Math 4650

9/8/25

Ex: Let $a_n = \frac{1}{n}$

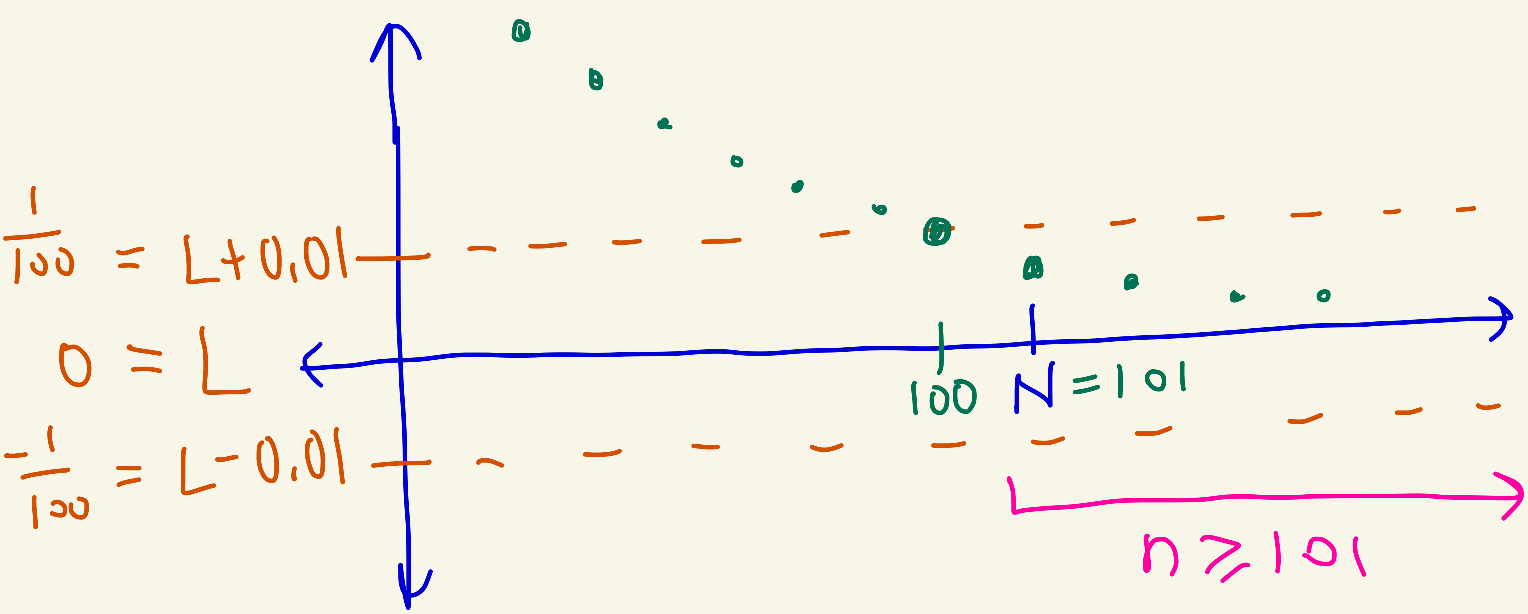
sequence: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$

We hypothesize that $\lim_{n \to \infty} \frac{1}{n} = 0$.

Before we prove this, let's calculate N for various ε to get a feel for the def of limit.

Let $\varepsilon = 0.01$.

$L = 0$



We want

$$\left|\frac{1}{n} - \underset{\substack{\uparrow \\ L=0}}{0}\right| < \underbrace{0.01}_{\varepsilon}$$

or

$$\frac{1}{n} < \frac{1}{100}$$

or

$$100 < n$$

Set $N = 101$.

if $n \geq N$, then $\left|\frac{1}{n} - 0\right| < \varepsilon$.

What if $\varepsilon = 0.00001 = \frac{1}{100000}$

Set $N = 100{,}001$.

If $n \geq N$, then

$$\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N} = \frac{1}{100{,}001} < \varepsilon$$

Claim: $\displaystyle\lim_{n \to \infty} \frac{1}{n} = 0.$

proof:

Let $\varepsilon > 0$.

We want to find N where if $n \geq N$, then $\left|\frac{1}{n} - 0\right| < \varepsilon$.

We have

$$\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| \underset{\boxed{n \geq 1}}{=} \frac{1}{n}$$

We want $\frac{1}{n} < \varepsilon$.

This is the same as $\frac{1}{\varepsilon} < n$.

