Theorem (Cauchy-Riemann Equations)

Suppose $f: A \to \mathbb{C}$ where $A \subseteq \mathbb{C}$ is an open set.
Let $f(x+iy) = u(x,y) + iv(x,y)$. Let $z_0 = x_0 + iy_0 \in A$.
If $f'(z_0)$ exists, then $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist at (x_0, y_0)
and they satisfy the Cauchy-Riemann equations:

$$\boxed{\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \quad \text{and} \quad \frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0)}$$

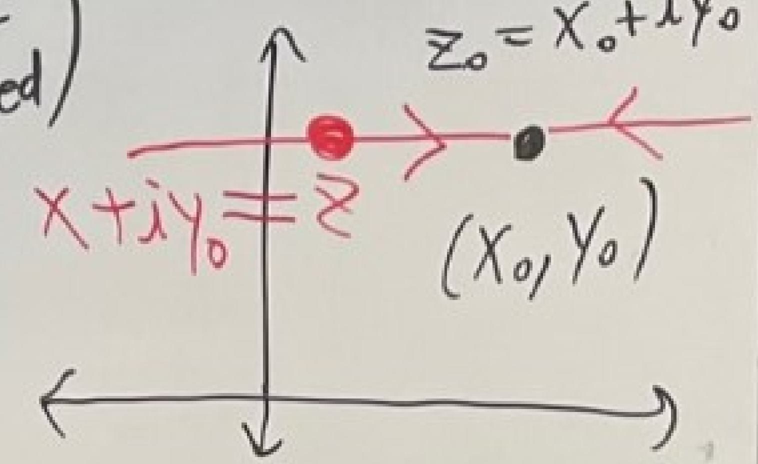
Moreover, $\boxed{f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0)}$

proof: Suppose $f'(z_0)$ exists.

Then, $f'(z_0) = \lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ exists.

Then we can approach z_0 in different directions and get $f'(z_0)$ for $\dfrac{f(z) - f(z_0)}{z - z_0}$ no matter what direction we approach it from.

Let's first approach in the x-direction. (ie only x will change the y_0 will stay fixed)



So,

$$f'(z_0) = \lim_{\substack{x+iy_0 \to x_0 + iy_0 \\ (x \to x_0)}} \frac{f(x+iy_0) - f(x_0 + iy_0)}{(x+iy_0) - (x_0 + iy_0)}$$

$$= \lim_{x \to x_0} \frac{u(x,y_0) + i\,v(x,y_0) - u(x_0,y_0) - i\,v(x_0,y_0)}{x - x_0}$$

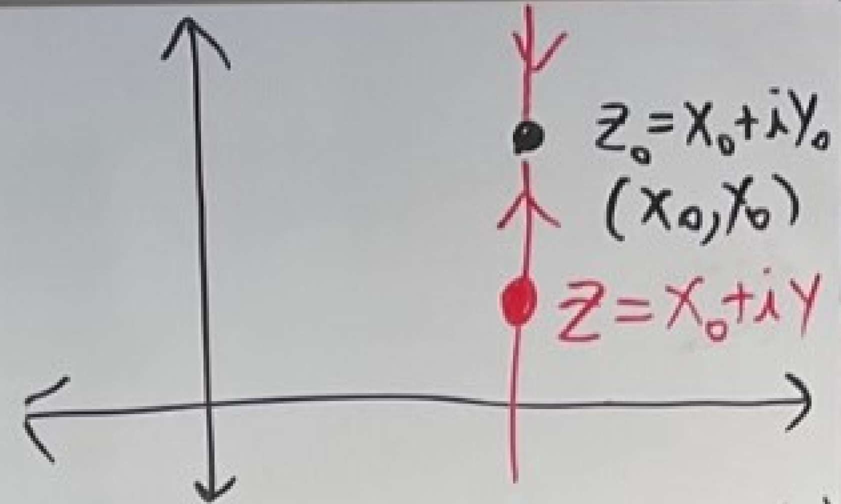
$$= \lim_{x \to x_0} \frac{u(x,y_0) - u(x_0,y_0)}{x - x_0} + i\left(\lim_{x \to x_0} \frac{v(x,y_0) - v(x_0,y_0)}{x - x_0}\right) = \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0)$$

Let's now approach in the y-direction

$$f'(z_0) = \lim_{y \to y_0} \frac{f(x_0 + iy) - f(x_0 + iy_0)}{(x_0 + iy) - (x_0 + iy_0)}$$

$[x_0 + iy \to x_0 + iy_0]$

$$\boxed{\frac{1}{i} = -i}$$



$$= \lim_{y \to y_0} \frac{u(x_0,y) + iv(x_0,y) - u(x_0,y_0) - iv(x_0,y_0)}{i(y-y_0)} = \lim_{y \to y_0} \frac{-iu(x_0,y) + v(x_0,y) + iu(x_0,y_0) - v(x_0,y_0)}{y - y_0}$$

$$= \lim_{y \to y_0} \frac{v(x_0,y) - v(x_0,y_0)}{y - y_0} - i \lim_{y \to y_0} \frac{u(x_0,y) - u(x_0,y_0)}{y-y_0} = \frac{\partial v}{\partial y}(x_0,y_0) - i \frac{\partial u}{\partial y}(x_0,y_0)$$

So, $f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i \frac{\partial v}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) + i\left(-\frac{\partial u}{\partial y}(x_0,y_0)\right)$

