1 - P(\bar{E}) = 1 - \frac{1}{16} = \boxed{\frac{15}{16}} \approx \boxed{0.9375}$$

$\boxed{\underline{\text{Thm}}\text{: } P(E) = 1 - P(\bar{E})}$