Let N be a natural number with $\frac{1}{\varepsilon} < N$.

Then if $n \geq N$ we have

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} \underset{n \geq N}{\leq} \frac{1}{N} \underset{N > \varepsilon}{<} \varepsilon$$

Thus, if $n \geq N$, then

$$\left| \frac{1}{n} - 0 \right| < \varepsilon$$

So, $\lim\limits_{n \to \infty} \frac{1}{n} = 0$ $\blacksquare$

<u>Ex:</u> Let c be a constant.

Let's show $\lim_{n\to\infty} c = c$.

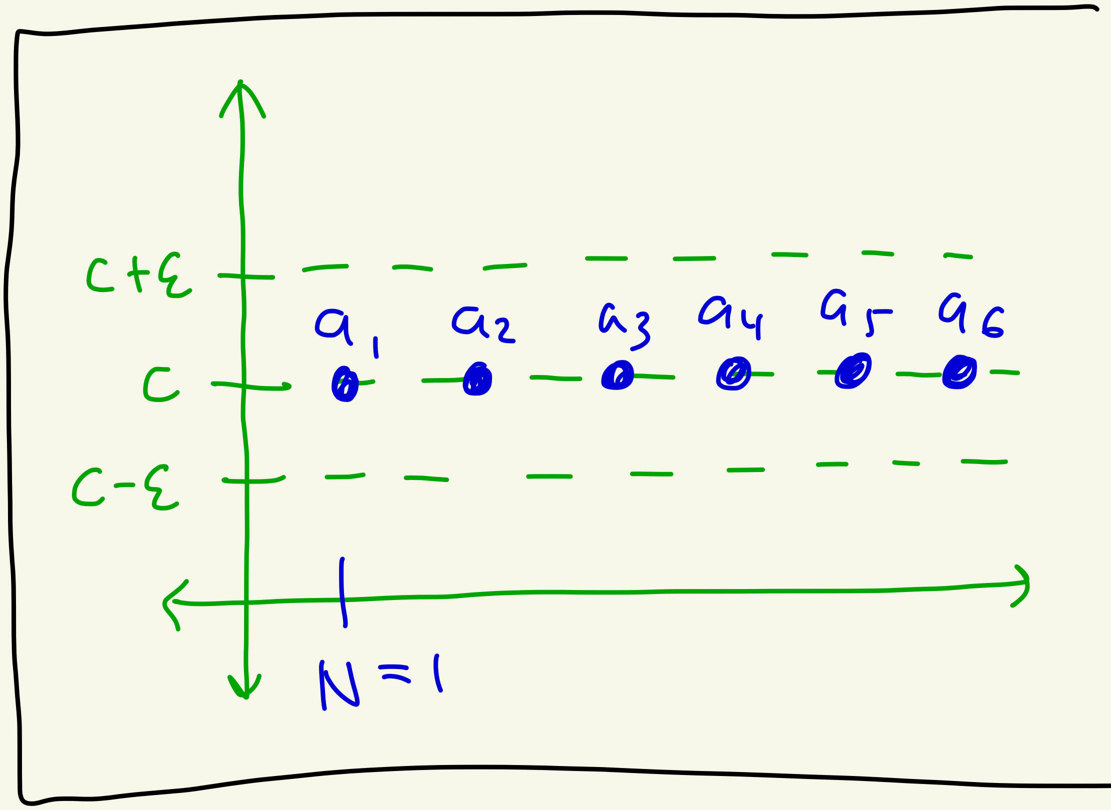
<u>proof:</u>

Let $\varepsilon > 0$.

Let

$$a_n = c$$

for all $n \geq 1$.



Set $N = 1$.

If $n \geq N$, then

$$|a_n - c| = |c - c| = 0 < \varepsilon$$

So, $\lim_{n\to\infty} a_n = c$. $\boxed{//}$

Ex: Let's show that

$$\lim_{n\to\infty} \frac{n}{n+1} = 1.$$

proof:

Let $\varepsilon > 0$.

Goal: We want to find N where if $n \geq N$ then $\left|\frac{n}{n+1} - 1\right| < \varepsilon$.

We have that

$$\left|\frac{n}{n+1} - 1\right| = \left|\frac{n-(n+1)}{n+1}\right|$$

$$= \left|\frac{-1}{n+1}\right|$$

$$= \frac{|-1|}{|n+1|}$$

$$= \frac{1}{n+1}$$

We want $\frac{1}{n+1} < \varepsilon$

This is when $\frac{1}{\varepsilon} - 1 < n$.