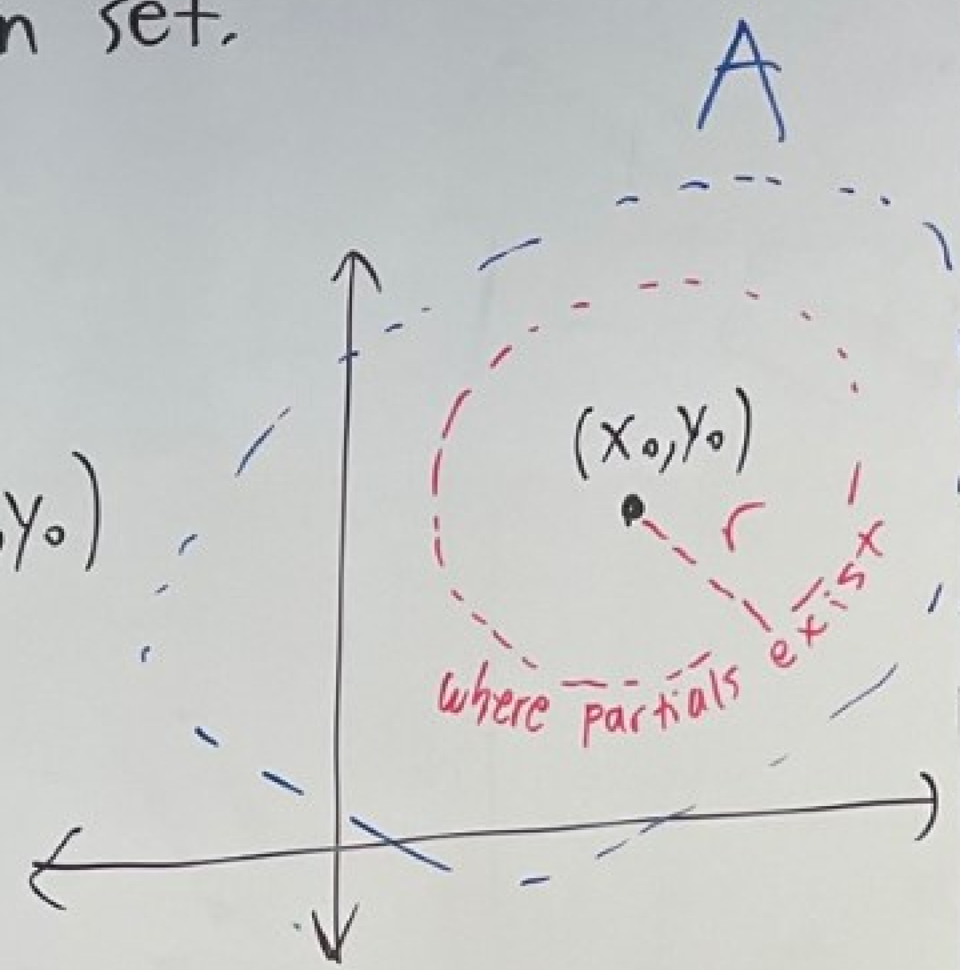
So, $\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0)$ and $\frac{\partial v}{\partial x}(x_0,y_0) = -\frac{\partial u}{\partial y}(x_0,y_0)$ ▨

Converse direction

Let $f: A \to \mathbb{C}$ where $A \subseteq \mathbb{C}$ is an open set.

Suppose $f(x+iy) = u(x,y) + iv(x,y)$.

Suppose $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist in some disk $D(\underbrace{(x_0,y_0)}_{x_0+iy_0}; r)$ and are continuous at (x_0,y_0)



If $\boxed{\begin{aligned} &\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \\ \text{and } &\frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0) \end{aligned}}$ Cauchy-Riemann equations

then $f'(z_0)$ exists where $z_0 = x_0 + iy_0$

Ex: Let $f(z) = z^2$. Where does f' exist? Find a formula for f'

$$f(x+iy) = (x+iy)^2 = \underbrace{(x^2-y^2)}_{u(x,y)=x^2-y^2} + i\,\underbrace{2xy}_{v(x,y)=2xy}$$

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= 2x \\ \frac{\partial u}{\partial y} &= -2y \\ \frac{\partial v}{\partial x} &= 2y \\ \frac{\partial v}{\partial y} &= 2x \end{aligned}}$$

- $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist for all (x,y) and are continuous for all (x,y)
- CR-equations:

  $\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$ for all (x,y)

So, $f'(z)$ exists for all $z = x+iy$ and

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + i(2y) = 2(x+iy) = 2z$$

Ex: Let $f(z) = \overline{z}$. Where does $f'(z)$ exist?

$f(x+iy) = x - iy$ ← $\boxed{u(x,y) = x \text{ and } v(x,y) = -y}$

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= 1 \\ \frac{\partial u}{\partial y} &= 0 \\ \frac{\partial v}{\partial x} &= 0 \\ \frac{\partial v}{\partial y} &= -1 \end{aligned}}$$

① $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist and are continuous for all (x,y)

② CR-equations

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= 1 \\ \frac{\partial v}{\partial y} &= -1 \end{aligned}\right\} \text{never equal} \qquad \left.\begin{aligned} \frac{\partial u}{\partial y} &= 0 \\ -\frac{\partial v}{\partial x} &= 0 \end{aligned}\right\} \text{equal everywhere}$$

CR-equations are not satisfied for any (x,y).
So, $f'(z)$ does not exist anywhere