Pick $N > \frac{1}{\varepsilon} - 1$.

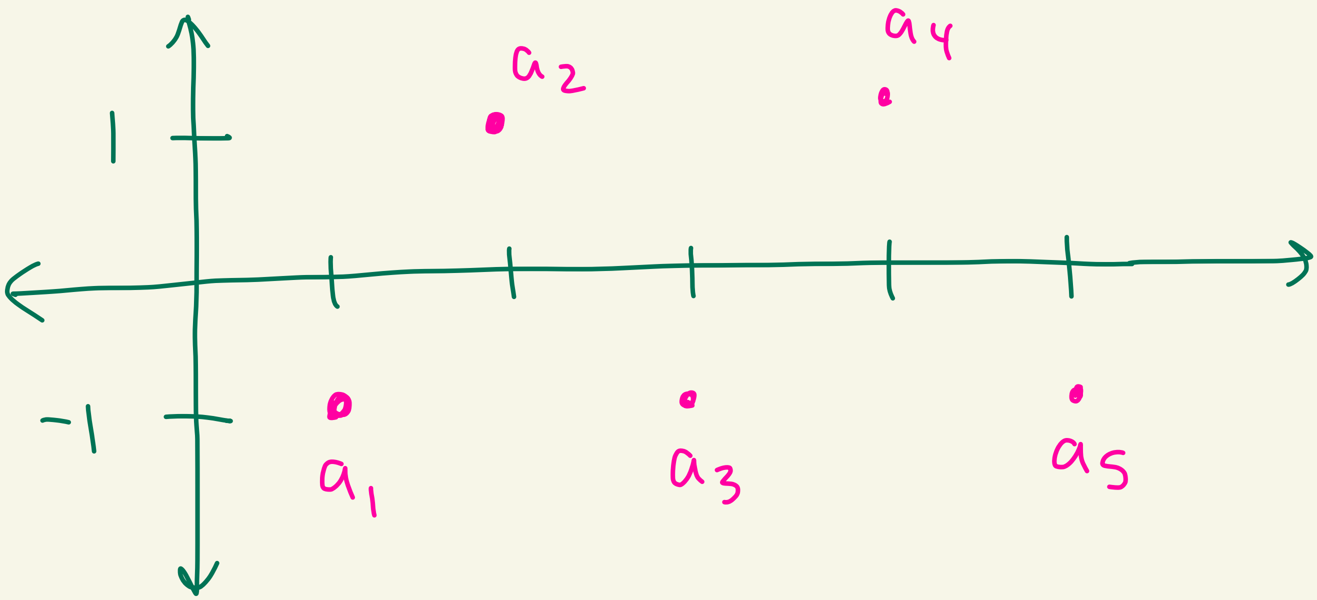
Then if $n \geq N$, then

$$\left| \frac{n}{n+1} - 1 \right| = \frac{1}{n+1} < \varepsilon$$

$$\boxed{\frac{1}{n+1} \leq \frac{1}{N+1} < \frac{1}{\left(\frac{1}{\varepsilon} - 1\right) + 1} = \frac{1}{\left(\frac{1}{\varepsilon}\right)} = \varepsilon}$$

So, $\lim_{n \to \infty} \frac{n}{n+1} = 1$. ▨

Ex: Consider $a_n = (-1)^n$.



We will show that $\lim\limits_{n\to\infty} (-1)^n$ does not exist.

We prove this by contradiction.

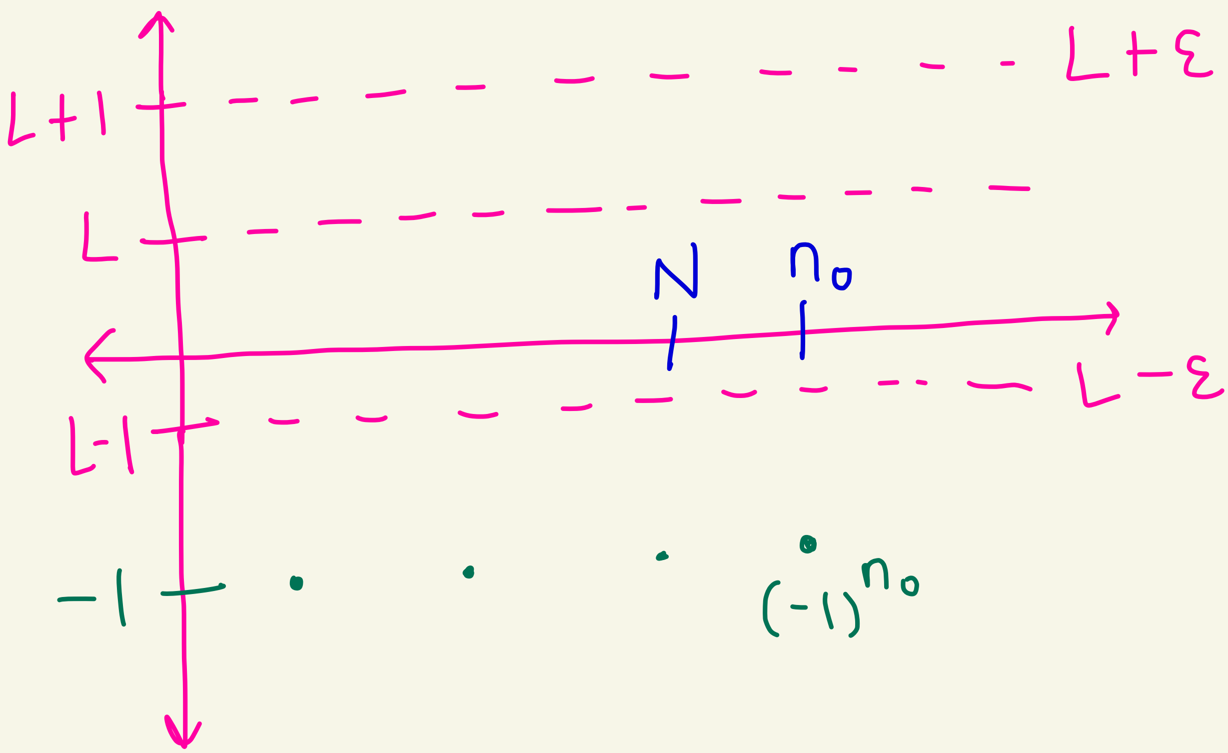
Suppose $\lim\limits_{n\to\infty} (-1)^n = L$ exists.

Let $\varepsilon = 1$.

Then since $\lim\limits_{n\to\infty} (-1)^n = L$ there must exist N where if

$n \geq N$ then $|(-1)^n - L| < \underbrace{1}_{\varepsilon = 1}$.

<u>Case 1:</u> Suppose $L \geq 0$.



Pick $n_0 \geq N$ with n_0 odd.

Then,

$$1 \underset{\boxed{\text{from above}}}{>} |(-1)^{n_0} - L| = |-1-L|$$

$$\underset{\boxed{\substack{L \geq 0 \\ -1-L<0}}}{=} -(-1-L) = 1+L$$

$$\geq 1+0 = 1$$

We get $1 > 1$.

Contradiction.

<u>Case 2:</u> Suppose $L < 0$.

Pick an even $n_e \geq N$.

Then,

$$1 > |(-1)^{n_e} - L| = |1 - L|$$

$$= 1 - L > 1 + 0 = 1.$$

from above

$1 - L > 0$ since $L < 0$

$L < 0$
$-L > 0$

So, $1 > 1$.

Contradiction.

In either case we get a contradiction. So, $\lim_{n\to\infty} (-1)^n$ does not exist. ☑

Theorem: Limits are unique.

That is, if $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} a_n = L_2$, then $L_1 = L_2$.

proof:

Let $\varepsilon > 0$.

Since $\lim_{n\to\infty} a_n = L_1$ there exists N_1 where if $n \geq N_1$, then $|a_n - L_1| < \varepsilon/2$.

Since $\lim_{n\to\infty} a_n = L_2$ there exists N_2 where if $n \geq N_2$ then $|a_n - L_2| < \varepsilon/2$.

Pick some $n_0 \geq \max\{N_1, N_2\}$.

That is, $n_0 \geq N_1$ and $n_0 \geq N_2$.

Then,

$$
\begin{aligned}
|L_1 - L_2| &= |L_1 - a_{n_0} + a_{n_0} - L_2| \\
&\leq |L_1 - a_{n_0}| + |a_{n_0} - L_2| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$

$\boxed{\begin{array}{l} \Delta\text{-inequality} \\ |x+y| \leq |x| + |y| \end{array}}$

So, $|L_1 - L_2| < \varepsilon$.

Since $\varepsilon > 0$ was arbitrary,

by HW this implies $|L_1 - L_2| = 0$.

So, $L_1 - L_2 = 0$

So, $L_1 = L_2$.